Proof of the Limits of Sine and Cosine at Infinity

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ABSTRACT

We develop a representation of complex numbers separate from the Cartesian and polar representations and define a representing functional for converting between representations. We define the derivative of a function of a complex variable with respect to each representation and then we examine the variation within the definition of the derivative. After studying the transformation law for the variation between representations of complex numbers, we show that the new representation has special properties which allow for a consistent modification to the transformation law for the variation which preserves the definition of the derivative. We refute a common proof that the limits of sine and cosine at infinity cannot exist. Then we use the newly defined modified variation in the definition of the derivative to compute the limits of sine and cosine at infinity.

Contents

1 Development of $\hat{\mathbb{C}}$

1.1 Properties of real numbers $\mathbb{R}$

Definition 1.1.1

A real number $x \in \mathbb{R}$ is a cut in the real number line.

Definition 1.1.2

A cut in a line separates one line into two pieces. For the real number line, a number separates all real numbers onto a set of "smaller" real numbers and a set of "larger" real numbers.

Theorem 1.1.3

All functions of the form

$$f \ : \ \mathbb{R} \ \ \rightarrow \ \ \mathbb{R} \ \ , \qquad \text{with} \qquad f(x) = mx + b \ \ , \ \ m, b \in \mathbb{R} \ \ , \ \ m \neq 0 \ \ ,$$

are one-to-one.

Proof:

We say f is a one-to-one function when

$$f(x_1) = f(x_2) \qquad \Longleftrightarrow \qquad x_1 = x_2 \ \ .$$

Evaluation of $f(x)$ at x_1 and x_2 yields

$$mx_1 + b = mx_2 + b \qquad \Longleftrightarrow \qquad x_1 = x_2 \ \ .$$

All such functions f are one-to-one.

1.2 Properties of extended real numbers $\overline{\mathbb{R}}$

Definition 1.2.1

The extended real numbers are

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 1.2.2

The ∞ symbol is such that for $n \in \mathbb{N}$ and $x_n > 0$

$$x_n \in \mathbb{R} \quad : \quad \lim_{n\to\infty} x_n = \text{diverges} \qquad \longrightarrow \qquad x_n \in \overline{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \infty \quad .$$

Definition 1.2.3

The additive absorptive properties of $\pm\infty$ are such that

$$\forall \quad b \in \mathbb{R} \quad \exists \quad \pm\infty \in \overline{\mathbb{R}} \quad : \quad \pm\infty + b = \pm\infty \quad .$$

Definition 1.2.4

The multiplicative absorptive properties of $\pm\infty$ are such that

$$\forall \quad b \in \mathbb{R} \quad , \quad b > 0 \quad \exists \quad \pm\infty \in \overline{\mathbb{R}} \quad : \quad \pm\infty \times b = \pm\infty \quad .$$

Definition 1.2.5

∞ does not have an additive inverse so

$$\infty - \infty = \text{undefined} \quad .$$

Definition 1.2.6

∞ does not have a multiplicative inverse so

$$\frac{\infty}{\infty} = \text{undefined} \quad .$$

Remark

Even while ∞ does not have the inverse composition properties of the real numbers, $\overline{\mathbb{R}}$ has the useful property that one may use numbers on both sides of divergent limits. ∞ is a special number that $\overline{\mathbb{R}}$ was conceived to accommodate.

Theorem 1.2.7

Not all functions of the form

$$f \; : \; \mathbb{R} \;\; \to \;\; \overline{\mathbb{R}} \;\; , \qquad\qquad \text{with} \qquad\qquad f(x) = mx + b \;\; , \;\; m, b \in \overline{\mathbb{R}} \;\; , \;\; m \neq 0 \;\; ,$$

are one-to-one.

Proof

To show a contradiction with the definition of a one-to-one function, consider $m = \infty$. By the absorptive properties of ∞

$$f(x_1) = \infty\, x_1 + b = \infty \;\; , \qquad\qquad \text{and} \qquad\qquad f(x_2) = \infty\, x_2 + b = \infty \;\; ,$$

but

$$\infty = \infty \qquad\qquad \not\Leftrightarrow \qquad\qquad x_1 = x_2 \;\; .$$

Therefore, functions of this type are not always one-to-one. We might show the same contradiction with $b = \infty$.

1.3 Properties of modified extended real numbers $\widehat{\mathbb{R}}$

Definition 1.3.1

Later we will show that a certain class of $\widehat{\mathbb{R}}$ are also $\mathbb{R}$ numbers (Main Theorem 1.3.10.) To distinguish what are usually called $\mathbb{R}$ numbers from this new class of $\mathbb{R}$ numbers, define $\mathbb{R}_0$ as real numbers which can be written without including an infinity symbol.

Definition 1.3.2

Modified extended real numbers are

$$\widehat{\mathbb{R}} \quad \equiv \quad \{\pm\widehat{\infty} + b : b \in \mathbb{R}_0, b \neq 0\} \quad .$$

They have the properties that

$$\forall \quad x \in \widehat{\mathbb{R}} \quad \exists \quad b \in \mathbb{R}_0 \quad , \quad b \neq 0 \qquad : \quad x = \pm\widehat{\infty} + b \quad ,$$

and

$$x_n \in \mathbb{R} \quad , \quad x > 0 \quad : \quad \lim_{n\to\infty} x_n = \text{diverges} \qquad \longrightarrow \qquad x_n \in \mathbb{R} \cup \{\widehat{\infty}\} \quad : \quad \lim_{n\to\infty} x_n = \widehat{\infty} \quad .$$

Definition 1.3.3

The hat symbol on $\widehat{\infty}$ is an instruction to delay the additive absorptive operation of ∞ as long as possible within the freedom allowed by the order of operations.

Remark

Numbers of the form

$$x = a\,\widehat{\infty} + b \quad , \qquad\qquad \text{with} \qquad\qquad a, b \in \mathbb{R}_0 \quad , \quad a \neq \pm 1 \quad , \quad b \neq 0 \quad ,$$

can be recast as $\widehat{\mathbb{R}}$ numbers by applying the multiplicative absorptive properties of $\widehat{\infty}$. To the contrary, numbers of the form

$$x = a\,\widehat{\infty} + b \quad , \qquad\qquad \text{with} \qquad\qquad b = 0 \quad ,$$

cannot be cast as $\widehat{\mathbb{R}}$ numbers.

Example 1.3.4

By delaying additive absorption we may define unique numbers of the form

$$x = \pm\widehat{\infty} + b \quad ,$$

but there are certain instances in which additive absorption cannot be postponed. Consider two series

$$x_n = \sum_{n=1}^{n} n \quad , \qquad\qquad \text{and} \qquad\qquad y_n = c_0 + \sum_{n=1}^{n} n \quad ,$$

where is c_0 is some non-zero real number. Applying the definition of $\widehat{\infty}$ (Definition 1.3.2) we obtain

$$\lim_{n\to\infty} x_n = \widehat{\infty} \quad , \qquad\qquad \text{and} \qquad\qquad \lim_{n\to\infty} y_n = \widehat{\infty} \quad ,$$

but we may also write y_n as

$$y_n = c_0 + x_n \quad .$$

Then we may take the limit as

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_0 + x_n = c_0 + \widehat{\infty} \quad .$$

This delivers a contradiction to implicit uniqueness of $\widehat{\mathbb{R}}$ numbers through

$$\widehat{\infty} = \widehat{\infty} + c_0 \quad .$$

In this instance, we take the instruction from Definition 1.3.3 and notice it is no longer possible to delay the additive absorption. Therefore, we must remove the hat such that

$$\widehat{\infty} = \widehat{\infty} + 1 \qquad\qquad \longrightarrow \qquad\qquad \infty = \infty + 1 \quad ,$$

and the contradiction is avoided by additive absorptive property of ∞ (Definition 1.2.3.)

Remark

Since we have removed the hat at the end of Example 1.3.4, there is no absolute need to replace Definition 1.2.2 with the limit property of Definition 1.3.2. We do so only for simplicity of notation: every infinity in $\widehat{\mathbb{R}}$ is $\pm\widehat{\infty}$. We might take two different infinites $\widehat{\infty}$ and ∞ such that the former is never additively absorptive and latter always is but for the purposes of finding the limits of sine and cosine at infinity it will be sufficient to use a single symbol $\widehat{\infty}$ with the instruction to delay additive absorption as long as possible. If is no contradiction is induced by avoiding the additive absorption operation altogether then the instruction in Definition 1.3.3 tells us to avoid it altogether.

Definition 1.3.5

$\widehat{\mathbb{R}}$ numbers are such that

$$x_n \in \overline{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \infty \qquad\qquad \longrightarrow \qquad\qquad x_n \in \widehat{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \text{diverges} \quad ,$$

because $\widehat{\infty} \notin \widehat{\mathbb{R}}$.

Remark

Since $\widehat{\infty} \notin \widehat{\mathbb{R}}$, the issue demonstrated by Example 1.3.4 cannot be a problem in the analysis of $\widehat{\mathbb{R}}$. If we wanted to infinitely continue $\widehat{\mathbb{R}} \to \overline{\widehat{\mathbb{R}}}$ in the fashion of $\mathbb{R} \to \overline{\mathbb{R}}$ such that

$$\widehat{\mathbb{R}} \;:\; \lim_{n\to\infty} x_n = \text{diverges} \qquad\qquad \longrightarrow \qquad\qquad \overline{\widehat{\mathbb{R}}} \;:\; \lim_{n\to\infty} x_n = \widehat{\infty} \quad ,$$

then we would mirror the extension of

$$\mathbb{R} \quad \to \quad \mathbb{R} \cup \{\pm\infty\} \quad ,$$

with

$$\{\pm\widehat{\infty} + b : b \in \mathbb{R}, b \neq 0\} \quad \to \quad \{\pm\widehat{\infty} + b : b \in \mathbb{R}\} \quad ,$$

where the case of $b = 0$ defines a special number $\widehat{\infty}$ without an additive inverse.

Definition 1.3.6

The operations $\widehat{\infty} - \widehat{\infty}$ and $\widehat{\infty}/\widehat{\infty}$ are undefined.

Definition 1.3.7

Since we have not defined a composition law for $\widehat{\infty} + b$, expressions of the form $\widehat{\infty} + b$ are defined by self-identity.

Definition 1.3.8

$\widehat{\mathbb{R}}$ is defined such that for any positive real numbers $a, b \in \mathbb{R}_0$

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad\qquad \Longleftrightarrow \qquad\qquad a < b \ \ .$$

Definition 1.3.9

The number $\widehat{\infty}$ is a cut in the modified extended real number line because it separates all modified extended real numbers into two groups as

$$\widehat{\infty} - b < \widehat{\infty} < \widehat{\infty} + b \ \ , \qquad\qquad \text{where} \qquad\qquad b \in \mathbb{R}_0 \ \ , \ \ \ b > 0 \ \ ,$$

and trivially

$$-\widehat{\infty} \pm b < \widehat{\infty} - b \ \ .$$

The numbers ∞ and $\widehat{\infty}$ are defined to be the same number.

Main Theorem 1.3.10

$\widehat{\mathbb{R}}$ numbers of the form

$$x = \big(\pm\widehat{\infty} \mp b\big) \ \ , \qquad\qquad \text{with} \qquad\qquad b \in \mathbb{R}_0 \ \ , \ \ \ b > 0 \ \ ,$$

are $\mathbb{R}$ numbers.

Proof

By Definition 1.3.8, $\widehat{\mathbb{R}}$ is defined such that for any $a, b > 0$ with $a, b \in \mathbb{R}_0$

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad\qquad \Longleftrightarrow \qquad\qquad a < b \;\; .$$

We can add a to both sides to derive

$$\widehat{\infty} > \big(\widehat{\infty} - b + a\big) \;\; , \qquad\qquad \text{and} \qquad\qquad a < b \quad \Longleftrightarrow \quad \big(b - a\big) > 0$$

Therefore, we see that numbers of the form

$$x = \big(\pm\widehat{\infty} \mp b\big) \;\; , \qquad\qquad \text{with} \qquad\qquad b \in \mathbb{R} \;\; , \;\; b > 0 \;\; ,$$

are cuts in the extended real number line $\overline{\mathbb{R}}$ which are not equal to infinity. By Definitions 1.1.1 and 1.1.2, all such x are in $\mathbb{R}$.)

Definition 1.3.11

The multiplicative absorptive properties of $\pm\widehat{\infty}$ are such that

$$\forall \quad b \in \mathbb{R}_0 \;\; , \;\; b > 0 \quad \exists \quad \pm\widehat{\infty} \in \overline{\mathbb{R}} \quad : \quad \pm\widehat{\infty} \times b = \pm\widehat{\infty} \;\; .$$

Definition 1.3.12

$\widehat{\mathbb{R}}$ numbers are such that

$$\widehat{\infty} + a = \widehat{\infty} + b \qquad\qquad \Longleftrightarrow \qquad\qquad a = b \;\; .$$

Definition 1.3.13

The additive composition law for $\widehat{\mathbb{R}} + \mathbb{R}_0$ is

$$\big(\pm\widehat{\infty} + a\big) + b = \pm\widehat{\infty} + \big(a + b\big) \;\; .$$

Definition 1.3.14

The additive composition laws for $\widehat{\mathbb{R}} + \widehat{\mathbb{R}}$ are

$$\big(\pm\widehat{\infty} + a\big) + \big(\pm\widehat{\infty} + b\big) = \pm 2\,\widehat{\infty} + \big(a+b\big) = \pm\widehat{\infty} + \big(a+b\big)$$

$$\big(\widehat{\infty} + a\big) + \big(-\widehat{\infty} + b\big) = a + b \quad ,$$

where

$$2\,\widehat{\infty} = \widehat{\infty} \quad ,$$

follows from the absorptive properties of $\widehat{\infty}$ (Definition 1.3.11.)

