Proof of the Limits of Sine and Cosine at Infinity

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ABSTRACT

We develop a representation of complex numbers separate from the Cartesian and polar representations and define a representing functional for converting between representations. We define the derivative of a function of a complex variable with respect to each representation and then we examine the variation within the definition fo the derivative. After studying the transformation law for the variation between representations of complex numbers, we show that the new representation has special properties which allow for a consistent modification to the transformation law for the variation which preserves the definition of the derivative. We refute a common proof that the limits of sine and cosine at infinity cannot exist. Then we use the newly defined modified variation in the definition of the derivative to compute the limits of sine and cosine at infinity.

Contents

1 Development of $\hat{\mathbb{C}}$

1.1 Properties of real numbers $\mathbb{R}$

Theorem 1.1.1

All functions of the form

$$f \quad : \quad \mathbb{R} \quad \rightarrow \quad \mathbb{R} \quad , \qquad \text{with} \qquad f(x) = mx + b \quad , \quad m, b \in \mathbb{R} \quad ,$$

are one-to-one.

Proof:

We say f is a one-to-one function when

$$f(x_1) = f(x_2) \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

Evaluation of $f(x)$ yields

$$mx_1 + b = mx_2 + b \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

All such functions f are one-to-one.

1.2 Properties of extended real numbers $\overline{\mathbb{R}}$

Definition 1.2.1

The extended real numbers are

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 1.2.2

The additive absorptive properties of $\pm\infty$ are such that

$$\forall \quad b \in \mathbb{R} \quad \exists \quad \pm\infty \in \overline{\mathbb{R}} \ , \qquad \text{such that} \qquad \pm\infty + b = \pm\infty \ .$$

Definition 1.2.3

The multiplicative absorptive properties of $\pm\infty$ are such that

$$\forall \quad b \in \mathbb{R} \ , \ b \neq 0 \quad \exists \quad \pm\infty \in \overline{\mathbb{R}} \ , \qquad \text{such that} \qquad \pm\infty \times b = \pm\infty \ .$$

Definition 1.2.4

The ∞ symbol is such that

$$x_n \in \mathbb{R} \ : \ \lim_{n\to\infty} x_n = \text{diverges} \qquad \longrightarrow \qquad x_n \in \overline{\mathbb{R}} \ : \ \lim_{n\to\infty} x_n = \infty \ ,$$

for $n \in \mathbb{N}$. (All further instances of n are implicitly $n \in \mathbb{N}$.)

Theorem 1.2.5

There is no additive inverse defined for ∞.

Proof:

Consider two $\overline{\mathbb{R}}$ sequences

$$x_n = 2n \ , \qquad \text{and} \qquad y_n = n \ ,$$

such that

$$\lim_{n\to\infty} x_n = \infty \ , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty \ .$$

It is an identity of limits that

$$\lim_{n\to\infty} \big(x_n - y_n\big) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n \ .$$

We obtain a contradiction with

$$\lim_{n\to\infty}\big(x_n - y_n\big) = \lim_{n\to\infty} y_n = \infty$$

$$\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = \infty - \infty = 0 \quad .$$

Therefore,

$$\infty - \infty = \text{undefined} \quad .$$

Definition 1.2.6

∞ does not have a multiplicative inverse so

$$\frac{\infty}{\infty} = \text{undefined} \quad .$$

Remarks 1.2.7

Even while ∞ does not have the inverse composition properties of the real numbers, $\overline{\mathbb{R}}$ has the useful property that one may use numbers on both sides of divergent limits. ∞ is a special number that $\overline{\mathbb{R}}$ was conceived to accommodate.

Theorem 1.2.8

Not all functions of the form

$$f \;:\; \mathbb{R} \;\to\; \overline{\mathbb{R}} \;, \qquad \text{with} \qquad f(x) = mx + b \;\;,\;\; m, b \in \overline{\mathbb{R}} \;\;,$$

are one-to-one.

Proof:

To show a contradiction with the definition of a one-to-one function, consider $m = \infty$. By the absorptive properties of ∞

$$f(x_1) = \infty\, x_1 + b = \infty \;\;, \qquad \text{and} \qquad f(x_2) = \infty\, x_2 + b = \infty \;\;,$$

but

$$\infty = \infty \qquad \not\Leftrightarrow \qquad x_1 = x_2 \quad .$$

We might show the same contradiction with $b = \infty$. Therefore, functions of this type are not always one-to-one.

Definition 1.2.9

For any $n \geq 1$ we have

$$\infty^n = \infty \quad .$$

1.3 Properties of modified extended real numbers $\widehat{\mathbb{R}}$

Definition 1.3.1

Modified extended real numbers are

$$\widehat{\mathbb{R}} \quad \equiv \quad \{\pm\widehat{\infty} + b : b \in \mathbb{R}, b \neq 0\} \quad .$$

They have the properties that

$$\forall \quad x \in \widehat{\mathbb{R}} \quad \exists \quad b \in \mathbb{R} \ , \quad b \neq 0 \ , \qquad \text{such that} \qquad x = \pm\widehat{\infty} + b \ ,$$

and

$$x_n \in \mathbb{R} \ \ : \ \ \lim_{n\to\infty} x_n = \text{diverges} \qquad \longrightarrow \qquad x_n \in \mathbb{R} \cup \{\widehat{\infty}\} \ \ : \ \ \lim_{n\to\infty} x_n = \widehat{\infty} \quad .$$

Remarks 1.3.2

The main difference between ∞ and $\widehat{\infty}$ is that we suppress the additive absorptive property of ∞ for $\widehat{\infty}$. In multiplication, we see that $\widehat{\infty}$ does not absorb -1 and when we make the extension to complex numbers it will not absorb $\pm i$.

Theorem 1.3.3

For any $x \in \widehat{\mathbb{R}}$

$$x = a\,\widehat{\infty} + b \qquad \Longleftrightarrow \qquad a, b \in \mathbb{R} \;\;,\;\; a = \pm 1 \;\;,\;\; b \neq 0 \;\;.$$

Proof:

Proof follows from the definition of $\widehat{\mathbb{R}}$.

Remarks 1.3.4

Numbers of the form

$$x = a\,\widehat{\infty} + b \;\;, \qquad \text{with} \qquad a \neq \pm 1 \;\;,\;\; b \neq 0 \;\;,$$

can be recast as $\widehat{\mathbb{R}}$ numbers by applying the multiplicative absorptive properties of $\widehat{\infty}$. To the contrary, numbers of the form

$$x = a\,\widehat{\infty} + b \;\;, \qquad \text{with} \qquad b = 0 \;\;,$$

cannot be cast as $\widehat{\mathbb{R}}$ numbers.

Definition 1.3.5

The operations $\widehat{\infty} - \widehat{\infty}$ and $\widehat{\infty}/\widehat{\infty}$ are undefined.

Definition 1.3.6

The additive absorptive properties of $\pm\widehat{\infty}$ are such that

$$\forall \quad b \in \mathbb{R} \;\;,\;\; b \neq 0 \quad \exists \quad \pm\widehat{\infty} \in \overline{\mathbb{R}} \;\;, \qquad \text{such that} \qquad \pm\widehat{\infty} + b \neq \pm\widehat{\infty} \;\;.$$

Expressions of the form $\widehat{\infty} + b$ are defined by self-identity.

Definition 1.3.7

The multiplicative absorptive properties of $\pm\widehat{\infty}$ are such that

$$\forall \quad b \in \mathbb{R} \ , \quad b \neq 0 \quad \exists \quad \pm\widehat{\infty} \in \overline{\mathbb{R}} \ , \qquad \text{such that} \qquad \pm\widehat{\infty} \times b = \pm\widehat{\infty} \ .$$

Definition 1.3.8

$\widehat{\mathbb{R}}$ numbers are such that

$$\widehat{\infty} + a = \widehat{\infty} + b \qquad \Longleftrightarrow \qquad a = b \ .$$

Definition 1.3.9

The additive composition law for $\widehat{\mathbb{R}} + \mathbb{R}$ is

$$\big(\pm\widehat{\infty} + a\big) + b = \pm\widehat{\infty} + \big(a + b\big) \ .$$

Definition 1.3.10

The additive composition laws for $\widehat{\mathbb{R}} \pm \widehat{\infty}$ are

$$\big(\pm\widehat{\infty} + a\big) \pm \widehat{\infty} = \big(\pm\widehat{\infty} + a\big)$$

$$\big(\pm\widehat{\infty} + a\big) \mp \widehat{\infty} = a \ .$$

Definition 1.3.11

The additive composition laws for $\widehat{\mathbb{R}} + \widehat{\mathbb{R}}$ are

$$\big(\pm\widehat{\infty} + a\big) + \big(\pm\widehat{\infty} + b\big) = \pm 2\,\widehat{\infty} + \big(a + b\big) = \pm\widehat{\infty} + \big(a + b\big)$$

$$\big(\widehat{\infty} + a\big) + \big(-\widehat{\infty} + b\big) = a + b \ ,$$

where

$$2\,\widehat{\infty} = \widehat{\infty} \quad ,$$

follows from the absorptive properties of $\widehat{\infty}$ (Definition 1.3.7.)