Definition 1.3.15

The additive composition laws for $\widehat{\mathbb{R}} \pm \widehat{\infty}$ are

$$\big(\pm\widehat{\infty} + a\big) \pm \widehat{\infty} = \big(\pm\widehat{\infty} + a\big)$$

$$\big(\pm\widehat{\infty} + a\big) \mp \widehat{\infty} = a \quad .$$

Remark

A good way to visualize modified extended real numbers is to write

$$x \in \mathbb{R}_0 \qquad\qquad \Longleftrightarrow \qquad\qquad x \quad \equiv \quad \widehat{0} + x \quad ,$$

where x measures distance from the origin $\widehat{0}$ with the hat as an instruction not to let 0 be absorbed. We may transfinitely extend the real number line to include the points at infinity and an interval beyond such that $\pm\widehat{\infty}$ are the origins of $\widehat{\mathbb{R}}$. By Definition 1.3.2 we have $\pm\widehat{\infty} \notin \widehat{\mathbb{R}}$ so they are not origins in the traditional sense but in analogy we have

$$x \in \widehat{\mathbb{R}} \qquad\qquad \Longleftrightarrow \qquad\qquad x \quad \equiv \quad \pm\widehat{\infty} + b \quad ,$$

where b measures distance from $\pm\widehat{\infty}$ located infinitely far away from the Cartesian origin $\widehat{0}$. In particular, this makes a lot of sense for the additive identity (Definition 1.3.14)

$$\big(\widehat{\infty} + a\big) + \big(\widehat{\infty} + b\big) = \widehat{\infty} + \big(a + b\big) \quad .$$

Also note that we have mentioned functions of the form

$$y(x) = mx + b \quad ,$$

because the function which shifts the origin $\widehat{0} \to \widehat{\infty}$

$$f \quad : \quad b \quad \to \quad \widehat{\infty} + b \quad ,$$

is a case of the same.

Theorem 1.3.16

The additive composition laws for $\widehat{\mathbb{R}}$ do not require an additive inverse for $\widehat{\infty}$.

Proof

Consider the additive composition of two $\widehat{\mathbb{R}}$ numbers

$$x_1 = \widehat{\infty} + b_1 \quad , \qquad \text{and} \qquad x_2 = -\widehat{\infty} + b_2 \quad ,$$

such that

$$x_1 + x_2 = 0 \quad .$$

The case of $b_1 = -b_2 = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$ (Definition 1.3.2.)

Theorem 1.3.17

All $\widehat{\mathbb{R}}$ numbers have an additive inverse.

Proof

Consider the case of $b = -a$ in the identity (Definition 1.3.14)

$$\big(\widehat{\infty} + a\big) + \big(-\widehat{\infty} + b\big) = a + b \quad .$$

Then

$$\forall \quad x = \widehat{\infty} + a \quad \exists \quad x' = -\widehat{\infty} - a \quad : \quad x + x' = 0 \quad .$$

This is the definition of the additive inverse.

Remark

Operations of the form $\widehat{\mathbb{R}} + \widehat{\infty} - \widehat{\infty}$ are undefined because $\widehat{\infty} - \widehat{\infty}$ is not defined. Add $\widehat{\infty}$ to both sides of the identity (Definition 1.3.15)

$$\big(\widehat{\infty} + a\big) - \widehat{\infty} = a \quad ,$$

to obtain

$$\big(\widehat{\infty} + a\big) - \widehat{\infty} + \widehat{\infty} = a + \widehat{\infty} \quad .$$

By adding the quantity in parentheses to either of $\pm\widehat{\infty}$ first, and then adding $\mp\widehat{\infty}$, we may obtain two different values

$$\Big[\big(\widehat{\infty} + a\big) - \widehat{\infty}\Big] + \widehat{\infty} = a + \widehat{\infty} \quad , \qquad \text{and} \qquad \Big[\big(\widehat{\infty} + a\big) + \widehat{\infty}\Big] - \widehat{\infty} = a \quad .$$

To the contrary of $\widehat{\mathbb{R}} + \widehat{\infty} - \widehat{\infty}$, expressions like $\big(\widehat{\mathbb{R}} + \widehat{\infty}\big) - \widehat{\infty}$ and $\big(\widehat{\mathbb{R}} - \widehat{\infty}\big) + \widehat{\infty}$ are perfectly well defined because the order of operations is specified by the bracketing.

Theorem 1.3.18

All functions of the form

$$f \quad : \quad \mathbb{R} \quad \rightarrow \quad \widehat{\mathbb{R}} \ , \qquad\qquad \text{with} \qquad\qquad f(x) = mx + b \ \ , \ \ m, b \in \mathbb{R} \cup \widehat{\mathbb{R}} \ \ , \ \ m \neq 0 \ \ ,$$

are one-to-one.

Proof

Consider $m = \widehat{\infty} + a_1$ and $b = \widehat{\infty} + a_2$. By the additive and multiplicative properties of $\widehat{\infty}$ we find that

$$f(x_1) = \big(\widehat{\infty} + a_1\big)x_1 + \big(\widehat{\infty} + a_2\big) = \big(\widehat{\infty} + a_1x_1\big) + \big(\widehat{\infty} + a_2\big) = \widehat{\infty} + \big(a_1x_1 + a_2\big)$$

$$f(x_2) = \big(\widehat{\infty} + a_1\big)x_2 + \big(\widehat{\infty} + a_2\big) = \big(\widehat{\infty} + a_1x_2\big) + \big(\widehat{\infty} + a_2\big) = \widehat{\infty} + \big(a_1x_2 + a_2\big) \quad .$$

By Definition 1.3.12, we have

$$\widehat{\infty} + \big(a_1x_1 + a_2\big) = \widehat{\infty} + \big(a_1x_2 + a_2\big) \qquad\qquad \Longleftrightarrow \qquad\qquad x_1 = x_2 \quad .$$

The case of $m, b \in \mathbb{R}$ was treated in Theorem 1.1.3, so we have shown that all such functions are one-to-one.

1.4 Properties of modified extended complex numbers $\widehat{\mathbb{C}}$

Definition 1.4.1

Complex numbers are

$$\mathbb{C} \quad \equiv \quad \{x + iy : x, y \in \mathbb{R}\} \quad .$$

Definition 1.4.2

Define a class of complex numbers

$$\mathbb{C}_0 \quad \equiv \quad \{x + iy : x, y \in \mathbb{R}_0\} \quad .$$

Definition 1.4.3

As ∞ and $\widehat{\infty}$ do not absorb -1, they do not absorb $\pm i$. We have four distinct compound symbols $\{\pm\widehat{\infty}, \pm i\,\widehat{\infty}\}$ when positive real infinity is written as $+\widehat{\infty}$.

Definition 1.4.4

Extended complex numbers are

$$\overline{\mathbb{C}} \quad \equiv \quad \{x + iy : x, y \in \overline{\mathbb{R}}\} \quad .$$

Definition 1.4.5

Modified extended complex numbers are such that

$$\widehat{\mathbb{C}} \quad \equiv \quad \{\widehat{\infty} \pm i\,\widehat{\infty} + Z, -\widehat{\infty} \pm i\,\widehat{\infty} + Z : Z \in \mathbb{C}_0, \mathrm{Im}(Z) \neq 0, \mathrm{Re}(Z) \neq 0\} \quad .$$

Definition 1.4.6

Infinitely continued modified extended complex numbers $\overline{\overline{\widehat{\mathbb{C}}}}$ are such that

$$\overline{\overline{\widehat{\mathbb{C}}}} \quad \equiv \quad \widehat{\mathbb{C}} \cup \{\mathrm{Im}(Z) = 0, \mathrm{Re}(Z) = 0\} \quad .$$

Definition 1.4.7

The additive composition laws for $\widehat{\mathbb{C}} + \mathbb{C}_0$ are

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) + z = \widehat{\infty} \pm i\,\widehat{\infty} + \big(Z + z\big)$$

$$\big(-\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) + z = -\widehat{\infty} \pm i\,\widehat{\infty} + \big(Z + z\big) \quad .$$

Definition 1.4.8

The additive composition laws for $\widehat{\mathbb{C}} \pm \widehat{\infty}$ and $\widehat{\mathbb{C}} \pm i\,\widehat{\infty}$ are

$$\left(\widehat{\infty} \pm i\,\widehat{\infty} + Z\right) \pm \widehat{\infty} = \widehat{\infty} \pm i\,\widehat{\infty} + Z$$

$$\left(\widehat{\infty} \pm i\,\widehat{\infty} + Z\right) \mp \widehat{\infty} = \pm i\,\widehat{\infty} + Z$$

$$\left(\widehat{\infty} \pm i\,\widehat{\infty} + Z\right) \pm i\,\widehat{\infty} = \widehat{\infty} \pm i\,\widehat{\infty} + Z$$

$$\left(\widehat{\infty} \pm i\,\widehat{\infty} + Z\right) \mp i\,\widehat{\infty} = \widehat{\infty} + Z \quad .$$

Remark

The additive properties of $\widehat{\mathbb{C}} + \widehat{\mathbb{C}}$ are implicit in the other composition laws.

Definition 1.4.9

For two modified extended complex numbers

$$z_1 = \widehat{\infty} + i\,\widehat{\infty} + Z_1 \quad , \qquad \text{and} \qquad z_2 = \widehat{\infty} + i\,\widehat{\infty} + Z_2 \quad ,$$

we have

$$z_1 = z_2 \qquad \Longleftrightarrow \qquad Z_1 = Z_2 \quad .$$

Definition 1.4.10

Any sequence of the form

$$z_n \in \mathbb{C} \quad , \quad z_n = x_n + iy_n \quad , \qquad \text{with} \qquad x_n, y_n \in \mathbb{R} \quad , \quad x_n, y_n > 0 \quad ,$$

is such that

$$z_n \in \mathbb{C} \quad : \quad \begin{cases} \lim\limits_{n\to\infty} x_n = \text{diverges} \\ \\ \lim\limits_{n\to\infty} y_n = \text{diverges} \end{cases} \quad , \qquad \longrightarrow \qquad z_n \in \overline{\widehat{\mathbb{C}}} \quad : \quad \lim_{n\to\infty} z_n = \widehat{\infty} + i\,\widehat{\infty} \quad .$$

Corollary 1.4.11

$\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on the Riemann sphere $\mathbb{S}^2$.

Proof

A is the complement of B on $\mathbb{S}^2$ if and only if

$$\mathbb{S}^2 \quad \equiv \quad A \cup B \quad , \qquad\qquad \text{and} \qquad\qquad A \cap B = 0 \quad .$$

The Riemann sphere is obtained from $\mathbb{C}$ by adding a point for infinity to both ends of the real and imaginary axes and then imposing conditions

$$\pm\infty = \infty \quad , \qquad\qquad \text{and} \qquad\qquad \pm\, i\infty = \infty \quad .$$

When we say, "$\widehat{\mathbb{C}}$ on the Riemann sphere," we mean the limit of $\widehat{\mathbb{C}}$ where $\pm\widehat{\infty}, \pm i\,\widehat{\infty} \to \infty$. Considering the usual association between the Riemann sphere and the extended complex numbers $\overline{\mathbb{C}}$ (Definition 1.4.4)

$$\overline{\mathbb{C}} \quad \equiv \quad \{x + iy : x, y \in \overline{\mathbb{R}}\} \quad .$$

it follows that

$$\mathbb{S}^2 \quad \equiv \quad \{\overline{\mathbb{C}} : \pm\infty \to \infty, \pm i\infty \to \infty\} \quad .$$

Imposing these conditions on $\widehat{\mathbb{C}}$ as defined by Definition 1.4.5, namely

$$\widehat{\mathbb{C}} \quad \equiv \quad \{\widehat{\infty} \pm i\,\widehat{\infty} + Z, -\widehat{\infty} \pm i\,\widehat{\infty} + Z : Z \in \mathbb{C}_0, \mathrm{Im}(Z) \neq 0, \mathrm{Re}(Z) \neq 0\} \quad ,$$

gives

$$\{\widehat{\mathbb{C}} : \pm\widehat{\infty} \to \infty, \pm i\,\widehat{\infty} \to \infty\} \quad \equiv \quad \{\infty\} \ .$$

Since it is the definition of the Riemann sphere that

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\infty\} \ ,$$

we can use the definition of the complement to write

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\widehat{\mathbb{C}} : \pm\widehat{\infty} \to \infty, \pm i\,\widehat{\infty} \to \infty\} \ , \qquad \text{and} \qquad \mathbb{C} \cap \{\infty\} = 0 \ .$$

Therefore, $\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on the Riemann sphere.