Theorem 1.3.12

The additive composition laws for $\widehat{\mathbb{R}}$ do not require an additive inverse for $\widehat{\infty}$.

Proof:

Consider the additive composition of two $\widehat{\mathbb{R}}$ numbers

$$x_1 = \widehat{\infty} + b_1 \quad , \qquad\qquad \text{and} \qquad\qquad x_2 = \pm\widehat{\infty} + b_2 \quad .$$

The case of $b_1 = b_2 = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$.

Theorem 1.3.13

All $\widehat{\mathbb{R}}$ numbers have an additive inverse.

Proof

Consider the case of $b = -a$ in the identity

$$\big(\widehat{\infty} + a\big) + \big(-\widehat{\infty} + b\big) = a + b \quad .$$

Then

$$\forall \quad x = \widehat{\infty} + a \quad \exists \quad x' = -\widehat{\infty} - a \quad , \qquad\qquad \text{such that} \qquad\qquad x + x' = 0 \quad .$$

This is the definition of the additive inverse.

Remarks 1.3.14

We can extract a multiplicative composition law from the absorptive properties of $\widehat{\infty}$

$$b \times \widehat{\infty} = \widehat{\infty} \quad ,$$

but we cannot extract the law for division because multiplying both sides by $\widehat{\infty}^{-1}$ yields the undefined expression $\widehat{\infty} \times \widehat{\infty}^{-1}$. If there was a multiplicative inverse for $\widehat{\infty}$ then we could use the rule for division to write

$$\frac{\widehat{\infty} + b}{\widehat{\infty}} \times \widehat{\infty} = b \times \widehat{\infty} \qquad \Longrightarrow \qquad \widehat{\infty} + b = \widehat{\infty} \quad .$$

This contradicts the additive property of $\widehat{\infty}$ that

$$\widehat{\infty} + b \neq \widehat{\infty} \quad ,$$

so $\widehat{\infty}/\widehat{\infty}$ must be undefined.

Remarks 1.3.15

Operations of the form $\widehat{\mathbb{R}} + \widehat{\infty} - \widehat{\infty}$ are undefined because $\widehat{\infty} - \widehat{\infty}$ is not defined. Add $\widehat{\infty}$ to both sides of

$$\big(\widehat{\infty} + a\big) - \widehat{\infty} = a \quad ,$$

to obtain

$$\big(\widehat{\infty} + a\big) - \widehat{\infty} + \widehat{\infty} = a + \widehat{\infty} \quad .$$

By adding the quantity in parentheses to either of $\pm\widehat{\infty}$ first and then adding $\mp\widehat{\infty}$, we may obtain two different values

$$\Big[\big(\widehat{\infty} + a\big) - \widehat{\infty}\Big] + \widehat{\infty} = a + \widehat{\infty} \quad , \qquad \text{and} \qquad \Big[\big(\widehat{\infty} + a\big) + \widehat{\infty}\Big] - \widehat{\infty} = a \quad .$$

To the contrary of $\widehat{\mathbb{R}}+\widehat{\infty}-\widehat{\infty}$, expressions like $\left(\widehat{\mathbb{R}}+\widehat{\infty}\right)-\widehat{\infty}$ and $\left(\widehat{\mathbb{R}}-\widehat{\infty}\right)+\widehat{\infty}$ are perfectly well defined because the order of operations is specified by the bracketing.

Definition 1.3.16

Infinity written as $\widehat{\infty}$ does not absorb infinity. In other words, for any $n > 1$

$$\widehat{\infty}^n \neq \widehat{\infty} \quad .$$

Definition 1.3.17

All composition laws written for $\widehat{\infty}$ apply for $\widehat{\infty}^n$. For example,

$$\left(\widehat{\infty}^2 + b\right) - \widehat{\infty}^2 = b$$

$$\left(\widehat{\infty}^2 + b\right) - \widehat{\infty} \neq b$$

$$b \times \widehat{\infty}^2 = \widehat{\infty}^2$$

$$\frac{b}{\widehat{\infty}^2} = 0 \quad .$$

Theorem 1.3.18

The additive inverse property of $\widehat{\mathbb{R}}$ is consistent with the definition of the limit.

Proof:

Due to the absorptive properties of ∞, limits in $\overline{\mathbb{R}}$ can have the form

$$\lim_{n\to\infty} x_n = \infty = \infty + a \quad , \qquad \text{with} \qquad x_n, a \in \mathbb{R} \quad ,$$

but there will never be a limit in $\widehat{\mathbb{R}}$ of the form

$$\lim_{n\to\infty} x_n = \widehat{\infty} = \widehat{\infty} + a \quad , \qquad \text{because} \qquad \widehat{\infty} + a \neq \widehat{\infty} \quad .$$

(This follows from Definition 1.3.6.) Therefore, one is not able to use the additive inverse properties of $\widehat{\mathbb{R}}$ to obtain a contradiction of the type used to prove Theorem 1.2.5.

Definition 1.3.19

$\widehat{\mathbb{R}}$ numbers are such that

$$x_n \in \overline{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \widehat{\infty} \qquad\qquad \longrightarrow \qquad\qquad x_n \in \widehat{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \text{diverges} \quad ,$$

because $\widehat{\infty} \notin \widehat{\mathbb{R}}$.

Remarks 1.3.20

If we wanted to infinitely continue $\widehat{\mathbb{R}} \to \overline{\widehat{\mathbb{R}}}$ in the fashion of $\mathbb{R} \to \overline{\mathbb{R}}$ such that

$$\widehat{\mathbb{R}} \quad : \quad \lim_{n\to\infty} x_n = \text{diverges} \qquad\qquad \longrightarrow \qquad\qquad \overline{\widehat{\mathbb{R}}} \quad : \quad \lim_{n\to\infty} x_n = \widehat{\infty} \quad ,$$

then we would mirror the extension of

$$\mathbb{R} \quad \to \quad \mathbb{R}\cup\{\pm\infty\} \ , \qquad\qquad \text{with} \qquad\qquad \{\pm\widehat{\infty}+b : b \in \mathbb{R}, b \neq 0\} \quad \to \quad \{\pm\widehat{\infty}+b : b \in \mathbb{R}\} \ ,$$

where the case of $b = 0$ defines a special number $\widehat{\infty}$ without an additive inverse.

Definition 1.3.21

$\widehat{\mathbb{R}}$ is defined such that for any $a, b > 0$

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad\qquad \Longleftrightarrow \qquad\qquad a < b \quad .$$

Remarks 1.3.22

A good way to visualize modified extended real numbers is to write

$$x \in \mathbb{R} \qquad\qquad \Longleftrightarrow \qquad\qquad x \quad \equiv \quad \widehat{0} + x \quad ,$$

where x measures distance from the origin $\widehat{0}$. We may transfinitely extend the real number line to include the points at infinity and an interval beyond such that $\pm\widehat{\infty}$ are the origins of $\widehat{\mathbb{R}}$. Then we have

$$x \in \widehat{\mathbb{R}} \qquad\qquad \Longleftrightarrow \qquad\qquad x = \pm\widehat{\infty} + b \quad ,$$

where b measures distance from another origin $\widehat{\infty}$ or $-\widehat{\infty}$ located infinitely far away from the Cartesian origin $\widehat{0}$. In particular, this makes a lot of sense for the additive identity (Definition 1.3.11)

$$\big(\widehat{\infty} + a\big) + \big(\widehat{\infty} + b\big) = \widehat{\infty} + \big(a + b\big)$$

We have mentioned functions of the form

$$y = mx + b \quad ,$$

because the function which shifts the origin

$$f \quad : \quad b \quad \to \quad \widehat{\infty} + b \quad ,$$

is a case of the same.

Theorem 1.3.23

All functions of the form

$$f \quad : \quad \mathbb{R} \quad \to \quad \widehat{\mathbb{R}} \quad , \qquad\qquad \text{with} \qquad\qquad f(x) = mx + b \quad , \quad m, b \in \mathbb{R} \cup \widehat{\mathbb{R}} \quad ,$$

are one-to-one.

Proof:

Consider $m = \widehat{\infty} + a_1$ and $b = \widehat{\infty} + a_2$. By the additive and multiplicative properties of $\widehat{\infty}$ we find that

$$f(x_1) = \big(\widehat{\infty} + a_1\big)x_1 + \big(\widehat{\infty} + a_2\big) = \big(\widehat{\infty} + a_1 x_1\big) + \big(\widehat{\infty} + a_2\big) = \widehat{\infty} + \big(a_1 x_1 + a_2\big)$$

$$f(x_2) = \big(\widehat{\infty} + a_1\big)x_2 + \big(\widehat{\infty} + a_2\big) = \big(\widehat{\infty} + a_1 x_2\big) + \big(\widehat{\infty} + a_2\big) = \widehat{\infty} + \big(a_1 x_2 + a_2\big) \quad .$$

By the non-absorptive additive properties of $\widehat{\infty}$

$$\widehat{\infty} + \big(a_1 x_1 + a_2\big) = \widehat{\infty} + \big(a_1 x_2 + a_2\big) \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

The case of $m, b \in \mathbb{R}$ was treated in Theorem 1.1.1, so we have shown that all such functions are one-to-one.

1.4 Properties of modified extended complex numbers $\widehat{\mathbb{C}}$

Definition 1.4.1

Complex numbers are

$$\mathbb{C} \quad \equiv \quad \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}\} \quad .$$

Definition 1.4.2

Extended complex numbers are

$$\overline{\mathbb{C}} \quad \equiv \quad \{x + iy : x \in \overline{\mathbb{R}}, y \in \overline{\mathbb{R}}\} \quad , \qquad \text{where} \qquad i\infty \neq \infty \quad .$$

Definition 1.4.3

Modified extended complex numbers are such that

$$\widehat{\mathbb{C}} \quad \equiv \quad \{\widehat{\infty} \pm i\,\widehat{\infty} + Z, -\widehat{\infty} \pm i\,\widehat{\infty} + Z : Z \in \mathbb{C}, \mathrm{Im}(Z) \neq 0, \mathrm{Re}(Z) \neq 0\} \quad .$$

Definition 1.4.4

Infinitely continued modified extended complex numbers $\overline{\widehat{\mathbb{C}}}$ are such that

$$\overline{\widehat{\mathbb{C}}} \quad \equiv \quad \widehat{\mathbb{C}} \cup \{\mathrm{Im}(z)=0, \mathrm{Re}(z)=0\} \quad .$$

Definition 1.4.5

$\widehat{\mathbb{C}}$ is such that $\{\pm\widehat{\infty}, \pm i\,\widehat{\infty}\}$ are four distinct symbols, all of which are compound symbols when we write "$+\widehat{\infty}$."

Definition 1.4.6

The additive composition laws for $\widehat{\mathbb{C}} + \mathbb{C}$ are

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) + z = \widehat{\infty} + \pm i\,\widehat{\infty} + \big(Z + z\big)$$

$$\big(-\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) + z = -\widehat{\infty} + \pm i\,\widehat{\infty} + \big(Z + z\big) \quad .$$

Definition 1.4.7

The additive composition laws for $\widehat{\mathbb{C}} \pm \widehat{\infty}$ and $\widehat{\mathbb{C}} \pm i\,\widehat{\infty}$ are

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) \pm \widehat{\infty} = \widehat{\infty} \pm i\,\widehat{\infty} + Z$$

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) \mp \widehat{\infty} = \pm i\,\widehat{\infty} + Z$$

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) \pm i\,\widehat{\infty} = \widehat{\infty} \pm i\,\widehat{\infty} + Z$$

$$\big(\widehat{\infty} \pm i\,\widehat{\infty} + Z\big) \mp i\,\widehat{\infty} = \widehat{\infty} + Z \quad .$$

Remarks 1.4.8

The additive properties of $\widehat{\mathbb{C}} + \widehat{\mathbb{C}}$ are implicit in the other composition laws.

Definition 1.4.9

The multiplicative properties of $\pm\widehat{\infty}$ are

$$-1 \times \pm\widehat{\infty} = \mp\widehat{\infty}$$

$$i \times \pm\widehat{\infty} = \pm i\,\widehat{\infty}$$

$$-i \times \pm\widehat{\infty} = \mp i\,\widehat{\infty}$$

$$\big(1 + i\big) \times \pm\widehat{\infty} = \text{undefined} \quad ,$$

and for any non-zero $b \in \mathbb{R}$

$$b \times \pm\widehat{\infty} = \text{sign}(b) \times \pm\widehat{\infty} \quad .$$

<u>Definition 1.4.10</u>

The multiplicative properties of $\pm i\,\widehat{\infty}$ follow from Definition 1.4.9.

<u>Definition 1.4.11</u>

The absorptive properties of $\pm\widehat{\infty}$ are

$$\forall \quad x \in \mathbb{R} \ , \ \ x > 0 \quad \exists \quad \widehat{\infty} \in \widehat{\mathbb{R}} \ , \qquad \text{such that} \qquad x\big(\widehat{\infty}\big) = \big(\widehat{\infty}\big)$$

$$\forall \quad x \in \mathbb{R} \ , \ \ x < 0 \quad \exists \quad \widehat{\infty} \in \widehat{\mathbb{R}} \ , \qquad \text{such that} \qquad x\big(\widehat{\infty}\big) = \big(-\widehat{\infty}\big) \quad .$$

<u>Definition 1.4.12</u>

The absorptive properties of $\pm i\,\widehat{\infty}$ are

$$\forall \quad x \in \mathbb{R} \ , \ \ x > 0 \ , \ \ i\,\widehat{\infty} \in \widehat{\mathbb{R}} \ , \qquad \text{such that} \qquad x\big(i\,\widehat{\infty}\big) = \big(i\,\widehat{\infty}\big)$$

$$\forall \quad x \in \mathbb{R} \ , \ \ x < 0 \ , \ \ i\,\widehat{\infty} \in \widehat{\mathbb{R}} \ , \qquad \text{such that} \qquad x\big(i\,\widehat{\infty}\big) = \big(-i\,\widehat{\infty}\big) \quad .$$

Theorem 1.4.13

Infinity $\widehat{\infty}$ does not obey the distributive property of multiplication.

Proof:

If $\widehat{\infty}$ had a distributive multiplicative property then

$$\big(1+i\big)\widehat{\infty} = \widehat{\infty} + i\,\widehat{\infty} \quad .$$

This contradicts the the multiplicative properties of $\widehat{\infty}$ (Definition 1.4.9).

Definition 1.4.14

For two modified extended complex numbers

$$z_1 = \widehat{\infty} + i\,\widehat{\infty} + Z_1 \quad , \qquad\qquad \text{and} \qquad\qquad z_2 = \widehat{\infty} + i\,\widehat{\infty} + Z_2 \quad ,$$

we have

$$z_1 = z_2 \qquad\qquad \Longleftrightarrow \qquad\qquad Z_1 = Z_2 \quad .$$

Definition 1.4.15

Any sequence of the form

$$z_n \in \mathbb{C} \quad , \quad z_n = x_n + iy_n \quad , \qquad\qquad \text{with} \qquad\qquad x_n, y_n \in \mathbb{R} \quad , \quad x_n, y_n > 0 \quad ,$$

is such that

$$z_n \in \mathbb{C} \quad : \quad \begin{cases} \lim\limits_{n\to\infty} x_n = \text{diverges} \\ \\ \lim\limits_{n\to\infty} y_n = \text{diverges} \end{cases} \quad , \qquad \longrightarrow \qquad z_n \in \overline{\overline{\mathbb{C}}} \quad : \quad \lim_{n\to\infty} z_n = \widehat{\infty} + i\,\widehat{\infty} \quad .$$

Corollary 1.4.16

$\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on the Riemann sphere $\mathbb{S}^2$.

Proof

A is the complement of B on $\mathbb{S}^2$ if

$$\mathbb{S}^2 \quad \equiv \quad A \cup B \quad .$$

The Riemann sphere is obtained from $\mathbb{C}$ by adding a point for infinity to both ends of the real and imaginary axes and then imposing, in addition to the preexisting properties, new conditions

$$\pm\infty = \infty \quad , \qquad\qquad \text{and} \qquad\qquad \pm i\infty = \infty \quad .$$

It follows that

$$\mathbb{S}^2 \quad \equiv \quad \{\overline{\mathbb{C}} : \pm\infty \to \infty, \pm i\infty \to \infty\} \quad .$$

Imposing these conditions on $\widehat{\mathbb{C}}$ (Definition 1.4.3) gives

$$\widehat{\mathbb{C}} \quad \to \quad \{\infty\} \quad .$$

Since it is the definition of the Riemann sphere that

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\infty\} \quad ,$$

we can use the definition of the complement to write

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\widehat{\mathbb{C}} : \pm\widehat{\infty} \to \infty, \pm i\,\widehat{\infty} \to \infty\} \quad .$$

1.5 Properties of modified complex numbers $\hat{\mathbb{C}}$

Definition 1.5.1

Modified complex numbers $\hat{\mathbb{C}}$ shall be such that

$$z \in \hat{\mathbb{C}} \qquad \Longrightarrow \qquad z = \begin{cases} x + iy^+ & \text{for} \quad \mathrm{Im}(z) > 0 \\ x & \text{for} \quad \mathrm{Im}(z) = 0 \\ x - iy^- & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad ,$$

where

$$y^{\pm}(y) \quad : \quad \mathbb{R} \quad \rightarrow \quad \widehat{\mathbb{R}} \;\; ,$$

with

$$y^+(y) = \widehat{\infty} - y \;\; , \qquad \text{and} \qquad y^-(y) = \widehat{\infty} + y \;\; .$$

More broadly

$$\hat{\mathbb{C}} \quad \equiv \quad \mathbb{R} \cup \{x \pm iy^{\pm} : x \in \mathbb{R}, y^{\pm} \in \widehat{\mathbb{R}}\} \;\; .$$

Theorem 1.5.2

$\hat{\mathbb{C}}$ numbers are such that

$$z \in \hat{\mathbb{C}} \;\; , \;\; z = x \pm iy^{\pm} \qquad \Longrightarrow \qquad 0 < y^{\pm} < \infty \;\; .$$