1.5 Properties of modified complex numbers $\hat{\mathbb{C}}$

Definition 1.5.1

Modified complex numbers $\hat{\mathbb{C}}$ shall be such that for every $z' \in \mathbb{C}_0$ of the form

$$z' = x + iy \ ,$$

there is a corresponding number $z \in \hat{\mathbb{C}}$. Modified complex numbers are defined as

$$z = \begin{cases} x + iy^+ & \text{if} \quad y > 0 \\ x & \text{if} \quad y = 0 \\ x - iy^- & \text{if} \quad y < 0 \end{cases} \ , \qquad \text{where} \qquad y^{\pm}(y) \ : \ \mathbb{R}_0 \ \to \ \widehat{\mathbb{R}} \ ,$$

with

$$y^+(y) = \widehat{\infty} - y \ , \qquad \text{and} \qquad y^-(y) = \widehat{\infty} + y \ .$$

Theorem 1.5.2

$\hat{\mathbb{C}}$ numbers are such that

$$z \in \hat{\mathbb{C}} \quad , \quad z = x \pm iy^{\pm} \qquad \Longrightarrow \qquad 0 < y^{\pm} < \infty \quad .$$

Proof

Definition 1.3.14 gives

$$||\widehat{\infty}|| = \infty \quad .$$

$y^{\pm}$ are such that

$$y^{\pm} \;:\; \mathbb{R}_0 \quad \to \quad \widehat{\mathbb{R}} \quad , \qquad \text{and} \qquad y^{\pm} = \big(\widehat{\infty} \mp y\big) \quad , \quad y \in \mathbb{R}_0 \quad .$$

By the definition of $\widehat{\mathbb{R}}$, $y^{\pm} = \widehat{\infty}$ and $y^{\pm} = 0$ are not allowed. For any $a, b \in \mathbb{R}_0$ with $a, b > 0$ we have from Definition 1.3.8

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad \Longleftrightarrow \qquad a < b \quad ,$$

wherein $a, b > 0$ is required by the restriction of the domain of $y^{\pm}(y)$ in $z = x + iy^{\pm}$ (Definition 1.5.1). This shows that $y^{\pm}$ increases as $||y||$ decreases. Therefore,

$$\sup y^{\pm} = y^{\pm}(\inf ||y||) \quad .$$

$y \in \mathbb{R}_0$ gives

$$\inf ||y|| = 0 \qquad \Longrightarrow \qquad \sup y^{\pm} = \widehat{\infty} - 0 = \widehat{\infty} \quad .$$

$y^{\pm} \in \widehat{\mathbb{R}}$ and $\widehat{\infty} \notin \widehat{\mathbb{R}}$. To show that $y^{\pm}$ is always greater than zero, consider that

$$\forall \quad b \in \mathbb{R} \quad , \quad \widehat{\infty} > b \qquad \Longrightarrow \qquad \widehat{\infty} - b > 0 \quad .$$

Remark

When we use

$$z \in \mathbb{C} \qquad \Longrightarrow \qquad z = re^{i\theta} \quad , \quad r, \theta \in \mathbb{R} \quad ,$$

where

$$r(x,y) = \sqrt{x^2 + y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \quad ,$$

we do not need to define an entire new class of analysis with some variant of $\mathbb{C}$, call it $\mathbb{C}'$, to distinguish it from

$$z \in \mathbb{C} \qquad \Longrightarrow \qquad z = x + iy \quad , \quad x, y \in \mathbb{R} \quad .$$

In $\widehat{\mathbb{C}}$, we did not add the point at infinity to $\mathbb{C}$ but we did take away the points along the real and imaginary axes of $\mathbb{C}$ because $y^{\pm}(\pm\widehat{\infty}) = \widehat{\infty} - \widehat{\infty}$ is undefined. Therefore, a unique construction requires a unique label. With regards to $\hat{\mathbb{C}}$, however, we have neither added the point at infinity nor taken away any points so there is an argument to be made that

$$\hat{\mathbb{C}} \quad \equiv \quad \mathbb{C} \quad .$$

2 Properties of $\mathbb{C}$

2.1 Definition of a representation of complex numbers $\mathbb{C}$

Definition 2.1.1

$(\!(x,y)\!)$ is the Cartesian representation of $\mathbb{C}$ in which

$$z(x,y) = x + iy \quad .$$

We say

$$(\!(x,y)\!) \quad \equiv \quad z(x,y) \quad \equiv \quad x + iy \quad .$$

$z(x, y)$ is a complex valued function of two real variables and we call $x+iy$ the analytic form of the Cartesian representation of $\mathbb{C}$.

Definition 2.1.2

$(\!(x_2, y_2)\!)$ is a representation of $\mathbb{C}$ if and only if $(\!(x_1, y_1)\!)$ is a representation of $\mathbb{C}$ and there exist two conversion functions

$$x_2 = x_2(x_1, y_1) \quad , \qquad \text{and} \qquad y_2 = y_2(x_1, y_1) \quad ,$$

whose domains are all of $\mathbb{C}$.

Theorem 2.1.3

$(\!(r, \theta)\!)$ is a representation of $\mathbb{C}$.

Proof

$(\!(x, y)\!)$ is a representation of $\mathbb{C}$ and we have two conversion functions

$$r(x, y) = \sqrt{x^2 + y^2} \quad , \qquad \text{and} \qquad \theta(x, y) = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if} \quad x \neq 0 \\ \dfrac{\pi}{2} & \text{if} \quad x = 0 \quad , \quad y > 0 \\ -\dfrac{\pi}{2} & \text{if} \quad x = 0 \quad , \quad y < 0 \end{cases} \quad .$$

$(\!(r, \theta)\!)$ is a representation of $\mathbb{C}$ because all of $\mathbb{C}$ is in the domain of the conversion functions.

Remark

Due to the freedom to choose the sign of $\sqrt{y^2}$ we might add a rule to the definition of $\theta(x, y)$ to be more explicit and we might even define different representations for the sign permutations.

Definition 2.1.4

If $(\!(x_1, y_1)\!)$ and $(\!(x_2, y_2)\!)$ are two representations of $\mathbb{C}$ then there exists a representing functional of two conversion functions

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)] \quad : \quad (\!(x_1, y_1)\!) \quad \rightarrow \quad (\!(x_2, y_2)\!) \quad ,$$

where

$$x_1(x_2, y_2) \;:\; (\mathbb{R}, \mathbb{R}) \quad \rightarrow \quad \mathbb{R} \quad , \qquad\qquad \text{and} \qquad\qquad y_1(x_2, y_2) \;:\; (\mathbb{R}, \mathbb{R}) \quad \rightarrow \quad \mathbb{R} \quad ,$$

are the two conversion functions.

Remark

The purpose of the representing functional is to convert the analytic form of one representation into the analytic form of another.

Definition 2.1.5

The representing functional has two conversion functions in its domain, each a function of two real variables.

Definition 2.1.6

For a representing functional $z_B[A]$, A is the known analytic representation and B is the analytic form of the representation into which we convert.

Definition 2.1.7

For a representing functional $z_B[A]$, the conversion functions must be written in the form $x_A(x_B, y_B)$ and $y_A(x_B, y_B)$.

Definition 2.1.8

The rules for constructing the representing functional with the conversion functions $x_1(x_2, y_2)$ and $y_1(x_2, y_2)$ are:

- Choose a representing functional

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)]$$

- Replace the known representation $(\!(x_1, y_1)\!)$ with its analytic form

$$z_{(\!(x_2,y_2)\!)}[z(x_1, y_1)]$$

- Replace the known real variables x_1 and y_1 with their conversion functions

$$z_{(\!(x_2,y_2)\!)}[z(x_1(x_2, y_2), y_1(x_2, y_2))]$$

- Simplify in terms of x_2 and y_2 to get the analytic form of

$$(\!(x_2, y_2)\!)$$

Example 2.1.9

Here we use the representing functional

$$z_{(\!(r,\theta)\!)}[(\!(x, y)\!)] = (\!(r, \theta)\!) \quad ,$$

to construct the polar representation of $\mathbb{C}$ from its Cartesian representation. The conversion functions are

$$x(r, \theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r, \theta) = r\sin(\theta) \quad .$$

The representing functional is

$$z_{(\!(r,\theta)\!)}[(\!(x, y)\!)] = z_{(\!(r,\theta)\!)}[x + iy] = r\cos(\theta) + ir\sin(\theta) = re^{i\theta} \quad .$$

Therefore,

$$(\!(r,\theta)\!) = re^{i\theta} \quad .$$

Example 2.1.10

Here we use the representing functional

$$z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] = (\!(x,y)\!) \quad .$$

to construct the Cartesian representation of $\mathbb{C}$ from its polar representation. The conversion functions are

$$r(x,y) = \sqrt{x^2+y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\frac{y}{x} \quad .$$

For $x \neq 0$, the representing functional is

$$\begin{aligned}
z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] &= z_{(\!(x,y)\!)}[re^{i\theta}] \\
&= \sqrt{x^2+y^2}\, e^{i\tan^{-1}(y/x)} \\
&= \sqrt{x^2+y^2}\, \cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + i\sqrt{x^2+y^2}\, \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \sqrt{x^2+y^2}\left(\frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) + i\sqrt{x^2+y^2}\left(\frac{\left(\frac{y}{x}\right)}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) \\
&= x+iy \quad .
\end{aligned}$$

For $x = 0$ the representing functional is

$$z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] = z_{(\!(x,y)\!)}[re^{i\theta}] = \sqrt{x^2+y^2}\, e^{\pm i\pi/2} = \pm i\sqrt{y^2}$$

Within the freedom to choose the positive or negative root of y^2 we take

$$\pm i\sqrt{y^2} = iy \quad ,$$

and, therefore,

$$(\!(x, y)\!) = x + iy \quad .$$

Remark

The polar representation relies on incorporation of the number e so we should consider other representations that include different numbers such as $\widehat{\infty}$.

Definition 2.1.11

If we have a representation

$$(\!(f(x_2), g(y_2))\!) = z(x_2, y_2) \quad ,$$

then the rules for constructing

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1, y_1)\!)] = (\!(f(x_2), g(y_2))\!) \quad ,$$

are:

- Choose a representing functional
$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1, y_1)\!)]$$
- Replace the known representation $(\!(x_1, y_1)\!)$ with its analytic form
$$z_{(\!(f(x_2),g(y_2))\!)}[z(x_1, y_1)]$$
- Map the x_1 and y_1 variables through the functions $f(x)$ and $g(x)$ respectively
$$z_{(\!(f(x_2),g(y_2))\!)}[z(f(x_1), g(y_1))]$$

- Replace the known real variables x_1 and y_1 with their conversion functions

$$z_{(\!(f(x_2),g(y_2))\!)}[z(f(x_1(x_2,y_2)),g(y_1(x_2,y_2)))]$$

- Simplify in terms of x_2 and y_2 to get the analytic form of

$$(\!(f(x_2),g(y_2))\!)$$

Theorem 2.1.12

The representation of $\mathbb{C}$ corresponding to $\hat{\mathbb{C}}$ is

$$(\!(x_2,\{\emptyset,\pm\widehat{\infty}-y^{\pm}\})\!)=z(x_2,\{0,y^{\pm}\})\quad ,$$

with Cartesian conversion functions

$$x(x_2,\{0,y^{\pm}\})=x_2\quad ,\qquad \text{and}\qquad y(x_2,\{0,y^{\pm}\})=\begin{cases}\widehat{\infty}-y^{+} & \text{if}\quad \mathrm{Im}(z)>0\\ 0 & \text{if}\quad \mathrm{Im}(z)=0\\ \widehat{\infty}+y^{-} & \text{if}\quad \mathrm{Im}(z)<0\end{cases}\quad .$$

Proof

All of $\mathbb{C}$ is in the domain of these functions. $\hat{\mathbb{C}}$ is piecewise defined so it suffices to show that the pieces satisfy the definitions. For $(\!(x,\emptyset)\!)$ we have conversion functions

$$x(x_2,0)=x_2\quad ,\qquad \text{and}\qquad y(x_2,0)=0\quad ,$$

such that Definition 2.1.8 gives

$$z_{(\!(x_2,\emptyset)\!)}[(\!(x,y)\!)]=z_{(\!(x_2,\emptyset)\!)}[x+iy]=x(x_2,0)+iy(x_2,0)=x_2\quad .$$

Therefore,

$$(\!(x_2, \emptyset)\!) = x_2 \quad , \qquad \text{where} \qquad x_2 \quad \equiv \quad x \quad .$$

For $(\!(x_2, \widehat{\infty} - y^+)\!)$ we have

$$f(x) = x \quad , \qquad \text{and} \qquad g(y) = \widehat{\infty} - y \quad .$$

with conversion functions

$$x(x_2, y^+) = x_2 \quad , \qquad \text{and} \qquad y(x_2, y^+) = \widehat{\infty} - y^+ \quad .$$

Definition 2.1.11 gives

$$\begin{aligned}
z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)] &= z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[x+iy] \\
&= z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[f(x)+ig(y)] \\
&= z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[x+i\big(\widehat{\infty}-y\big)] \\
&= x(x_2,y^+)+i\big(\widehat{\infty}-y(x_2,y^+)\big) \\
&= x_2+i\big[\widehat{\infty}-\big(\widehat{\infty}-y^+\big)\big]
\end{aligned}$$

Since $y^+ \notin \mathbb{R}$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number and the quantity in square brackets is not formatted for an additive composition $\widehat{\infty} - \widehat{\mathbb{R}}$. Substitute $y^+ = \widehat{\infty} - y$ (Definition 1.5.1) so that

$$z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)] = x_2 + i\big\{\widehat{\infty} - \big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big]\big\} \quad .$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\mathbb{R}} + \widehat{\infty}$ so

$$z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)] = x_2 + i\big(\widehat{\infty} - y\big) = x_2 + iy^+ \quad .$$

Therefore,

$$(\!(x_2, \widehat{\infty} - y^+)\!) = x_2 + iy^+ \quad .$$

The final case is $(\!(x_2, -\widehat{\infty} - y^-)\!)$. We have

$$f(x) = x \quad , \qquad\qquad \text{and} \qquad\qquad g(y) = -\widehat{\infty} - y \quad .$$

with conversion functions are

$$x(x_2, y^-) = x_2 \quad , \qquad\qquad \text{and} \qquad\qquad y(x_2, y^-) = y^- - \widehat{\infty} \quad .$$

Definition 2.1.11 gives

$$\begin{aligned}
z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] &= z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[x+iy] \\
&= z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[f(x)+ig(y)] \\
&= z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[x+i\big(-\widehat{\infty}-y\big)] \\
&= x(x_2,y^-)+i\big(-\widehat{\infty}-y(x_2,y^-)\big) \\
&= x_2+i\big[-\widehat{\infty}-\big(y^- - \widehat{\infty}\big)\big] \quad .
\end{aligned}$$

Since $y^- \notin \mathbb{R}$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number and the quantity in square brackets is not formatted for an additive composition $\widehat{\infty} - \widehat{\mathbb{R}}$. Substitute $y^- = \widehat{\infty} + y$ (Definition 1.5.1) so that

$$z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] = x_2 + i\Big\{-\widehat{\infty} - \Big[\big(\widehat{\infty}+y\big) - \widehat{\infty}\Big]\Big\} \quad .$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\mathbb{R}} + \widehat{\infty}$ so

$$z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] = x_2 + i\big(-\widehat{\infty}-y\big) = x_2 - i\big(\widehat{\infty}+y\big) = x_2 - iy^- \quad .$$

Therefore,

$$(\!(x_2, -\widehat{\infty} - y^-)\!) = x_2 - iy^- \quad .$$

We have proven that

$$(\!(x_2, \{\emptyset, \pm\widehat{\infty} - y^\pm\})\!) = \begin{cases} x_2 + iy^+ & \text{for} \quad \mathrm{Im}(z) > 0 \\ x_2 & \text{for} \quad \mathrm{Im}(z) = 0 \\ x_2 - iy^- & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad ,$$

and the theorem follows from $x \equiv x_2$.

Example 2.1.13

In this example we show a case in which the representing functional correctly recovers the Cartesian representation from the $\hat{\mathbb{C}}$ representation. The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

and the representing functional is

$$\begin{aligned} z_{(\!(x,y)\!)}\big[(\!(x, \widehat{\infty} - y^+)\!)\big] &= z_{(\!(x,y)\!)}\big[x + i\big(\widehat{\infty} - y^+\big)\big] \\ &= x(x,y) + i\big(\widehat{\infty} - y^+(x,y)\big) \\ &= x + i\big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big] \\ &= x + iy \quad . \end{aligned}$$

We have shown that the representing functional takes the $\hat{\mathbb{C}}$ representation and returns the Cartesian representation.

Remark

At this point, the reader hopefully is asking, "What is this convoluted notation for?" We introduce the rigorous representation to quantify what we mean by phrases like "the Cartesian representation of $\mathbb{C}$," "the polar representation of $\mathbb{C}$," or even "the $\hat{\mathbb{C}}$ representation of $\mathbb{C}$." For instance, we might wish to state precisely that the conversion functions of the Cartesian representation to the polar representation are analytic but the conversion functions of the Cartesian representation to the $\hat{\mathbb{C}}$ representation are one-to-one.

Corollary 2.1.14

As an illustration of the high significance of conversion functions, consider the Gaussian integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \quad .$$

This integral is analytically intractable in the Cartesian representation of $\mathbb{C}$ (except by quadrature) but it can be solved easily in the polar representation. We write canonically

$$I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \times \int_{-\infty}^{\infty} dx\, e^{-x^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)} \quad ,$$

and then insert the conversion function

$$r(x, y) = \sqrt{x^2 + y^2} \quad .$$

We obtain the infinitesimal element of polar area from the conversion functions

$$x(r, \theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad x(r, \theta) = r\cos(\theta) \quad ,$$

via

$$dx = \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial \theta}\, d\theta \quad , \qquad \text{and} \qquad dy = \frac{\partial y}{\partial r}\, dr + \frac{\partial y}{\partial \theta}\, d\theta \quad .$$

Then

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, re^{-r^2} \qquad \Longleftrightarrow \qquad I(z) = \sqrt{\pi} \quad .$$

2.2 Definition of the representational derivative d/dz_1

Remark

To prove the limits of sine and cosine at infinity, we will use the definition of the derivative. First, we will compare the conventions for derivatives with respect to

$$z = x + iy \quad , \qquad \text{and} \qquad z = re^{i\theta} \quad ,$$

and then we will define derivatives with respect to the cases of

$$z = \begin{cases} x + iy^+ & \text{for} \quad \mathrm{Im}(z) > 0 \\ \\ x & \text{for} \quad \mathrm{Im}(z) = 0 \\ \\ x - iy^- & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad .$$

We will use the definition of the representation to increase the specificity of the distinctions that we will make.

Definition 2.2.1

The forward derivative of a complex-valued function is

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad ,$$

where

$$\Delta z = z + \big(h - z\big) \quad , \qquad \text{and} \qquad h \in \mathbb{C} \quad .$$

Theorem 2.2.2

The function $f(z) = e^z$ is an eigenfunction of the d/dz operator with unit eigenvalue.

Proof

Using the definition of the derivative we find that

$$\frac{d}{dz}\,e^z = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{x+iy+\Delta x+i\Delta y} - e^{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{e^{x+iy+i\Delta y} - e^{x+iy}}{i\Delta y}$$

$$\stackrel{*}{=} \lim_{\Delta y \to 0} \frac{ie^{x+iy+i\Delta y}}{i}$$

$$= e^z \quad .$$

(The $\stackrel{*}{=}$ symbol denotes an application of L'Hôpital's rule.)

Remark

The derivatives with respect to the polar and Cartesian representations are

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

and

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

with

$$\Delta z = z + \big(h - z\big) \quad , \qquad \text{and} \qquad h \in \mathbb{C} \quad .$$

There is usually no distinguishing between the two distinct instances of d/dz. We will be doing some tricks with these distinctions so it will be useful to distinguish the derivative with respect the each individual representation of complex numbers.

Definition 2.2.3

The representational derivative

$$\frac{d}{dz_1} f(z_2) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_2 + \Delta z_2) - f(z_2)}{\Delta z_2}$$

is such that the variables of the z_1 representation appear in the limit while the variables of z appear in the limiting function. For instance, when $\hat{\mathbb{C}}$ is a representation of $\mathbb{C}$ even while $(\!(x, \emptyset)\!)$, $(\!(x, \pm\widehat{\infty} - y^{\pm})\!)$ are individually not, we have

$$\frac{d}{dz} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad , \qquad \text{for} \qquad z(x, y) = x + iy$$

$$\frac{d}{dz'} f(z') = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z' + \Delta z') - f(z')}{\Delta z'} \quad , \qquad \text{for} \qquad z'(r, \theta) = re^{i\theta}$$

$$\frac{d}{dz^+} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad , \qquad \text{for} \qquad z^+(x, y^+) = x + iy^+$$

$$\frac{d}{dz^-} f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad , \qquad \text{for} \qquad z^-(x, y^-) = x - iy^-$$

$$\frac{d}{dz^\emptyset} f(z^\emptyset) = \lim_{\Delta x \to 0} \frac{f(z^\emptyset + \Delta z^\emptyset) - f(z^\emptyset)}{\Delta z^\emptyset} \quad , \qquad \text{for} \qquad z^\emptyset(x, \emptyset) = x \quad .$$

We will continue to use the labeling conventions on the right to describe the five main representations of $\mathbb{C}$: Cartesian, Polar, and the three pieces of the $\hat{\mathbb{C}}$ representation.

2.3 Definition of the representational variation Δz_1

Definition 2.3.1

The variation of a $\mathbb{C}$ number in the definition of the representational derivative is

$$\Delta z_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h_1 \in \mathbb{C} \quad , \qquad \text{where} \qquad h \to 0 \quad .$$

The variation with respect to each representation has its own h_1. Call Δz_1 the representational variation.

Remark

The variation Δz appears in each application of the representational derivative operator

$$\frac{d}{dz_1} f(z_1) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_1 + \Delta z_1) - f(z_1)}{\Delta z_1} \quad .$$

However, if we wish to compute a representational derivative of the form

$$\frac{d}{dz_1} f(z_2) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_2 + \Delta z_2) - f(z_2)}{\Delta z_2} \quad ,$$

then we will have to define Δz_2 in terms of the limiting variables x_1 and x_2.

Definition 2.3.2

The transformation law for the representational variation is

$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1} \Delta x_1 + \frac{\partial z_2}{\partial y_1} \Delta y_1 \quad .$$

Definition 2.3.3

$\Delta z_2(x_1, y_1)$ is the variation of the z_2 representation of $\mathbb{C}$ written in terms of the variables of the z_1 representation so that we may directly take limits of the form

$$\frac{d}{dz_1} f(z_2) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_2 + \Delta z_2(x_1, y_1)) - f(z_2)}{\Delta z_2} \quad ,$$

Definition 2.3.4

The rules for computing the transformed variation are:

- Choose a transformation

$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1} \Delta x_1 + \frac{\partial z_2}{\partial y_1} \Delta y_1$$

- Write out the analytic form of z_2

$$\Delta z_2(x_1, y_1) = \frac{\partial}{\partial x_1} \big(z_2(x_2, y_2)\big) \Delta x_1 + \frac{\partial}{\partial y_1} \big(z_2(x_2, y_2)\big) \Delta y_1$$

- Replace x_2 and y_2 with their conversion functions

$$\Delta z_2(x_1, y_1) = \frac{\partial}{\partial x_1} \big(z_2(x_2(x_1, y_1), y_2(x_1, y_1))\big) \Delta x_1 + ...$$

$$... + \frac{\partial}{\partial y_1} \big(z_2(x_2(x_1, y_1), y_2(x_1, y_1))\big) \Delta y_1$$

- Simplify in terms of x_1 and y_1 to get the analytic form of

$$\Delta z_2(x_1, y_1)$$

Remark

Examples 2.3.5 - 2.3.10 compute the transformed variation is several cases, some of which we will refer to later.

Example 2.3.5

For the case of

$$\frac{d}{dz}\,f(z^+) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad ,$$

we have

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z^+(x,y) &= \frac{\partial z^+}{\partial x}\,\Delta x + \frac{\partial z^+}{\partial y}\,\Delta y \\
&= \frac{\partial}{\partial x}\big(x + iy^+\big)\Delta x + \frac{\partial}{\partial y}\big(x + iy^+\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

Example 2.3.6

For the case of

$$\frac{d}{dz}\,f(z^-) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad ,$$

we have

$$z^-(x, y^-) = x - iy^- \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z^-(x,y) &= \frac{\partial z^-}{\partial x}\,\Delta x + \frac{\partial z^-}{\partial y}\,\Delta y \\
&= \frac{\partial}{\partial x}\big(x - iy^-\big)\Delta x + \frac{\partial}{\partial y}\big(x - iy^-\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

<u>**Example 2.3.7**</u>

For the case of

$$\frac{d}{dz^+}\,f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(x,y^+) = x \quad , \qquad\qquad \text{and} \qquad\qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is

$$
\begin{aligned}
\Delta z(x, y^+) &= \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^+}\,\Delta y^+ \\
&= \frac{\partial}{\partial x}\big(x + iy\big)\Delta x + \frac{\partial}{\partial y^+}\big(x + iy\big)\Delta y^+ \\
&= \frac{\partial}{\partial x}\big[x + i\big(\widehat{\infty} - y^+\big)\big]\Delta x + \frac{\partial}{\partial y^+}\big[x + i\big(\widehat{\infty} - y^+\big)\big]\Delta y^+ \\
&= \Delta x - i\Delta y^+ \quad .
\end{aligned}
$$

Example 2.3.8

For the case of

$$
\frac{d}{dz^-}\, f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad ,
$$

we have

$$
z(x, y) = x + iy \quad ,
$$

with two conversion functions

$$
x(x, y^-) = x \quad , \qquad \text{and} \qquad y(x, y^-) = y^- - \widehat{\infty} \quad .
$$

The transformation law for the variation is

$$
\begin{aligned}
\Delta z(x, y^-) &= \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^-}\,\Delta y^- \\
&= \frac{\partial}{\partial x}\big(x + iy\big)\Delta x + \frac{\partial}{\partial y^-}\big(x + iy\big)\Delta y^- \\
&= \frac{\partial}{\partial x}\big[x + i\big(y^- - \widehat{\infty}\big)\big]\Delta x + \frac{\partial}{\partial y^-}\big[x + i\big(y^- - \widehat{\infty}\big)\big]\Delta y^-
\end{aligned}
$$

$$= \Delta x + i\Delta y^- \quad .$$

Remark

Notice that the variation is the same between the two cases of z^+ but the sign changes between the conversions to and from z^-.