Proof:

Theorem is proven with

$$|\widehat{\infty}| = \infty \;\; .$$

$y^{\pm}$ are such that

$$y^{\pm} \quad : \quad \mathbb{R} \quad \rightarrow \quad \widehat{\mathbb{R}} \;\; , \qquad \text{and} \qquad y^{\pm} = \big(\widehat{\infty} \mp y\big) \;\; , \;\; y \in \mathbb{R} \;\; .$$

By the definition of $\widehat{\mathbb{R}}$, $y^{\pm} = \widehat{\infty}$ and $y^{\pm} = 0$ are not allowed. For any $a, b \in \mathbb{R}$ with $a, b > 0$ (Definition 1.3.22) we have

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad\qquad \Longleftrightarrow \qquad\qquad a < b \quad ,$$

wherein $a, b > 0$ follows from the restriction of the domain of $y^{\pm}(y)$ in $z = x + iy^{\pm}$ (Definition 1.5.1). This shows that $y^{\pm}$ increases as $|y|$ decreases. Therefore,

$$\sup y^{\pm} = y^{\pm}(\inf |y|) \quad .$$

$y \in \mathbb{R}$ gives

$$\inf |y| = 0 \qquad\qquad \Longrightarrow \qquad\qquad \sup y^{\pm} = \widehat{\infty} - 0 = \widehat{\infty} \quad .$$

$y^{\pm} < \widehat{\infty}$ follows because $\widehat{\infty} \notin \widehat{\mathbb{R}}$. To show that $y^{\pm}$ is always greater than zero, consider that

$$\forall \quad b \in \mathbb{R} \quad , \quad \widehat{\infty} > b \qquad\qquad \Longrightarrow \qquad\qquad \widehat{\infty} - b > 0 \quad .$$

Theorem 1.5.3

The $y^{\pm}(y)$ functions which generate $\hat{\mathbb{C}}$ are analytic.

Proof:

It suffices to show that the functions satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \qquad\qquad \text{and} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad .$$

For

$$z = x \pm iy^{\pm} = x \pm i\big(\widehat{\infty} \pm y\big) \quad ,$$

we have

$$u = x \quad , \qquad\qquad \text{and} \qquad\qquad v = \pm\big(\widehat{\infty} \pm y\big) \quad .$$

Analyticity follows by evaluation.

Remarks 1.5.4

When we use

$$z \in \mathbb{C} \qquad \Longrightarrow \qquad z = re^{i\theta} \quad , \quad r, \theta \in \mathbb{R} \quad ,$$

where

$$r(x,y) = \sqrt{x^2 + y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \quad ,$$

we do not need to define an entire new class of analysis with some variant of $\mathbb{C}'$ to distinguish it from

$$z \in \mathbb{C} \qquad \Longrightarrow \qquad z = x + iy \quad , \quad x, y \in \mathbb{R} \quad .$$

In $\widehat{\mathbb{C}}$, we did not add the point at infinity to $\mathbb{C}$ but we did take away the points along the real and imaginary axes of $\mathbb{C}$ because $\widehat{\infty} - \widehat{\infty}$ is not defined. Therefore, a unique construction requires a unique label. With regards to $\hat{\mathbb{C}}$, however, we have neither added the point at infinity nor taken away any points so there is an argument to be made that

$$\hat{\mathbb{C}} \quad \equiv \quad \mathbb{C} \quad .$$

2 Properties of $\mathbb{C}$

2.1 Definition of a representation of complex numbers $\mathbb{C}$

Definition 2.1.1

$(\!(x, y)\!)$ is the Cartesian representation of $\mathbb{C}$ in which

$$z(x,y) = x + iy \quad .$$

We say

$$(\!(x, y)\!) \quad \equiv \quad z(x, y) \quad \equiv \quad x + iy \quad .$$

Definition 2.1.2

$(\!(x_2, y_2)\!)$ is a representation of $\mathbb{C}$ if and only if $(\!(x_1, y_1)\!)$ is a representation of $\mathbb{C}$ and there exist two conversion functions

$$x_2 = x_2(x_1, y_1) \quad , \qquad \text{and} \qquad y_2 = y_2(x_1, y_1) \quad ,$$

whose domains are all of $\mathbb{C}$.

Definition 2.1.3

For any function of a complex variable $f(z)$

$$f \quad : \quad \mathbb{C} \quad \rightarrow \quad \mathbb{C} \qquad \Longleftrightarrow \qquad f(z_1) \quad : \quad (\!(x_1, y_1)\!) \quad \rightarrow \quad (\!(x_1, y_1)\!) \quad .$$

Theorem 2.1.4

$(\!(r, \theta)\!)$ is a representation of $\mathbb{C}$.

Proof:

$(\!(x, y)\!)$ is a representation of $\mathbb{C}$ and we have two conversion functions

$$r(x, y) = \sqrt{x^2 + y^2} \quad , \qquad \text{and} \qquad \theta(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \quad .$$

$(\!(r, \theta)\!)$ is a representation of $\mathbb{C}$ because all of $\mathbb{C}$ is in the domain of the conversion functions.

Definition 2.1.4

If $(\!(x_1, y_1)\!)$ and $(\!(x_2, y_2)\!)$ are two representations of $\mathbb{C}$ then there exists a representing functional of two conversion functions

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1,y_1)\!)] \quad : \quad (\!(x_1,y_1)\!) \quad \rightarrow \quad (\!(x_2,y_2)\!) \quad ,$$

where $x_1(x_2,y_2), y_1(x_2,y_2)$ are the two implicit conversion functions. The rule for constructing the representing functional with the conversion functions $x_1(x_2,y_2)$ and $y_1(x_2,y_2)$ is that

$$(\!(x_1,y_1)\!) \quad \equiv \quad z(x_1,y_1) \qquad \longrightarrow \qquad (\!(x_2,y_2)\!) \quad \equiv \quad z[x_1(x_2,y_2), y_1(x_2,y_2)] \quad .$$

Example 2.1.5

Here we use the representing functional

$$z_{(\!(r,\theta)\!)}[(\!(x,y)\!)] = (\!(r,\theta)\!) \quad .$$

to construct the polar representation of $\mathbb{C}$ from its Cartesian representation. The conversion functions are

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .$$

The representing functional is

$$z_{(\!(r,\theta)\!)}[(\!(x,y)\!)] = z_{(\!(r,\theta)\!)}[x+iy] = r\cos(\theta) + ir\sin(\theta) = re^{i\theta} \quad .$$

Therefore,

$$(\!(r,\theta)\!) = re^{i\theta} \quad .$$

Example 2.1.6

Here we use the representing functional

$$z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] = (\!(x,y)\!) \quad .$$

to construct the Cartesian representation of $\mathbb{C}$ from its polar representation. The conversion functions are

$$r(x,y) = \sqrt{x^2+y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\frac{y}{x} \quad .$$

The representing functional is

$$\begin{aligned} z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] &= z_{(\!(x,y)\!)}[re^{i\theta}] \\ &= \sqrt{x^2+y^2}\, e^{i\tan^{-1}(y/x)} \\ &= \sqrt{x^2+y^2}\, \cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + i\sqrt{x^2+y^2}\, \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\ &= \sqrt{x^2+y^2}\left(\frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) + i\sqrt{x^2+y^2}\left(\frac{\left(\frac{y}{x}\right)}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) = x+iy \quad . \end{aligned}$$

Therefore,

$$(\!(x,y)\!) = x+iy \quad .$$

Remarks 2.1.7

The polar representation requires incorporation of the number e so we should consider other representations that include different numbers such as $\widehat{\infty}$.

Definition 2.1.8

If we have a representation

$$(\!(f(x_2), g(y_2))\!) \quad \equiv \quad z(x_2, y_2) \quad ,$$

then the rule for constructing

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)] = (\!(f(x_2), g(y_2))\!) \quad ,$$

is

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)] \quad \equiv \quad z_{(\!(f(x_2),g(y_2))\!)}[f(x_1),g(y_1)] \quad .$$

Example 2.1.9

To see that the rule for representations labeled with functions is consistent with the definition of the representation, consider

$$f(x)=x \quad , \qquad\qquad \text{and} \qquad\qquad g(y)=y \quad ,$$

so that

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)] = z_{(\!(f(x_2),g(y_2))\!)}[f(x_1),g(y_1)] \quad .$$

To define the quantity in square brackets we need to know the form of $z_1=(x_1,y_1)$. Let $(\!(x_1,y_1)\!)$ be the Cartesian representation so that

$$\begin{aligned} z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)] &= z_{(\!(f(x_2),g(y_2))\!)}[f(x_1)+ig(y_1)] \\ &= f(x_1(x_2,y_2))+ig(y_1(x_2,y_2)) \\ &= x_1(x_2,y_2)+iy_1(x_2,y_2) \quad . \end{aligned}$$

$x_1(x_2,y_2)$ and $y_1(x_2,y_2)$ are the conversion functions of the Cartesian representation $(\!(x_1,y_1)\!)$ such that

$$z_1(x_1,y_1) \quad \equiv \quad x_1+iy_1 \qquad\qquad \longrightarrow \qquad\qquad z_1(x_2,y_2) \quad \equiv \quad x_1(x_2,y_2)+iy_1(x_2,y_2) \quad .$$

Theorem 2.1.10

The representation of $\mathbb{C}$ corresponding to $\hat{\mathbb{C}}$ is

$$(\!(x_2,\{\emptyset,\pm\widehat{\infty}-y^{\pm}\})\!) \quad \equiv \quad z(x_2,\{0,y^{\pm}\}) \quad ,$$

with Cartesian conversion functions

$$x(x_2, y^+) = x_2 \quad , \qquad\qquad \text{and} \qquad\qquad y(x_2, y^+) = \begin{cases} \widehat{\infty} - y^+ & \text{for} \quad \text{Im}(z) > 0 \\ 0 & \text{for} \quad \text{Im}(z) = 0 \\ \widehat{\infty} + y^- & \text{for} \quad \text{Im}(z) < 0 \end{cases} \quad .$$

Proof:

All of $\mathbb{C}$ is in the domain of these functions. $\hat{\mathbb{C}}$ is piecewise defined so it suffices to show that the pieces satisfy the definitions. For $(\!(x, \emptyset)\!)$ we have conversion functions

$$x(x_2, y^+) = x_2 \quad , \qquad\qquad \text{and} \qquad\qquad y(x_2, y^+) = 0 \quad ,$$

such that

$$z_{(\!(x_2,\emptyset)\!)}[(\!(x, y)\!)] = z_{(x,y)\to(\!(x_2,\emptyset)\!)}[x + iy] = x(x_2, y^+) + iy(x_2, y^+) = x_2 \quad .$$

Therefore,

$$(\!(x_2, \emptyset)\!) = x_2 \quad , \qquad\qquad \text{where} \qquad\qquad x_2 \quad \equiv \quad x \quad .$$

For $(\!(x_2, \widehat{\infty} - y^+)\!)$ we have

$$f(x) = x \quad , \qquad\qquad \text{and} \qquad\qquad g(y) = \widehat{\infty} - y \quad .$$

Therefore,

$$f(x_1(x_2, y_2)) + ig(y_1(x_2, y_2)) = x_1(x_2, y_2) + i\big(\widehat{\infty} - y_1(x_2, y_2)\big) \quad .$$

For $z \in \hat{\mathbb{C}}$ with $\text{Im}(z) > 0$, our conversion functions are

$$x(x_2, y^+) = x_2 \quad , \qquad\qquad \text{and} \qquad\qquad y(x_2, y^+) = \widehat{\infty} - y^+ \quad .$$

The representing functional of the conversion functions is

$$z_{(\!(x_2,\widehat{\infty}-y^+)\!)}\big[(\!(x,y)\!)\big] = z_{(x,y)\to(x,y^+)}\big[x+ig(y)\big]$$

$$= x(x_2,y^+) + i\big(\widehat{\infty} - y(x_2,y^+)\big)$$

$$= x_2 + i\big[\widehat{\infty} - \big(\widehat{\infty} - y^+\big)\big] \quad .$$

Since $y^+ \notin \mathbb{R}$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number and the quantity in square brackets is not formatted for an additive composition $\widehat{\infty} - \widehat{\mathbb{R}}$. Substitute $y^+ = \widehat{\infty} - y$ so that

$$z_{(\!(x_2,\widehat{\infty}-y^+)\!)}\big[(\!(x,y)\!)\big] = x_2 + i\big\{\widehat{\infty} - \big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big]\big\} \quad .$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\mathbb{R}} + \widehat{\infty}$ so

$$z_{(\!(x_2,\widehat{\infty}-y^+)\!)}\big[(\!(x,y)\!)\big] = x_2 + i\big(\widehat{\infty} - y\big)$$

$$= x_2 + iy^+ \quad .$$

Therefore,

$$(\!(x_2,\widehat{\infty} - y^+)\!) = x_2 + iy^+ \quad .$$

The final case is $(\!(x_2,-\widehat{\infty} - y^-)\!)$. We have

$$f(x) = x \quad , \qquad\qquad \text{and} \qquad\qquad g(y) = -\widehat{\infty} - y \quad .$$

Therefore,

$$f(x_1(x_2,y_2)) + ig(y_1(x_2,y_2)) = x_1(x_2,y_2) + i\big(-\widehat{\infty} - y_1(x_2,y_2)\big) \quad .$$

For $z \in \hat{\mathbb{C}}$ with $\mathrm{Im}(z) < 0$, our conversion functions are

$$x(x_2, y^-) = x_2 \quad , \qquad\qquad \text{and} \qquad\qquad y(x_2, y^-) = y^- - \widehat{\infty} \quad .$$

The representing functional is

$$\begin{aligned} z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] &= z_{(\!(x_2,\widehat{\infty}+y^-)\!)}[x + ig(y)] \\ &= x(x_2, y^+) - i\big(\widehat{\infty} + y(x_2, y^-)\big) \\ &= x_2 - i\big[\widehat{\infty} + \big(y^- - \widehat{\infty}\big)\big] \quad . \end{aligned}$$

Since $y^- \notin \mathbb{R}$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number. The quantity in square brackets is not formatted for an additive composition $\widehat{\infty} - \widehat{\mathbb{R}}$. Substitute $y^- = \widehat{\infty} + y$ so that

$$z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] = x_2 - i\big\{\widehat{\infty} + \big[\big(\widehat{\infty} + y\big) - \widehat{\infty}\big]\big\} \quad .$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\mathbb{R}} + \widehat{\infty}$ so

$$\begin{aligned} z_{(\!(x_2,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] &= x_2 - i\big(\widehat{\infty} + y\big) \\ &= x_2 - iy^- \quad . \end{aligned}$$

Therefore,

$$(\!(x_2, -\widehat{\infty} - y^-)\!) = x_2 - iy^- \quad .$$

We have proven that

$$(\!(x_2, \{\emptyset, \widehat{\infty} \mp y^\pm\})\!) \quad \equiv \quad \begin{cases} x_2 + iy^+ & \text{for} \quad \mathrm{Im}(z) > 0 \\ x_2 & \text{for} \quad \mathrm{Im}(z) = 0 \\ x_2 - iy^- & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad .$$

Example 2.1.11

In this example we show that the representing functional correctly recovers the Cartesian representation from the $\hat{\mathbb{C}}$ representation. The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

and the representing functional is

$$\begin{aligned} z_{(\!(x,y)\!)}\big[(\!(x, \widehat{\infty} - y^+)\!)\big] &= z_{(\!(x,y)\!)}\big[x + i\big(\widehat{\infty} - y^+\big)\big] \\ &= x(x,y) + i\big(\widehat{\infty} - y^+(x,y)\big) \\ &= x + i\big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big] \\ &= x + iy \quad . \end{aligned}$$

We have shown that the representing functional takes the $\hat{\mathbb{C}}$ representation and returns the Cartesian representation.

Remarks 2.1.12

At this point, the reader hopefully is asking, "What is this convoluted notation for?" We introduce the rigorous representation to quantify what we mean by phrases like "the Cartesian representation of $\mathbb{C}$," or "the polar representation of $\mathbb{C}$," or even "the $\hat{\mathbb{C}}$ representation of $\mathbb{C}$." For instance, we might wish to state precisely that the conversion functions of the Cartesian representation to the polar representation are analytic but the conversion functions of the Cartesian representation to the $\hat{\mathbb{C}}$ representation are one-to-one. Not only that, $y^{\pm}(x,y)$ *are* analytic functions while $x_2(x,y) = \mathrm{Re}(z(x,y))$ is not a complex analytic function in any representation of $\mathbb{C}$.

Example 2.1.13

Consider a complex number

$$z_0^+ = \alpha + i\,\widehat{\infty} \quad .$$

expressed in the z^+ piece of the $\hat{\mathbb{C}}$ representation. This number can never appear in the conversion from the Cartesian representation to the $\hat{\mathbb{C}}$ representation due to the piecewise definitions for $\mathrm{Im}(z) = 0$. However, we might begin with a number in the $\hat{\mathbb{C}}$ representation and wish to express it in the Cartesian representation. If we plug z_0^+ directly into the representing functional then we will obtain an undefined expression. The representing functional is

$$
\begin{aligned}
z_{(\!(x,y)\!)}[(\!(x,\widehat{\infty}-y^+)\!)] &= z_{(\!(x,y)\!)}[x+i\big(\widehat{\infty}-y^+\big)] \\
&= z_{(\!(x,y)\!)}[\alpha+i\big(\widehat{\infty}-\widehat{\infty}\big)] \quad .
\end{aligned}
$$

but we would have no way to evaluate the undefined expression. Since α is real, we can not put it inside the parentheses to form and $\widehat{\mathbb{R}}$ number. Therefore, the representing functional is not defined for $y^\pm = \widehat{\infty}$. However, if we require $y^\pm \in \widehat{\mathbb{R}}$ then we will never encounter the values $y^\pm = \widehat{\infty}$.