Example 2.3.9

For the case of

$$\frac{d}{dz} f(z') = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z' + \Delta z') - f(z')}{\Delta z'} \quad ,$$

we have

$$z'(r, \theta) = re^{i\theta} \quad ,$$

with two conversion functions

$$r(x, y) = \sqrt{x^2 + y^2} \quad , \qquad \text{and} \qquad \theta(x, y) = \tan^{-1} \frac{y}{x} \quad .$$

The transformation law for the variation is

$$\Delta z'(x, y) = \frac{\partial z'}{\partial x} \Delta x + \frac{\partial z'}{\partial y} \Delta y$$

$$= \frac{\partial}{\partial x}\big(re^{i\theta}\big)\Delta x + \frac{\partial}{\partial y}\big(re^{i\theta}\big)\Delta y$$

We have shown in Example 2.1.10 that the conversion functions yield $x + iy$ so

$$\Delta z'(x, y) = \frac{\partial}{\partial x}\big(x + iy\big)\Delta x + \frac{\partial}{\partial y}\big(x + iy\big)\Delta y$$

$$= \Delta x + i\Delta y \quad .$$

Example 2.3.10

For the case of

$$\frac{d}{dz'} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(r,\theta) = r\cos(\theta) \quad , \qquad\qquad \text{and} \qquad\qquad y(r,\theta) = r\sin(\theta) \quad .$$

The transformation law for the variation is

$$\begin{aligned} \Delta z(r,\theta) &= \frac{\partial z}{\partial r}\,\Delta r + \frac{\partial z}{\partial \theta}\,\Delta\theta \\ &= \frac{\partial}{\partial r}\big(x+iy\big)\Delta r + \frac{\partial}{\partial \theta}\big(x+iy\big)\Delta\theta \quad . \end{aligned}$$

We have shown in Example 2.1.9 that the conversion functions yield $re^{i\theta}$ so

$$\begin{aligned} \Delta z(r,\theta) &= \frac{\partial}{\partial r}\big(re^{i\theta}\big)\Delta r + \frac{\partial}{\partial \theta}\big(re^{i\theta}\big)\Delta\theta \\ &= e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta \quad . \end{aligned}$$

Remark

Examples 2.3.11 - 2.3.14 all consider the same function in four different representational derivatives and then, for breadth, we will demonstrate the derivative of the logarithm in Example 2.3.15 and case of the chain rule in Example 2.3.16.

Example 2.3.11

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad .$$

The conversion functions are

$$x(x, y) = x \quad , \qquad \text{and} \qquad y^+(x, y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is (Example 2.3.5)

$$\Delta z^+(x, y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz} 3\big(z^+\big)^2 &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big(z^+ + \Delta z^+\big)^2 - 3\big(z^+\big)^2}{\Delta z^+} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big(x + iy^+ + \Delta z^+\big)^2 - 3\big(x + iy^+\big)^2}{\Delta z^+} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big[x + i\big(\widehat{\infty} - y\big) + \Delta x - i\Delta y\big]^2 - 3\big[x - i\big(\widehat{\infty} - y\big)\big]^2}{\Delta x - i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{3\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]^2 - 3\big[x + i\big(\widehat{\infty} - y\big)\big]^2}{-i\Delta y} \\
&\overset{*}{=} \lim_{\Delta y \to 0} \frac{-6i\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]}{-i} \\
&= 6\big[x + i\big(\widehat{\infty} - y\big)\big] = 6\big(x + iy^+\big) = 6z^+ \quad .
\end{aligned}$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.12

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz'} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad .$$

The conversion functions are

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .$$

The transformation law for the variation is (Example 2.3.10)

$$\Delta z(r,\theta) = e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta$$

Converting to polar gives

$$\begin{aligned}
\frac{d}{dz'} 3z^2 &= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{3\big(z(r,\theta) + \Delta z(r,\theta)\big)^2 - 3\big(z(r,\theta)\big)^2}{\Delta z'} \\
&= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{3\big(re^{i\theta} + e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta} \\
&= \lim_{\Delta\theta \to 0} \frac{3\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{ire^{i\theta}\,\Delta\theta} \\
&\overset{*}{=} \lim_{\Delta\theta \to 0} \frac{6ire^{i\theta}\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)}{ire^{i\theta}} \\
&= 6\big(re^{i\theta}\big) = 6z \quad .
\end{aligned}$$

We have the correct transformation law for Δz.

Example 2.3.13

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz} f(z^{\emptyset}) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^{\emptyset} + \Delta z^{\emptyset}) - f(z^{\emptyset})}{\Delta z^{\emptyset}} \quad .$$

The conversion functions are

$$x(x, \emptyset) = x \quad , \qquad \text{and} \qquad y^{\emptyset}(x, \emptyset) = 0 \quad .$$

The transformation law for the variation is

$$\Delta z^{\emptyset}(x, y) = \frac{\partial}{\partial x}\big(x\big)\Delta x = \Delta x \quad .$$

Evaluation yields

$$\frac{d}{dz} 3\big(z^{\emptyset}\big)^2 = \lim_{\Delta x \to 0} \frac{3\big(x + \Delta x\big)^2 - 3\big(x\big)^2}{\Delta x} \overset{*}{=} \lim_{\Delta x \to 0} 6\big(x + \Delta x\big) = 6x = 6z^{\emptyset} \quad .$$

Example 2.3.14

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz^{-}} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^{-} \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad ,$$

The conversion functions are

$$x(x, y^{-}) = x \quad , \qquad \text{and} \qquad y(x, y^{-}) = y^{-} - \widehat{\infty} \quad .$$

The transformation law for the variation is (Example 2.3.8)

$$\Delta z(x, y^-) = \Delta x + i\Delta y^- \quad .$$

Evaluation yields

$$\frac{d}{dz^-}\, 3\big(z\big)^2 = \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}} \frac{3\big(z+\Delta z\big)^2 - 3\big(z\big)^2}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}} \frac{3\big(x+iy+\Delta z\big)^2 - 3\big(x+iy\big)^2}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}} \frac{3\big[x+i\big(y^- - \widehat{\infty}\big)+\Delta x + i\Delta y\big]^2 - 3\big[x-i\big(y^- - \widehat{\infty}\big)\big]^2}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y^-\to 0} \frac{3\big[x+i\big(y^- - \widehat{\infty}\big)+i\Delta y\big]^2 - 3\big[x+i\big(y^- - \widehat{\infty}\big)\big]^2}{i\Delta y}$$

$$\stackrel{*}{=} \lim_{\Delta y^-\to 0} \frac{6i\big[x+i\big(y^- - \widehat{\infty}\big)+i\Delta y\big]}{i}$$

$$= 6\big[x+i\big(y^- - \widehat{\infty}\big)\big] = 6\big(x+iy\big) = 6z \quad .$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.15

Consider the function $f(z) = \ln(z)$ and its representational derivative

$$\frac{d}{dz}\, f(z^-) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad .$$

The conversion functions are

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is (Example 2.3.6)

$$\Delta z^-(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\frac{d}{dz}\ln(z^-) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\ln(z^- + \Delta z^-) - \ln(z^-)}{\Delta z^-}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\ln(x - iy^- + \Delta x - i\Delta y) - \ln(x - iy^-)}{\Delta x - i\Delta y}$$

$$= \lim_{\Delta x\to 0} \frac{\ln(x - iy^- + \Delta x) - \ln(x - iy^-)}{\Delta x}$$

$$\overset{*}{=} \lim_{\Delta x\to 0} \frac{1}{x - iy^- + \Delta x} = \frac{1}{x - iy^-} = \frac{1}{z^-} \quad .$$

Example 2.3.16

Consider the derivative of $f(z)=3ze^{2z}$ in the form

$$\frac{d}{dz^+} f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y^+\to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad .$$

The conversion functions are

$$x(x,y^+) = x \quad , \qquad \text{and} \qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is (Example 2.3.7)

$$\Delta z(x, y^+) = \Delta x - i\Delta y^+ \quad .$$

Evaluation yields

$$\frac{d}{dz^+}\, 3ze^{2z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{3\big(z + \Delta z\big)\, e^{2(z+\Delta z)} - 3ze^{2z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{3\big(x + iy + \Delta z\big)\, e^{2(x+iy+\Delta z)} - 3\big(x+iy\big)e^{2(x+iy)}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \left[\frac{3\big[x + i\big(\widehat{\infty} - y^+\big) + \Delta x - i\Delta y^+\big]\, e^{2[x+i(\widehat{\infty}-y^+)+\Delta x - i\Delta y^+]}}{\Delta x - i\Delta y^+} - \ldots \right.$$

$$\left. \ldots - \frac{3\big[x + i\big(\widehat{\infty} - y^+\big)\big]\, e^{2[x+i(\widehat{\infty}-y^+)]}}{\Delta x - i\Delta y^+} \right]$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]} \lim_{\Delta y^+ \to 0} \frac{3\big[x + i\big(\widehat{\infty} - y^+\big) - i\Delta y^+\big]\, e^{-2i\Delta y^+} - 3\big[x + i\big(\widehat{\infty} - y^+\big)\big]}{-i\Delta y^+}$$

$$\stackrel{*}{=} e^{2[x+i(\widehat{\infty}-y^+)]} \lim_{\Delta y^+ \to 0} \frac{-3ie^{-2i\Delta y^+} - 6i\big[x + i\big(\widehat{\infty} - y^+\big) - i\Delta y^+\big]\, e^{-2i\Delta y^+}}{-i}$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]} \big\{3 + 6\big[x + i\big(\widehat{\infty} - y^+\big)\big]\big\}$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]} \big[3 + 6\big(x + iy\big)\big] = e^{2z}\big(3 + 6z\big) \quad .$$

Theorem 2.3.17

The complex exponential function e^z is an eigenfunction of the representational derivative operator d/dz_1.

Proof

It suffices to show that

$$\frac{d}{dz_1}\, e^{z_1} = e^{z_1} \quad , \qquad\qquad \text{and} \qquad\qquad \frac{d}{dz_1}\, e^{z_2} = e^{z_2} \quad ,$$

where z_1 and z_2 are two representations of $\mathbb{C}$. The first condition is satisfied by Theorem 2.2.2. For the second condition, consider

$$\frac{d}{dz}\, f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is (Example 2.3.5)

$$\Delta z^+(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz}\, e^{z^+} &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{z^+ + \Delta z^+} - e^{z^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{x + iy^+ + \Delta z^+} - e^{x+iy^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{x + i(\widehat{\infty} - y) + \Delta x - i\Delta y} - e^{x+i(\widehat{\infty}-y)}}{\Delta x - i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{e^{x + i(\widehat{\infty} - y) + \Delta x} - e^{x+i(\widehat{\infty}-y)}}{\Delta x} \\
&\overset{*}{=} \lim_{\Delta x \to 0} e^{x + i(\widehat{\infty} - y) + \Delta x} \\
&= e^{x+i(\widehat{\infty}-y)} = e^{x+iy^+} = e^{z^+} \quad .
\end{aligned}$$

All other cases of z_1, z_2 follow directly.

Remark

We have shown that the transformation law for the variation with respect to each representation produces the correct derivative. Since the formula

$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1}\,\Delta x_1 + \frac{\partial z_2}{\partial y_1}\,\Delta y_1 \quad ,$$

is totally standard, it is as expected. In the following subsection, we will examine the definition of the variation and propose a modified variation which obeys a separate transformation law. We will show that the two transformations do not always agree and that the transformation of the modified variation does not always work for the definition of the derivative. Then we will show that the transformation of the modified variation *does* always produce the correct derivative when the transformation is between the z and $z^{\pm}$ representations. This will be due to the composition laws of $\widehat{\mathbb{R}}$ and the properties of $\widehat{\infty}$.