Corollary 2.1.14

As an illustration of the high significance of conversion functions, consider the Gaussian integral

$$
I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \quad .
$$

This integral is analytically intractable in the Cartesian representation of $\mathbb{C}$ (except by quadrature) but it can be solved easily in the polar representation. We write canonically

$$
I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \times \int_{-\infty}^{\infty} dx\, e^{-x^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)} \quad ,
$$

and then insert the conversion function

$$
r(x,y) = \sqrt{x^2+y^2} \quad .
$$

We obtain the infinitesimal element of polar area from the conversion functions

$$
x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad x(r,\theta) = r\cos(\theta) \quad ,
$$

via

$$dx = \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial \theta}\, d\theta \quad , \qquad \text{and} \qquad dy = \frac{\partial y}{\partial r}\, dr + \frac{\partial y}{\partial \theta}\, d\theta \quad .$$

Then

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, re^{-r^2} \qquad \Longleftrightarrow \qquad I(z) = \sqrt{\pi} \quad .$$

2.2 Definition of the representational derivative d/dz_1

Remarks 2.2.1

To prove the limits of sine and cosine at infinity, we will use the definition of the derivative. First, we will compare the conventions for derivatives with respect to

$$z = x + iy \quad , \qquad \text{and} \qquad z = re^{i\theta} \quad ,$$

and then we will define derivatives with respect to

$$z = \begin{cases} x + iy^+ & \text{for} \quad \mathrm{Im}(z) > 0 \\ x & \text{for} \quad \mathrm{Im}(z) = 0 \\ x - iy^- & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad .$$

We will use the definition of the representation to increase the specificity of the distinctions that we make.

Definition 2.2.2

The forward derivative of a complex-valued function is

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad , \qquad \text{with} \qquad \Delta z = z + \big(h - z\big) \ \ , \ \ h \in \mathbb{C} \ \ , \ \ h \to 0 \ \ .$$

Theorem 2.2.3

The function $f(z) = e^z$ is an eigenfunction of the d/dz operator with unit eigenvalue.

Proof:

Using the definition of the derivative we find that

$$\frac{d}{dz} e^z = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{x+iy+\Delta x+i\Delta y} - e^{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{e^{x+iy+i\Delta y} - e^{x+iy}}{i\Delta y}$$

$$\stackrel{*}{=} \lim_{\Delta y \to 0} \frac{ie^{x+iy+i\Delta y^+}}{i}$$

$$= e^z \quad .$$

(The $\stackrel{*}{=}$ symbol denotes an application of L'Hôpital's rule.)

Remarks 2.2.4

The derivatives with respect to the polar and Cartesian representations are

$$\frac{d}{dz} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad , \qquad \text{and} \qquad \frac{d}{dz} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

with

$$\Delta z = z + \big(h - z\big) \quad , \quad h \in \mathbb{C} \quad , \quad h \to 0 \quad .$$

There is usually no distinguishing between the two distinct instances of d/dz. We will be doing some tricks with these distinctions so it will be useful to distinguish the derivative with respect the each individual representation of complex numbers.

Definition 2.2.5

The representational derivative

$$\frac{d}{dz_1} f(z) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

is such that the variables of the z_1 representation appear in the limit while the variables of z appear in the limiting function. For instance, when $\hat{\mathbb{C}}$ is a representation of $\mathbb{C}$ even while $(\!(x,\emptyset)\!)$, $(\!(x, \pm\widehat{\infty} - y^{\pm})\!)$ are individually not, we have

$$\frac{d}{dz} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad , \qquad \text{for} \qquad z(x,y) = x + iy$$

$$\frac{d}{dz'} f(z') = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z'+\Delta z') - f(z')}{\Delta z'} \quad , \qquad \text{for} \qquad z'(r,\theta) = re^{i\theta} \quad .$$

$$\frac{d}{dz^+} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad , \qquad \text{for} \qquad z^+(x,y^+) = x + iy^+$$

$$\frac{d}{dz^\emptyset} f(z^\emptyset) = \lim_{\Delta x \to 0} \frac{f(z^\emptyset + \Delta z^\emptyset) - f(z^\emptyset)}{\Delta z^\emptyset} \quad , \qquad \text{for} \qquad z^\emptyset(x,\emptyset) = x$$

$$\frac{d}{dz^-} f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad , \qquad \text{for} \qquad z^-(x,y^-) = x - iy^- \quad .$$

2.3 Definition of the representational variation Δz_1

Definition 2.3.1

The variation of a $\mathbb{C}$ number in the definition of the representational derivative is

$$\Delta z_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h \in \mathbb{C} \quad , \quad h \to 0 \quad .$$

The variation with respect to each representation has its own h_1.

Definition 2.3.2

The representing functional for the variation is

$$\Delta z_{(\!(x_2,y_2)\!)}[(\!(x_1,y_1)\!)] \quad \equiv \quad \Delta z_1(x_2,y_2) = \frac{\partial z_1}{\partial x_2}\,\Delta x_2 + \frac{\partial z_1}{\partial y_2}\,\Delta y_2 \quad .$$

Remarks 2.3.3

The transformation law for writing the various Δz that appear in the representational derivatives is

$$\Delta z_2(x_1,y_1) = \frac{\partial z_2}{\partial x_1}\,\Delta x_1 + \frac{\partial z_2}{\partial y_1}\,\Delta y_1 \quad .$$

The variation Δz appears in each application of the representational derivative operator

$$\frac{d}{dz_1}\,f(z_1) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_1+\Delta z_1) - f(z_1)}{\Delta z_1} \quad .$$

One might wish to use the conversion functions to rewrite the definition of the derivative so that Δz is not expressed in terms of $(\Delta x_1, \Delta y_1)$. Therefore, we will give careful attention to representational derivatives of the mixed form

$$\frac{d}{dz_1}\,f(z_2) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_2+\Delta z_2) - f(z_2)}{\Delta z_2} \quad .$$

Remarks 2.3.4

For the representing functional

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1,y_1)\!)] = (\!(x_2,y_2)\!) \quad ,$$

we use the conversion functions

$$x_1(x_2,y_2) = f_x(x_2,y_2) \quad , \qquad \text{and} \qquad y_1(x_2,y_2) = f_y(x_2,y_2) \quad ,$$

but for the representing functional for the variation we use the reverse conversion functions

$$x_2(x_1, y_1) = f'_x(x_1, y_1) \quad , \qquad\qquad \text{and} \qquad\qquad y_2(x_1, y_1) = f'_y(x_1, y_1) \quad .$$

It is clear from the context which is which and we will give several examples. For conversion functions like $y^+ = \widehat{\infty} - y$, we can easily obtain the conversion in either direction from a single function.

Definition 2.3.5

For the case of

$$\frac{d}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad ,$$

we have

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x, y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^+(x, y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z_{(\!(x,y)\!)}[(\!(x, \widehat{\infty} - y^+)\!)] \quad \equiv \quad \Delta z^+(x, y) &= \frac{\partial z^+}{\partial x} \Delta x + \frac{\partial z^+}{\partial y} \Delta y \\
&= \frac{\partial}{\partial x}\big(x + iy^+\big)\Delta x + \frac{\partial}{\partial y}\big(x + iy^+\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

Definition 2.3.6

For the case of

$$\frac{d}{dz} f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad ,$$

we have

$$z^-(x, y^-) = x - iy^- \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z_{(\!(x,y)\!)}[(\!(x, -\widehat{\infty} - y^-)\!)] \quad \equiv \quad \Delta z^-(x,y) &= \frac{\partial z^-}{\partial x}\Delta x + \frac{\partial z^-}{\partial y}\Delta y \\
&= \frac{\partial}{\partial x}\big(x - iy^-\big)\Delta x + \frac{\partial}{\partial y}\big(x - iy^-\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

Definition 2.3.7

For the case of

$$\frac{d}{dz^+} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(x,y^+) = x \quad , \qquad \text{and} \qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)] \quad \equiv \quad \Delta z(x,y^+) = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^+}\,\Delta y^+$$

$$= \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y^+}\big(x+iy\big)\Delta y^+$$

$$= \frac{\partial}{\partial x}\big[x + i\big(\widehat{\infty} - y^+\big)\big]\Delta x + \frac{\partial}{\partial y^+}\big[x + i\big(\widehat{\infty} - y^+\big)\big]\Delta y^+$$

$$= \Delta x - i\Delta y^+ \quad .$$

Definition 2.3.8

For the case of

$$\frac{d}{dz^-}\,f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^- \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(x,y^-) = x \quad , \qquad \text{and} \qquad y(x,y^-) = y^- - \widehat{\infty} \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,-\widehat{\infty}-y^-)\!)}[(\!(x,y)\!)] \quad \equiv \quad \Delta z(x,y^-) = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^-}\,\Delta y^-$$

$$= \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y^-}\big(x+iy\big)\Delta y^-$$

$$= \frac{\partial}{\partial x}\big[x+i\big(y^- - \widehat{\infty}\big)\big]\Delta x + \frac{\partial}{\partial y^-}\big[x+i\big(y^- - \widehat{\infty}\big)\big]\Delta y^-$$

$$= \Delta x + i\Delta y^- \quad .$$

Remarks 2.3.9

Notice that the variation is the same between the two cases of z^+ but the sign changes between to the conversions to and from z^-. This is a manifestation of the minus sign in

$$z^- = x - iy^- \quad .$$

Definition 2.3.10

For the case of

$$\frac{d}{dz}\,f(z') = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z'+\Delta z') - f(z')}{\Delta z'} \quad ,$$

we have

$$z'(r,\theta) = re^{i\theta} \quad ,$$

with two conversion functions

$$r(x,y) = \sqrt{x^2+y^2} \quad , \qquad\qquad \text{and} \qquad\qquad \theta(x,y) = \tan^{-1}\frac{y}{x} \quad .$$