2.4 Definition of the modified representational variation $\widehat{\Delta z_1}$

Example 2.4.1

The 1D case of

$$\Delta x = x + \big(h - x\big) \quad , \quad h \in \mathbb{R} \quad , \quad h \to 0 \quad .$$

gives a good illustration of the meaning of the definition of the variation. Considering three sequential points $\{0, x, h\}$ along the real number line where

$$0 < x < h \quad .$$

We could shift the origin to any other $y \in \mathbb{R}$ and then write the definition of the variation with respect to those coordinates using three sequential point $\{0', x', h'\}$ where

$$0' < x' < h' \quad .$$

For instance, if we shift $h \to h'$ then $h \to 0$ no longer generates an appropriate variation because

$$\lim_{h\to 0} h' = \lim_{h\to 0} y + h = y \quad ,$$

is not vanishingly small when y is not vanishingly small. To get the correct derivative for arbitrary y, we need to take $h' \to 0$ which means h goes to whatever value we have used to shift the origin. By the symmetry of the real line, either of these representations of the 1D variation Δx, that built around the origin 0 and that built around the translated origin $0'$, are exactly the same. Therefore, define a representation of $\mathbb{C}$ such that

$$z_{(\!(x,y^\gamma)\!)}[(\!(x,y)\!)] = (\!(x,y^\gamma)\!) = x + iy^\gamma \quad .$$

Let two conversion functions be

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^\gamma(x,y) = \gamma - y \quad , \quad \gamma \in \mathbb{R} \quad ,$$

Note that the z^γ representation is like the z^+ representation with the origin of the imaginary axis shifted by a finite amount γ rather than the infinite amount of Definition 1.5.1. Applying the conversion functions to the Cartesian h

$$h = a + ib \quad , \qquad\qquad \text{yields} \qquad\qquad h_\gamma = x(a,b) + iy^\gamma(a,b) = a + i\big(\gamma - b\big) \quad .$$

The modified variation shall transform by inserting the conversion functions directly into the definition of the variation:

$$\begin{aligned}
\widehat{\Delta z}^{\gamma}(x,y) &= z^\gamma + \big(h_\gamma - z^\gamma\big) \\
&= z^\gamma(x,y) + \big(h_\gamma - z^\gamma(x,y)\big) \\
&= x(x,y) + iy^\gamma(x,y) + \big[a_\gamma + ib_\gamma - \big(x(x,y) + iy^\gamma(x,y)\big)\big] \\
&= x + i\big(\gamma - y\big) + \big\{a + i\big(\gamma - b\big) - \big[x + i\big(\gamma - y\big)\big]\big\} \\
&= x + i\gamma - iy + a + i\gamma - ib - x - i\gamma + iy
\end{aligned}$$

$$= \big(x + a - x\big) + \big(iy - ib + iy\big) + i\gamma$$

$$= \Delta x - i\Delta y + i\gamma \quad .$$

The transformation law for the canonical variation Δz (Definition 2.3.2) gives

$$\Delta z^{\gamma}(x,y) = \frac{\partial}{\partial x}\big[x + i\big(\gamma - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\gamma - y\big)\big]\Delta y = \Delta x - i\Delta y \quad .$$

We find

$$\widehat{\Delta z}^{\gamma}(x,y) = \Delta z^{\gamma}(x,y) + i\gamma \quad .$$

The two transformation laws do no produce the same transformed values.

Remark

The transformation law for the variation does not agree with our attempt to transform the modified variation by directly converting its elements with the conversion functions. We will show, however, that this not a problem in all cases.

Definition 2.4.2

The rules for computing the transformed modified variation are:

- Choose a modified transformation

$$\widehat{\Delta} z_2(x_1, y_1) = z_2 + \big(h_2 - z_2\big)$$

- Write out the analytic form of z_2 and h_2

$$\widehat{\Delta} z_2(x_1, y_1) = z_2(x_2, y_2) + \big(h_2(x_2, y_2) - z_2(x_2, y_2)\big)$$

Note: If $z_2 = z^- = x - iy^-$ then $h_2 = h^- = a - ib$

- Replace x_2 and y_2 with their conversion functions

  $\widehat{\Delta z}_2(x_1, y_1) = z_2(x_2(x_1, y_1), y_2(x_1, y_1)) + \big(h_2(x_2(a, b), y_2(a, b)) - z_2(x_2(x_1, y_1), y_2(x_1, y_1))\big)$

- Simplify in terms of x_1 and y_1 to get the analytic form of

$$\widehat{\Delta z}_2(x_1, y_1)$$

Example 2.4.3

The continuation of $\widehat{\Delta z}^{\gamma}$ cannot be used when $\widehat{\Delta z}^{-}$ appears because the signs in the conversion functions do not agree. Instead we need to consider two conversion functions

$$x(x, y) = x \quad , \qquad \text{and} \qquad y^{\gamma'}(x, y) = \gamma' + y \quad ,$$

In this case, $y^{\gamma'}$ has the same form as $y^{-} = \widehat{\infty} + y$. To mimic the form of z^{-}, we will choose for this example

$$z^{\gamma'}(x, y^{\gamma'}) = x - iy^{\gamma'} \quad .$$

The transformation law (Definition 2.4.2) is

$$\begin{aligned}
\widehat{\Delta z}^{\gamma'}(x, y) &= z^{\gamma'} + \big(h^{\gamma'} - z^{\gamma'}\big) \\
&= z^{\gamma'}(x, y^{\gamma'}) + \big(h_2(x, y^{\gamma'}) - z^{\gamma'}(x, y^{\gamma'})\big) \\
&= z^{\gamma'}(x(x, y), y^{\gamma'}(x, y)) + \big(h_2(x(a, b), y^{\gamma'}(a, b)) - z^{\gamma'}(x(x, y), y^{\gamma'}(x, y))\big) \quad .
\end{aligned}$$

Since we seek to replicate the z^{-} representation, we take $h^{\gamma'} = a^{\gamma'} - ib^{\gamma'}$ so that

$$\begin{aligned}
\widehat{\Delta z}^{\gamma'}(x, y) &= x(x, y) - iy^{\gamma'}(x, y) + \big[x(a, b) - iy^{\gamma'}(a, b) - \big(x(x, y) - iy^{\gamma'}(x, y)\big)\big] \\
&= x - i\big(\gamma' + y\big) + \big\{a - i\big(\gamma' + b\big) - \big[x - i\big(\gamma' + y\big)\big]\big\}
\end{aligned}$$

$$= x - i\gamma' - iy + a - i\gamma' - ib - x + i\gamma' + iy$$

$$= \big(x + a - x\big) - i\big(y + b - y\big) - i\gamma'$$

$$= \Delta x - i\Delta y - i\gamma'$$

This does not agree with the canonical transformation of the variation (Definition 2.3.2)

$$\Delta z^{\gamma'}(x,y) = \frac{\partial}{\partial x}\big[x - i\big(\gamma' + y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x - i\big(\gamma' + y\big)\big]\Delta y = \Delta x - i\Delta y \quad .$$

<u>Definition 2.4.4</u>

The modified representational variation of a $\mathbb{C}$ number is

$$\widehat{\Delta z}_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h \in \mathbb{C} \quad , \quad h \to 0 \quad ,$$

so it is identically the representational variation Δz_1 (Definition 2.3.1.) The difference between Δz_1 and $\widehat{\Delta z}_1$ is that they obey different transformation laws between representations.

<u>Definition 2.4.5</u>

The modified variation $\widehat{\Delta z}_1$ transforms by direct substitution of the conversion functions. The transformation law defined for

$$h_1 = a + ib \quad , \quad a,b \in \mathbb{R} \quad , \quad a,b \to 0 \quad ,$$

is

$$\widehat{\Delta z}_1(x_2,y_2) = z_1[x_1(x_2,y_2),y_1(x_2,y_2)] + \big(x_1(a,b) + iy_1(a,b) - z_1[x_1(x_2,y_2),y_1(x_2,y_2)]\big) \quad .$$

Remark

Take note of

$$\widehat{\Delta z}^{-}(x,y) = x(x,y) - iy^{-}(x,y) + \big[x(a,b) - iy^{-}(a,b) - \big(x(x,y) - iy^{-}(x,y)\big)\big] \quad .$$

Since z^{-} is a non-trivial representation, we may not directly decompose the $z_1[x_1, y_1]$ of Definition 2.4.5 into a general form $x_1(x_2, y_2) + iy_1(x_2, y_2)$. For this reason, Definition 2.4.5 specifies h in the Cartesian representation. There are other cases in which h will not have the form $a + ib$.

Remark

If we transform the modified variation directly with the $y^{\pm}(y)$ conversion functions of the $\hat{\mathbb{C}}$ representation then we will get an undefined expression because the $\pm i\gamma$ terms become $\pm i\,\widehat{\infty}$ terms while $i\,\widehat{\infty} - i\,\widehat{\infty}$ is undefined. For this reason, the infinite continuation is defined as in Definitions 2.4.6 and 2.4.7; we take the limit of the finite behavior rather than the infinite behavior of the transformation law. We may also obtain a contradiction directly from the definition of the variation. Consider two equivalent expressions such that

$$x + \big(h - x\big) = \big(x + h\big) - x$$

$$\big(\widehat{\infty} - x\big) + \big[\big(\widehat{\infty} - h\big) - \big(\widehat{\infty} - x\big)\big] = \big[\big(\widehat{\infty} - x\big) + \big(\widehat{\infty} - h\big)\big] - \big(\widehat{\infty} - x\big)$$

$$\big(\widehat{\infty} - x\big) + \big(x - h\big) = \big[\widehat{\infty} - \big(x + h\big)\big] - \big(\widehat{\infty} - x\big)$$

$$\big(\widehat{\infty} - h\big) = -h \quad .$$

This is a contradiction.

Remark

Although $\Delta z_1(x_2, y_2)$ and $\widehat{\Delta z}_1(x_2, y_2)$ do not always transform such that the resultant derivatives are identical (implied in Examples 2.4.1 and 2.4.3), there are cases in which they do produce the same derivatives. We will show that $\widehat{\Delta z}_1$ is always valid for the case of $z_1 \in \hat{\mathbb{C}}$ when we make a special rule.

Definition 2.4.6

When the modified variation appears in a derivative

$$\frac{d}{dz^+}f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z+\widehat{\Delta z}) - f(z)}{\widehat{\Delta z}} \quad ,$$

or

$$\frac{d}{dz}f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

then to avoid an undefined expression (per the above remark), instead evaluate

$$\frac{d}{dz^\gamma}f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^\gamma \to 0}} \frac{f(z+\widehat{\Delta z}(x,y^\gamma)) - f(z)}{\widehat{\Delta z}} \quad ,$$

or

$$\frac{d}{dz}f(z^\gamma) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^\gamma + \widehat{\Delta z}^\gamma(x,y)) - f(z^\gamma)}{\widehat{\Delta z}^\gamma} \quad ,$$

respectively. (z^γ is defined in Example 2.4.1.) Once the expression is simplified, take the limits $\gamma \to \widehat{\infty}$ and $y^\gamma \to y^+$.

Definition 2.4.7

When the modified variation appears in a derivative

$$\frac{d}{dz^-}f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z+\widehat{\Delta z}) - f(z)}{\widehat{\Delta z}} \quad ,$$

or

$$\frac{d}{dz}f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^- + \widehat{\Delta z}^-) - f(z^-)}{\widehat{\Delta z}^-} \quad ,$$

then to avoid an undefined expression (per the above remark), instead evaluate

$$\frac{d}{dz^{\gamma'}}f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^{\gamma'} \to 0}} \frac{f(z + \widehat{\Delta z}(x, y^{\gamma})) - f(z)}{\widehat{\Delta z}} \quad ,$$

or

$$\frac{d}{dz}f(z^{\gamma'}) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^{\gamma'} + \widehat{\Delta z}^{\gamma'}(x, y)) - f(z^{\gamma'})}{\widehat{\Delta z}^{\gamma'}} \quad ,$$

respectively. ($z^{\gamma'}$ is defined in Example 2.4.3.) Once the expression is simplified, take the limits $\gamma' \to \widehat{\infty}$ and $y^{\gamma'} \to y^-$.

Definition 2.4.8

In the case of Definitions 2.4.6 and 2.4.7, the rule for taking the derivative with the modified representational variation is to compute the derivative with γ and then let $\gamma \to \widehat{\infty}$ once the expression is simplified. Division by γ shall always be avoided through L'Hôpital's rule.

Definition 2.4.9

For functions of the form

$$f(z) = cz^n \quad , \qquad \text{with} \qquad z \in \mathbb{C} \quad , \quad n \in \mathbb{N} \quad ,$$

and where c is any constant, the modified representational variation always produces the correct derivative for transformations between the Cartesian and $\hat{\mathbb{C}}$ representations of $\mathbb{C}$.

Proof

A representational derivative of cz^n taken with the modified variation is

$$\frac{d}{dz^+}cz^n = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{c\big(z + \widehat{\Delta z}\big)^n - c\big(z\big)^n}{\widehat{\Delta z}} \quad .$$

We have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(x,y^\gamma) = x \quad , \qquad \text{and} \qquad y(x,y^\gamma) = \gamma - y^\gamma \quad .$$

(The conversion functions use γ instead of $\widehat{\infty}$ because we will make the substitution to y^+ only after we have evaluated the definition of the derivative with γ.) The transformation law for the modified variation is

$$\begin{aligned}
\widehat{\Delta z}(x,y^\gamma) &= x(x,y^\gamma) + iy(x,y^\gamma) + \big[x(a,b) + iy(a,b) - \big(x(x,y^\gamma) + iy(x,y^\gamma)\big)\big] \\
&= x + i\big(\gamma - y^\gamma\big) + \big\{a + i\big(\gamma - b\big) - \big[x + i\big(\gamma - y^\gamma\big)\big]\big\} \\
&= x + i\gamma - iy^\gamma + a + i\gamma - ib - x - i\gamma + iy^\gamma \\
&= \big(x + a - x\big) - i\big(y^\gamma + b - y^\gamma\big) + i\gamma \\
&= \Delta x - i\Delta y^\gamma + i\gamma \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dz^\gamma}\, cz^n &= \lim_{\substack{\Delta x \to 0 \\ \Delta y^\gamma \to 0}} \frac{c\big(z + \widehat{\Delta z}(x,y^\gamma)\big)^n - c\big(z\big)^n}{\widehat{\Delta z}} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y^\gamma \to 0}} \frac{c\big(x + iy + \Delta x - i\Delta y^\gamma + i\gamma\big)^n - c\big(x + iy\big)^n}{\Delta x - i\Delta y^\gamma + i\gamma} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y^\gamma \to 0}} \frac{c\big[x + i\big(\gamma - y^\gamma\big) + \Delta x - i\Delta y^\gamma + i\gamma\big]^n - c\big[x + i\big(\gamma - y^\gamma\big)\big]^n}{\Delta x - i\Delta y^\gamma + i\gamma} \\
&= \lim_{\Delta x \to 0} \frac{c\big[x + i\big(\gamma - y^\gamma\big) + \Delta x + i\gamma\big]^n - c\big[x + i\big(\gamma - y^\gamma\big)\big]^n}{\Delta x + i\gamma}
\end{aligned}$$

If we take the limit $\Delta x \to 0$, we will divide by γ but Definition 2.4.8 says that this is not allowed when using the modified variation. Instead, we need to use L'Hôpital's rule so that

$$\begin{aligned}\frac{d}{dz^\gamma}\,cz^n &\overset{*}{=} \lim_{\Delta x\to 0} nc\big[x+i\big(\gamma-y^\gamma\big)+\Delta x+i\gamma\big]^{n-1}\\ &= nc\big[x+i\big(\gamma-y^\gamma\big)+i\gamma\big]^{n-1}\\ &= nc\big(x-iy^\gamma+2i\gamma\big)^{n-1} \quad .\end{aligned}$$

Having computed the derivative with the modified variation for the finitely translated origin of the z^γ representation of $\mathbb{C}$, Definitions 2.4.6 and 2.4.8 tell us to obtain the derivative with respect to z^+ from the derivative with respect to z^γ by taking the limit $\gamma \to \widehat{\infty}$. This gives

$$\begin{aligned}\frac{d}{dz^+}\,cz^n = \lim_{\substack{\gamma\to\widehat{\infty}\\ y^\gamma\to y^+}} \frac{d}{dz^\gamma}\,cz^n &= \lim_{\gamma\to\widehat{\infty}} nc\big(x-iy^+ +2i\gamma\big)^{n-1}\\ &= nc\big(x-iy^+ +2i\widehat{\infty}\big)^{n-1}\\ &= nc\big[x-i\big(y^+ -\widehat{\infty}\big)\big]^{n-1} \quad .\end{aligned}$$

As in Theorem 2.1.12, the quantity in parentheses is not formatted for an additive composition $\widehat{\mathbb{R}}+\widehat{\infty}$. Substitute $y^+ = \widehat{\infty}-y$ so that

$$\big(y^+ -\widehat{\infty}\big) = \big[\big(\widehat{\infty}-y\big)-\widehat{\infty}\big] \quad .$$

By the additive composition laws of $\widehat{\mathbb{R}}+\widehat{\infty}$ (Definition 1.3.15)

$$\big[\big(\widehat{\infty}-y\big)-\widehat{\infty}\big] = -y \quad .$$

Therefore,

$$\frac{d}{dz^+}\,cz^n = nc\big[x-i\big(-y\big)\big]^{n-1} = nc\big(x+iy\big)^{n-1} = ncz^{n-1} \quad .$$

This is the correct derivative. We have demonstrated transformation law for the modified variation the case of the z^+ representational derivative applied to the Cartesian representation and all other iterations follow directly.

Example 2.4.10

For example, we follow directly with a case of

$$\frac{d}{dz}\big(z^-\big)^n = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{c\big(z^- + \widehat{\Delta z}^-\big)^n - c\big(z^-\big)^n}{\widehat{\Delta z}} \quad ,$$

because, among Examples 2.3.5 - 2.3.8, the odd case is

$$\Delta z(x,y^-) = \Delta x + i\Delta y^- \neq \Delta x_1 - i\Delta y_1 \quad ,$$

(Example 2.3.8.) We have

$$z^-(x,y^-) = x - iy^- \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^{\gamma'}(x,y) = \gamma' + y \quad .$$

(As in the proof of Theorem 2.4.9, the conversion functions use γ' instead of $\widehat{\infty}$ because, per Definition 2.4.7, we will make the substitution to y^- only after we have evaluated the definition of the derivative with γ'.) The transformation law for the modified variation is

$$\begin{aligned}
\widehat{\Delta z}^{\gamma'}(x,y) &= x(x,y) - iy^{\gamma'}(x,y) + \big[x(a,b) - iy^{\gamma'}(a,b) - \big(x(x,y) - iy^{\gamma'}(x,y)\big)\big] \\
&= x - i\big(\gamma' + y\big) + \big\{a - i\big(\gamma' + b\big) - \big[x - i\big(\gamma' + y\big)\big]\big\} \\
&= x - i\gamma' - iy + a - i\gamma' - ib - x + i\gamma' + iy \\
&= \big(x + a - x\big) - i\big(y + b - y\big) - i\gamma'
\end{aligned}$$

$$= \Delta x - i\Delta y - i\gamma' \quad .$$

Therefore,

$$\frac{d}{dz}\,c\big(z^{\gamma'}\big)^n = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{c\big(z^{\gamma'} + \widehat{\Delta z}^{\gamma'}(x,y)\big)^n - c\big(z^{\gamma'}\big)^n}{\widehat{\Delta z}^{\gamma'}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{c\big(x - iy^{\gamma'} + \Delta x - i\Delta y - i\gamma'\big)^n - c\big(x - iy^{\gamma'}\big)^n}{\Delta x - i\Delta y - i\gamma'}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{c\big[x - i\big(\gamma' + y\big) + \Delta x - i\Delta y^{\gamma} - i\gamma'\big]^n - c\big[x - i\big(\gamma' + y\big)\big]^n}{\Delta x - i\Delta y - i\gamma'}$$

$$= \lim_{\Delta x\to 0} \frac{c\big[x - i\big(\gamma' + y\big) + \Delta x - i\gamma'\big]^n - c\big[x - i\big(\gamma' + y\big)\big]^n}{\Delta x - i\gamma'} \quad .$$

Definition 2.4.8 says to use L'Hôpital's rule so that

$$\frac{d}{dz}\,c\big(z^{\gamma'}\big)^n \stackrel{*}{=} \lim_{\Delta x\to 0} nc\big[x - i\big(\gamma' + y\big) + \Delta x - i\gamma'\big]^{n-1}$$

$$= nc\big[x - i\big(\gamma' + y\big) - i\gamma'\big]^{n-1}$$

$$= nc\big(x - iy - 2i\gamma\big)^{n-1} \quad .$$

To obtain $y^{\gamma'} \to y^-$ we need to also take the limit $\gamma' \to \widehat{\infty}$. This yields

$$\frac{d}{dz}\,c\big(z^-\big)^n = \lim_{\substack{\gamma'\to\widehat{\infty}\\ y^{\gamma'}\to y^-}} \frac{d}{dz}\,c\big(z^{\gamma'}\big)^n = \lim_{\gamma'\to\widehat{\infty}} nc\big(x - iy - 2i\gamma\big)^{n-1}$$

$$= nc\big(x - iy - 2i\widehat{\infty}\big)^{n-1}$$

$$= nc\big[x - i\big(y + \widehat{\infty}\big)\big]^{n-1} \quad .$$

Substitute $y^- = \widehat{\infty} + y$ to obtain

$$\frac{d}{dz}\,c\big(z^-\big)^n = nc\big(x - iy^-\big)^{n-1} = nc\big(z^-\big)^{n-1} \quad .$$

This is the correct derivative and it supports the proof of Theorem 2.4.9. There are a few more permutations of the representations which we have not demonstrated to explicitly prove every case of Theorem 2.4.9 but, as stated above, the other cases follow directly.

Theorem 2.4.11

For functions of the form

$$f(x) = \sin(x) \quad , \quad g(x) = \cos(x) \quad , \qquad\qquad \text{where} \qquad\qquad x \in \mathbb{R} \quad ,$$

the modified representational variation always produces the correct derivative for transformations between the Cartesian and $\hat{\mathbb{C}}$ representations of $\mathbb{C}$.

Proof

For any two functions $F(z)$ and $G(z)$ The derivative operator is such that

$$\frac{d}{dz}\big(F(z) + G(z)\big) = \frac{d}{dx}\,F(z) + \frac{d}{dx}\,G(z) \quad .$$

Sine is defined as a series of terms of the form of $F(x) = cx^n$ with $n \in \mathbb{N}$. Theorem 2.4.9 proves that the derivative relying on the modified variation between the Cartesian and $\hat{\mathbb{C}}$ representations will produce the correct derivative of all such terms because $x = z$ for $\mathrm{Im}(z) = 0$. Cosine is defined as a constant 1 plus a similar series of terms. Since the derivative of 1 vanishes trivially, the modified representational derivative will likewise always produce the correct derivative of cosine.

Main Theorem 2.4.12

For functions of the form

$$f(x) = e^z \quad , \qquad\qquad \text{with} \qquad\qquad z \in \mathbb{R} \quad ,$$

the modified representational variation always produces the correct derivative for transformations between the Cartesian and $\hat{\mathbb{C}}$ representations of $\mathbb{C}$.

Proof

The exponential is defined as a constant plus a series of terms of the form of $F(x) = cx^n$ with $n \in \mathbb{N}$. Since the derivative of a constant vanishes trivially, proof follows from Theorem 2.4.9.

3 Proof of limits of sine and cosine at infinity

3.1 Refutation of proof of nonexistence of limits at infinity

Definition 3.1.1

We say that the limit of a sequence exists if and only if all of its subsequences converge to the same value.

Theorem 3.1.2

It is impossible to compute the limits

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(x) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \cos(x) \quad .$$

Proof *(Refuted)*

The definition of the limit requires that for a limit

$$\lim_{x\to\infty} F(x) = l \quad ,$$

to exist, the function F must converge to l in all of its subsequences. For proof by contradiction, consider two subsequences of x

$$x_n = 2n\pi + \frac{\pi}{2} \quad , \qquad \text{and} \qquad x_m = 2m\pi.$$

For any $n, m \in \mathbb{N}$, we have

$$\sin(x_n) = 1 \quad , \qquad \text{and} \qquad \sin(x_m) = 0 \quad .$$

It is, therefore, impossible for all subsequences of x to converge to some constant l.

Theorem 3.1.3

Existence of the limits

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(x) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \cos(x) \quad .$$

cannot be ruled out on the basis on non-convergent subsequences

$$x_n = 2n\pi + \frac{\pi}{2} \quad , \qquad \text{and} \qquad x_m = 2m\pi.$$

Proof

The convergence of the sequences are determined by the final n points, not the first n points. By Main Theorem 1.3.10, the final n points of an increasing sequence of $\mathbb{R}$, *i.e.*: any subsequence of x, will have the form

$$x = \widehat{\infty} - b \quad .$$

Since the points in x_n and x_m are evenly spaced by 2π and are such that for $m, n \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \infty \quad , \qquad \text{and} \qquad \lim_{m\to\infty} x_m = \infty \quad ,$$

we may recognize that ∞ and $\widehat{\infty}$ are the same endpoint of the extended real number line (Definition 1.3.14) to write the final n points of x_n and x_m as

$$x_{\widehat{\infty}-n'} = \widehat{\infty} - 2n'\pi \quad , \qquad \text{and} \qquad x_{\widehat{\infty}-m'} = \widehat{\infty} - 2m'\pi \quad .$$

where

$$n', m' \to 0 \quad , \qquad \text{as} \qquad n, m \to \infty \quad .$$

Since $\widehat{\infty} - n \neq \widehat{\infty}$, all of these points are distinct. It is obvious that both sequences converge to the same value.