The transformation law for the variation is

$$\Delta z'_{(\!(x,y)\!)}[(\!(r,\theta)\!)] \quad \equiv \quad \Delta z'(x,y) = \frac{\partial z'}{\partial x}\,\Delta x + \frac{\partial z'}{\partial y}\,\Delta y$$

$$= \frac{\partial}{\partial x}\big(re^{i\theta}\big)\Delta x + \frac{\partial}{\partial y}\big(re^{i\theta}\big)\Delta y$$

We have shown in Example 2.1.6 that the conversion functions yield $x + iy$ so

$$\Delta z'(x,y) = \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y}\big(x+iy\big)\Delta y$$

$$= \Delta x + i\Delta y \quad .$$

Definition 2.3.11

For the case of

$$\frac{d}{dz'}\,f(z) = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(r,\theta) = r\cos(\theta) \quad , \qquad\qquad \text{and} \qquad\qquad y(r,\theta) = r\sin(\theta) \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(r,\theta)\!)}[(\!(x,y)\!)] \quad \equiv \quad \Delta z(r,\theta) = \frac{\partial z}{\partial r}\,\Delta r + \frac{\partial z}{\partial \theta}\,\Delta\theta$$

$$= \frac{\partial}{\partial r}\big(x+iy\big)\Delta r + \frac{\partial}{\partial \theta}\big(x+iy\big)\Delta\theta$$

We have shown in Example 2.1.5 that the conversion functions yield $re^{i\theta}$ so

$$\Delta z'(x,y) = \frac{\partial}{\partial r}\big(re^{i\theta}\big)\Delta r + \frac{\partial}{\partial \theta}\big(re^{i\theta}\big)\Delta\theta$$

$$= e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta \quad .$$

Example 2.3.12

In this example we will consider the derivative of $3z^2$ with four different representational derivatives.

Example 2.3.12.1

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz}\, f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad .$$

The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)] \quad \equiv \quad \Delta z^+(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\frac{d}{dz}\, 3\big(z^+\big)^2 = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big(z^+ + \Delta z^+\big)^2 - 3\big(z^+\big)^2}{\Delta z^+}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big(x + iy^+ + \Delta z^+\big)^2 - 3\big(x + iy^+\big)^2}{\Delta z^+}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{3\big[x + i\big(\widehat{\infty} - y\big) + \Delta x - i\Delta y\big]^2 - 3\big[x - i\big(\widehat{\infty} - y\big)\big]^2}{\Delta x - i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]^2 - 3\big[x + i\big(\widehat{\infty} - y\big)\big]^2}{-i\Delta y}$$

$$\stackrel{*}{=} \lim_{\Delta y \to 0} \frac{-6i\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]}{-i}$$

$$= 6\big[x + i\big(\widehat{\infty} - y\big)\big] = 6\big(x + iy^+\big) = 6z^+ \quad .$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.12.2

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz'} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad .$$

The conversion functions are

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(r,\theta)\!)}\big[(\!(x,y)\!)\big] \quad \equiv \quad \Delta z(r,\theta) = e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta$$

Converting to polar gives

$$\frac{d}{dz'} 3z^2 = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{3\big(z(r,\theta) + \Delta z(r,\theta)\big)^2 - 3\big(z(r,\theta)\big)^2}{\Delta}$$

$$= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{3\big(re^{i\theta} + e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0} \frac{3\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{ire^{i\theta}\,\Delta\theta}$$

$$\stackrel{*}{=} \lim_{\Delta\theta \to 0} \frac{6ire^{i\theta}\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)}{ire^{i\theta}}$$

$$= 6\big(re^{i\theta}\big) = 6z \quad .$$

We have the correct transformation law for Δz.

Example 2.3.12.3

Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz}\, f(z^{\emptyset}) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^{\emptyset} + \Delta z^{\emptyset}) - f(z^{\emptyset})}{\Delta z^{\emptyset}} \quad .$$

The conversion functions are

$$x(x, \emptyset) = x \quad , \qquad \text{and} \qquad y^{\emptyset}(x, \emptyset) = 0 \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,y)\!)}[(\!(x,\emptyset)\!)] \quad \equiv \quad \Delta z^{\emptyset}(x,y) = \frac{\partial}{\partial x}\big(x\big)\Delta x = \Delta x \quad .$$

This case reduces to $\mathbb{R}$. Note that the non-analyticity of the conversion functions

$$x_2(x,y) = x \quad , \qquad \text{and} \qquad y^{+}(x,y) = \begin{cases} \widehat{\infty} - y & \text{for} \quad \mathrm{Im}(z) > 0 \\ 0 & \text{for} \quad \mathrm{Im}(z) = 0 \\ \widehat{\infty} + y & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad .$$

only occurs in the $(\!(x,\emptyset)\!)$ piece of the representation. This is exactly the piece which is more properly treated with real rather than complex analysis, and complex analyticity has nothing to do with real analysis. The conversion functions of the expansive $(\!(x,\{\emptyset,\pm\widehat{\infty}-y^\pm\})\!)$ regions are themselves analytic so it is highly likely that the $(\!(x,\{\emptyset,\pm\widehat{\infty}-y^\pm\})\!)$ representation shares all of the most favorable analytic properties of the z and z' representations while having the added benefit of being one-to-one. Evaluation yields

$$\frac{d}{dz}3\big(z^\emptyset\big)^2=\lim_{\Delta x\to 0}\frac{3\big(x+\Delta x\big)^2-3\big(x\big)^2}{\Delta x}\stackrel{*}{=}\lim_{\Delta x\to 0}6\big(x+\Delta x\big)=6x=6z^\emptyset\quad .$$

Example 2.3.12.4

Consider the function $f(z)=3z^2$ and its representational derivative

$$\frac{d}{dz^-}f(z)=\lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{f(z+\Delta z)-f(z)}{\Delta z}\quad ,$$

The conversion functions are

$$x(x,y^-)=x\quad ,\qquad\qquad \text{and}\qquad\qquad y(x,y^-)=y^- -\widehat{\infty}\quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,-\widehat{\infty}-y^-)\!)}\big[(\!(x,y^-)\!)\big]\quad\equiv\quad \Delta z(x,y^-)=\Delta x+i\Delta y^-\quad .$$

Evaluation yields

$$\begin{aligned}\frac{d}{dz^-}3\big(z\big)^2&=\lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big(z+\Delta z\big)^2-3\big(z\big)^2}{\Delta z}\\
&=\lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big(x+iy+\Delta z\big)^2-3\big(x+iy\big)^2}{\Delta z}\\
&=\lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big[x+i\big(y^- -\widehat{\infty}\big)+\Delta x+i\Delta y\big]^2-3\big[x-i\big(y^- -\widehat{\infty}\big)\big]^2}{\Delta x+i\Delta y}\end{aligned}$$

$$= \lim_{\Delta y^- \to 0} \frac{3\big[x + i\big(y^- - \widehat{\infty}\big) + i\Delta y\big]^2 - 3\big[x + i\big(y^- - \widehat{\infty}\big)\big]^2}{i\Delta y}$$

$$\stackrel{*}{=} \lim_{\Delta y^- \to 0} \frac{6i\big[x + i\big(y^- - \widehat{\infty}\big) + i\Delta y\big]}{i}$$

$$= 6\big[x + i\big(y^- - \widehat{\infty}\big)\big] = 6\big(x + iy\big) = 6z \quad .$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.13

Consider the function $f(z) = \ln(z)$ and its representational derivative

$$\frac{d}{dz} f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad .$$

The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,y)\!)}[(\!(x, -\widehat{\infty} - y)\!)] \quad \equiv \quad \Delta z^-(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\frac{d}{dz} \ln(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\ln(z^- + \Delta z^-) - \ln(z^-)}{\Delta z^-}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\ln(x - iy^- + \Delta x - i\Delta y) - \ln(x - iy^-)}{\Delta x - i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{\ln(x - iy^- + \Delta x) - \ln(x - iy^-)}{\Delta x}$$

$$\stackrel{*}{=} \lim_{\Delta x \to 0} \frac{1}{x - iy^- + \Delta x} = \frac{1}{x - iy^-} = \frac{1}{z^-} \quad .$$

Theorem 2.3.14

The representational derivative d/dz_1 obeys the chain rule.