Remark

We have shown that $\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on $\mathbb{S}^2$ in the limit where $\pm\widehat{\infty}, \pm i\widehat{\infty} \to \infty$ (Corollary 1.4.11.) Now we have reason to consider another complementary arrangement on $\mathbb{S}^2$ and we will consider a great circle $\mathbb{S}^1$ to simplify the statements. Since there are exactly as many $\widehat{\mathbb{R}}$ numbers of the form $\widehat{\infty}+b$ as there are non-zero $\mathbb{R}_0$ numbers of the form b, meaning that for every non-zero b there is a $\widehat{\infty}+b$, and every $\widehat{\mathbb{R}}$ number of this form is greater than every $\mathbb{R}_0$ number, we should set the infinity that $\mathbb{R}_0$ tends toward, call it $\widehat{\infty} - \varepsilon$, on the equator of $\mathbb{S}^2$ when $\widehat{0}$ and $\widehat{\infty}$ are at the two poles. Since there are as many points in the interval $[\infty - \varepsilon, \infty]$ as there are in $[0, \infty - \varepsilon]$, *i.e.*: infinity, one would favor the representation in which the area around the pole at infinity is stretched over an entire hemisphere because the density of numbers on the surface of the sphere is uniform when $b \in \mathbb{R}_0$ tends toward a value on the equator. $\widehat{\mathbb{R}}$ numbers of the form $\widehat{\infty} - b$ with $b > 0$ will also tend toward that same value for increasing $b \in \mathbb{R}$. The only number that increasing $\mathbb{R}_0$ and decreasing $\widehat{\mathbb{R}}$ can both tend to is $\widehat{\infty} - \varepsilon$ and there is an apparent condition that the metric along $\mathbb{R}_0$ cannot be the same as that on $\widehat{\mathbb{R}}$ when $\mathbb{R}_0$ tends toward $\widehat{\infty} - \varepsilon$ at the opposite pole of $\mathbb{S}^2$ from its origin. When $\mathbb{R}$ tends toward infinity at the opposite pole from its origin, then every $\widehat{\mathbb{R}}$ number is squeezed to one side of the sphere. Regarding the refutation of Theorem 3.1.2, all the points in the $\widehat{\mathbb{R}}$ hemisphere can take the same form $\widehat{\infty} - \varepsilon$ because the same limit at the equator is constrained to be adjacent to the pole when $\widehat{\mathbb{R}}$ is collapsed to a point. Indeed, $\widehat{\infty} - \varepsilon$ is exactly of the form $\widehat{\infty} - b$. Since any point $b \in \mathbb{R}$ is such that there is some other point b' where $b < b' < \infty$, there are an infinite number of values that can be assigned to ε, each one corresponding to an $\mathbb{R}$ number in the form $\widehat{\infty} - b$ with $b > 0$. There are some immediate features of interest in expanding the neighborhood of polar infinity to cover an entire hemisphere. By the symmetry of the sphere, and by the symmetry of there being exactly as many positive $\widehat{\mathbb{R}}$ numbers less than infinity as there are $\mathbb{R}$ numbers greater than zero, we can deduce that the limits of sine and cosine at $\widehat{\infty}$ should be the same as what they are at $\widehat{0}$; the sphere has mirror symmetry about it equator. Furthermore, since ε is vanishingly small, equatorial infinity is separated from polar infinity by a vanishingly small distance. We may deduce the behavior at the equator from the behavior at the pole because there is a representation in which equatorial infinity is adjacent to polar infinity (Corollary 1.4.11.) In the next section, we will use a totally different method to derive the behavior of sine and cosine at infinity but we will find that it is exactly like the behavior at zero. Also note, if increasing $\mathbb{R}_0$ tends toward $\widehat{\infty} - \varepsilon$ then increasing $\widehat{\mathbb{R}}$ tends toward $\widehat{\infty} - \varepsilon'$ where $\varepsilon > \varepsilon'$.

3.2 Proof of limits of sine and cosine at infinity

Theorem 3.2.1

The values of sine and cosine at infinity are

$$\sin(\infty) = 0 \quad , \qquad \text{and} \qquad \cos(\infty) = 1 \quad .$$

Proof

We have proven in Theorem 2.3.17 that

$$\frac{d}{dz_1} e^{z_2} = e^{z_2} \quad .$$

For $f(z) = e^z$ in the case of

$$\frac{d}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

we have proven in Main Theorem 2.4.12 that the definition of the derivative relying on the modified variation $\widehat{\Delta z}^+$ will produce the correct derivative. We have

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the modified variation requires us to use the z^γ representation (Definition 2.4.6) so that

$$\widehat{\Delta z}^\gamma(x,y) = \Delta x - i\Delta y + i\gamma \quad ,$$

(Example 2.4.1.) Evaluation yields

$$\frac{d}{dz}\,e^{z^\gamma} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^\gamma+\widehat{\Delta z}^{\gamma})-f(z^\gamma)}{\widehat{\Delta z}^{\gamma}}$$

$$= e^{x+iy^\gamma}\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{\Delta x - i\Delta y + i\gamma}-1}{\Delta x - i\Delta y + i\gamma}$$

$$= e^{x+iy^\gamma}\lim_{\Delta x\to 0} \frac{e^{\Delta x + i\gamma}-1}{\Delta x + i\gamma}$$

$$\stackrel{*}{=} e^{x+iy^\gamma}\lim_{\Delta x\to 0} e^{\Delta x + i\gamma}$$

$$= e^{x+iy^\gamma}e^{i\gamma} \quad .$$

To recover the z^+ representation from the the z^γ representation we take

$$\frac{d}{dz}\,e^{z^+} = \lim_{\substack{\gamma\to\widehat{\infty}\\ y^\gamma\to y^+}} e^{x+iy^\gamma}e^{i\gamma} = e^{x+iy^+}e^{i\widehat{\infty}} = e^{z^+}e^{i\widehat{\infty}} \quad .$$

The exponential is an eigenfunction of the derivative with unit eigenvalue (Theorem 2.2.2) so

$$1 = e^{i\widehat{\infty}} = \cos(\widehat{\infty}) + i\sin(\widehat{\infty}) \quad .$$

Equating the real and imaginary parts gives

$$\sin(\widehat{\infty}) = 0 \quad , \qquad\qquad \text{and} \qquad\qquad \cos(\widehat{\infty}) = 1 \quad .$$

Theorem is proven with Definition 1.3.14:

$$\sin(\widehat{\infty}) = \sin(\infty) \quad , \qquad\qquad \text{and} \qquad\qquad \cos(\widehat{\infty}) = \cos(\infty) \quad .$$

Main Theorem 3.2.2

The limits of sine and cosine at infinity are

$$\lim_{x\to\infty} \sin(x) = 0 \quad , \qquad \text{and} \qquad \lim_{x\to\infty} \cos(x) = 1 \quad .$$

Proof

A function has a limit l if and only if the function converges to l in any subsequence. Since the set of $x \in \mathbb{R}$ converges to ∞, all of its subsequences also converge to $\widehat{\infty}$. Therefore, for any subsequence x_n of x, we have

$$\lim_{n\to\infty} \sin(x_n) = \sin(\infty) \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \cos(x_n) = \cos(\infty) \quad .$$

Proof follows from Theorem 3.2.1: $\sin(\infty) = 0$ and $\cos(\infty) = 1$.

Theorem 3.2.3

Sine and cosine are continuous at infinity.

Proof

We say that a function is continuous at a point if

$$\lim_{x\to x_0} f(x) = f(x_0) \quad .$$

Sine and cosine are such that

$$\lim_{x\to\infty} \sin(x) = \sin(\infty) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} \cos(x) = \cos(\infty) \quad .$$

Both functions are continuous at infinity.

Theorem 3.2.4

The values of sine and cosine at ∞ preserve the odd- and evenness of sine and cosine respectively.

Proof

We have

$$z^{-}(x,y^{-}) = x - iy^{-} \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^{-}(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the modified variation requires us to use the $z^{\gamma'}$ representation (Definition 2.4.7) so that

$$\widehat{\Delta z}^{\gamma'}(x,y) = \Delta x - i\Delta y - i\gamma' \quad ,$$

(Example 2.4.3.) Evaluation yields

$$\frac{d}{dz}\, e^{z^{\gamma'}} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^{\gamma'} + \widehat{\Delta z}^{\gamma'}) - f(z^{\gamma'})}{\widehat{\Delta z}^{\gamma'}}$$

$$= e^{x - iy^{\gamma'}} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{\Delta x - i\Delta y - i\gamma'} - 1}{\Delta x - i\Delta y - i\gamma'}$$

$$= e^{x - iy^{\gamma'}} \lim_{\Delta x \to 0} \frac{e^{\Delta x - i\gamma'} - 1}{\Delta x + i\gamma'}$$

$$\overset{*}{=} e^{x - iy^{\gamma'}} \lim_{\Delta x \to 0} e^{\Delta x - i\gamma'}$$

$$= e^{x - iy^{\gamma'}} e^{-i\gamma'} \quad .$$

To recover the z^{-} representation from the the $z^{\gamma'}$ representation we take

$$\frac{d}{dz}\, e^{z^-} = \lim_{\substack{\gamma' \to \widehat{\infty} \\ y^{\gamma'} \to y^-}} e^{x - iy^{\gamma'}}\, e^{-i\widehat{\infty}} = e^{x-iy^-}\, e^{-i\widehat{\infty}} = e^{z^-}\, e^{-i\widehat{\infty}} \quad .$$

It follows that

$$1 = e^{-i\widehat{\infty}} = \cos(-\widehat{\infty}) + i \sin(-\widehat{\infty}) \quad .$$

Equating the real and imaginary parts gives

$$\cos(-\widehat{\infty}) = 1 \quad , \qquad \text{and} \qquad \sin(-\widehat{\infty}) = 0 \quad .$$

Therefore,

$$\cos(-\widehat{\infty}) = \cos(\widehat{\infty}) \quad , \qquad \text{and} \qquad \sin(-\widehat{\infty}) = -\sin(\widehat{\infty}) \quad .$$

Sine is an odd function and cosine is an even function.

Theorem 3.2.5

Sine and cosine satisfy the double angle identities at infinity.

Proof

The relevant identities are

$$\sin(2x) = 2\sin(x)\cos(x) \quad , \qquad \text{and} \qquad \cos(2x) = 1 - \sin^2(x) \quad .$$

These identities are satisfied trivially for $x = \widehat{\infty}$.

Remark

We have introduced a rule (Definition 2.4.8) such that the modified variation in the derivative requires an application of L'Hôpital's where it is not independently motivated. To avoid this, consider the definition

$$\frac{d}{dz^-}\,f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad .$$

We have written the canonical variation of $y^- \in \widehat{\mathbb{R}}$ as an $\mathbb{R}$ number when there is an argument to be made that the variation of a $\widehat{\mathbb{R}}$ number should not have the same form as the variation of an $\mathbb{R}$ number. In the interpretation where $\widehat{\mathbb{R}}$ numbers measure magnitude relative to the origin at infinity, the limit of small variation is

$$\Delta y^- \to \widehat{\infty} + 0 \quad .$$

To see how a small variation $\Delta y^- \to 0$ is in some way equivalent to $\Delta y^- \to \widehat{\infty}$, consider the composition laws of $\widehat{\mathbb{R}} + \mathbb{R}$ and $\widehat{\mathbb{R}} + \widehat{\infty}$:

$$\big(\widehat{\infty} + b\big) + 0 = \big(\widehat{\infty} + b\big) \quad , \qquad \text{and} \qquad \big(\widehat{\infty} + b\big) + \widehat{\infty} = \big(\widehat{\infty} + b\big) \quad .$$

We find that for $\widehat{\mathbb{R}}$ numbers, some of which are $\mathbb{R}$ numbers,

$$\lim_{\Delta x \to 0} \big(\widehat{\infty} + b\big) + \Delta x = \lim_{\Delta x \to \widehat{\infty}} \big(\widehat{\infty} + b\big) + \Delta x \quad .$$

This is highly relevant because the $\hat{\mathbb{C}}$ representation of $\mathbb{C}$ relies exclusively on $\widehat{\mathbb{R}}$ numbers to chart the imaginary axis. For the derivative $d/dz^-\,f(z)$ in the canonical variation we have (Example 2.3.8)

$$\Delta z(x, y^-) = \Delta x + i\Delta y^- \quad .$$

Therefore,

$$\frac{d}{dz^-}\,e^z = e^z \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to \widehat{\infty}}} \frac{e^{\Delta x + i\Delta y^-} - 1}{\Delta x + i\Delta y^-}$$

$$= e^z \lim_{\Delta y^- \to \widehat{\infty}} \frac{e^{i\Delta y^-} - 1}{i\Delta y^+}$$

$$= e^z \, \frac{e^{i\widehat{\infty}} - 1}{i\widehat{\infty}} = e^z \left[\frac{i\widehat{\infty} + \frac{\left(i\widehat{\infty}\right)^2}{2} + \frac{\left(i\widehat{\infty}\right)^3}{3!} + ...}{i\widehat{\infty}} \right] \quad .$$

By taking the limit, we have obtained an expression of the indeterminate form ∞/∞ which is an ordinary context for L'Hôpital's rule. Application of the rule yields

$$e^z \lim_{\Delta y^- \to \widehat{\infty}} \frac{e^{i\Delta y^-} - 1}{i\Delta y^+} \overset{*}{=} -ie^z \lim_{\Delta y^- \to \widehat{\infty}} ie^{i\Delta y^-} = e^z e^{i\widehat{\infty}} \quad .$$

The limits of sine and cosine at infinity may be derived from this expression too.