Proof:

For proof by example, consider the derivative of $f(z)\!=\!3ze^{2z}$ in the form

$$\frac{d}{dz^+}\, f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad .$$

The conversion functions are

$$x(x,y^+) = x \quad , \qquad\qquad \text{and} \qquad\qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is

$$\Delta z_{(\!(x,\widehat{\infty}-y^+)\!)}\big[(\!(x,y)\!)\big] \quad \equiv \quad \Delta z(x,y^+) = \Delta x - i\Delta y^+ \quad .$$

Evaluation yields

$$\frac{d}{dz^+}\, 3ze^{2z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{3\big(z+\Delta z\big)\, e^{2(z+\Delta z)} - 3ze^{2z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{3\big(x+iy+\Delta z\big)\, e^{2(x+iy+\Delta z)} - 3\big(x+iy\big)e^{2(x+iy)}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \left[\frac{3\big[x + i\big(\widehat{\infty} - y^+\big) + \Delta x - i\Delta y^+\big]\, e^{2[x+i(\widehat{\infty}-y^+)+\Delta x - i\Delta y^+]}}{\Delta x - i\Delta y^+} - ... \right.$$

$$\left. ... - \frac{3\big[x + i\big(\widehat{\infty} - y^+\big)\big]\, e^{2[x+i(\widehat{\infty}-y^+)]}}{\Delta x - i\Delta y^+} \right]$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]} \lim_{\Delta y^+ \to 0} \frac{3\big[x+i\big(\widehat{\infty}-y^+\big)-i\Delta y^+\big]\, e^{-2i\Delta y^+} - 3\big[x+i\big(\widehat{\infty}-y^+\big)\big]}{-i\Delta y^+} \quad .$$

$$\stackrel{*}{=} e^{2[x+i(\widehat{\infty}-y^+)]} \lim_{\Delta y^+ \to 0} \frac{-3ie^{-2i\Delta y^+} - 6i\big[x+i\big(\widehat{\infty}-y^+\big)-i\Delta y^+\big]\, e^{-2i\Delta y^+}}{-i}$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]}\Big\{3+6\big[x+i\big(\widehat{\infty}-y^+\big)\big]\Big\}$$

$$= e^{2[x+i(\widehat{\infty}-y^+)]}\big[3+6\big(x+iy\big)\big] = e^{2z}\big(3+6z\big) \quad .$$

We have shown that the representational derivative satisfies the chain rule.

Theorem 2.3.15

The complex exponential function e^z is an eigenfunction of the representational derivative operator d/dz_1.

Proof:

It suffices to show that

$$\frac{d}{dz_1}\, e^{z_1} = e^{z_1} \quad , \qquad \text{and} \qquad \frac{d}{dz_1}\, e^{z_2} = e^{z_2} \quad ,$$

where z_1 and z_2 are two representations of $\mathbb{C}$. The first condition is satisfied by Theorem 2.2.3. For the second condition, consider

$$\frac{d}{dz}\, f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is (Definition 2.3.5)

$$\Delta z_{(\!(x,y)\!)}[(\!(x,\widehat{\infty}-y^+)\!)] \quad \equiv \quad \Delta z^+(x,y)=\Delta x - i\Delta y \ \ .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz}\,e^{z^+} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{z^+ + \Delta z^+} - e^{z^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{x+iy^+ + \Delta z^+} - e^{x+iy^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{x+i(\widehat{\infty}-y)+\Delta x - i\Delta y} - e^{x+i(\widehat{\infty}-y)}}{\Delta x - i\Delta y} \\
&= \lim_{\Delta x\to 0} \frac{e^{x+i(\widehat{\infty}-y)+\Delta x} - e^{x+i(\widehat{\infty}-y)}}{\Delta x} \\
&\stackrel{*}{=} \lim_{\Delta x\to 0} e^{x+i(\widehat{\infty}-y)+\Delta x} \\
&= e^{x+i(\widehat{\infty}-y)} = e^{x+iy^+} = e^{z^+} \quad .
\end{aligned}$$

Definition 2.3.16

When we write the representational derivative, the limits are in the representation of the derivative and the function is in its own representation (which may or may not be the representation of the derivative). When we implement the transformation law for the variation, we are inserting the variation of the representation of the derivative into the other representation. Consider

$$\frac{d}{dz_2}\,f(z_1) = \lim_{\substack{\Delta x_2\to 0\\ \Delta y_2\to 0}} \left[\frac{f(z_1+\Delta z_1)-f(z_1)}{\Delta z_1}\right] \quad .$$

Even after the variation has been transformed $\Delta z_1 \to \Delta z_2$, everything in square brackets remains in the z_1 representation.

2.4 Definition of the modified representational variation $\widehat{\Delta z_1}$

Remarks 2.4.1

We have shown that we have the correct transformation law for the variation with respect to each representation. In this section, we will examine the definition of the variation and propose a modified variation which obeys a separate transformation law. We will show that the two transformations do not always agree, and that the transformation of the modified variation does not always work for the definition of the derivative. Then we will show that the transformation of the modified variation *does* always produce the correct derivative when the transformation is between the z and $z^{\pm}$ representations. This is due to the composition laws of $\widehat{\mathbb{R}}$ and the properties of $\widehat{\infty}$.

Example 2.4.2

The 1D case of

$$\Delta x = x + \big(h - x\big) \quad , \quad h \in \mathbb{R} \quad , \quad h \to 0 \quad .$$

gives a good illustration of the meaning of the definition of the variation. Considering three points $\{0, x, h\}$ along the real number line, we could shift the origin to any other $x \in \mathbb{R}$ and then write the definition of the variation with respect to those coordinates. For instance, if we shift $h \to h'$ then $h \to 0$ no longer generates an appropriate variation. We need to take $h' \to 0$ which means h goes to whatever value we have used to shift the origin. By the symmetry of the real line, either of these representations of the 1D variation Δx are exactly the same. Therefore, define a representation of $\mathbb{C}$ such that

$$z_{(\!(x,y^\gamma)\!)}\big[(\!(x,y)\!)\big] = (\!(x,y^\gamma)\!) \quad , \qquad \text{with} \qquad z^\gamma = x + iy^\gamma \quad .$$

We have two conversion functions

$$x(x,y^\gamma) = x \quad , \qquad \text{and} \qquad y(x,y^\gamma) = \gamma - y^\gamma \quad ,$$

so that

$$(\!(x,y^\gamma)\!) \quad \equiv \quad z^\gamma(x,y^\gamma) \quad , \qquad \text{and} \qquad (\!(x,y)\!) \quad \equiv \quad z(x,y) \quad .$$

(We have shifted the origin of y instead of x to mimic the structure of $\hat{\mathbb{C}}$.) For the $(\!(x,y)\!)$ Cartesian representation of $\mathbb{C}$, we have

$$h_1 = a_1 + ib_1 \quad , \qquad \text{and} \qquad h_\gamma = x(a,b) + iy(a,b) = a + i\big(\gamma - b\big) \quad .$$

Therefore, the modified variation transforms as

$$\begin{aligned}
\widehat{\Delta z}^{\gamma}(x,y) &= z^\gamma + \big(h_\gamma - z^\gamma\big) \\
&= z^\gamma(x,y) + \big(h_\gamma - z^\gamma(x,y)\big) \\
&= x(x,y) + iy^\gamma(x,y) + \big[a_\gamma + ib_\gamma - \big(x(x,y) + iy^\gamma(x,y)\big)\big] \\
&= x + i\big(\gamma - y\big) + \big\{a + i\big(\gamma - b\big) - \big[x + i\big(\gamma - y\big)\big]\big\} \\
&= x + i\gamma - iy + a + i\gamma - ib - x - i\gamma + iy \\
&= \big(x + a - x\big) + \big(iy - ib + iy\big) + i\gamma \\
&= \Delta x + i\big(\gamma - \Delta y\big) \quad .
\end{aligned}$$

The transformation law for the canonical variation Δz gives

$$\begin{aligned}
\Delta z^\gamma(x,y) &= \frac{\partial}{\partial x}\big[x + i\big(\gamma - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\gamma - y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

We find

$$\widehat{\Delta z}^{\gamma}(x,y) = \Delta z^\gamma(x,y) + i\gamma \quad .$$

Therefore, the transformation law for the variation does not agree with our attempt to transform the modified variation by directly converting its elements with the conversion functions. We will show, however, that this not a problem in all cases.

Definition 2.4.3

The modified representational variation of a $\mathbb{C}$ number is

$$\widehat{\Delta z}_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h \in \mathbb{C} \quad , \quad h \to 0 \quad ,$$

so it is identically the representational variation Δz_1. The difference between Δz_1 and $\widehat{\Delta z}_1$ is that they obey different transformation laws.

Definition 2.4.4

We say the representing functional for the modified variation is

$$\widehat{\Delta z}_{(\!(x_2,y_2)\!)}[(\!(x_1,y_1)\!)] \quad \equiv \quad \widehat{\Delta z}_1(x_2,y_2) \quad .$$

Definition 2.4.5

The modified variation $\widehat{\Delta z}_1$ transforms by direct substitution of the conversion functions. The transformation law defined for

$$h_1 = a + ib \quad , \quad a,b \in \mathbb{R} \quad , \quad a,b \to 0 \quad ,$$

is

$$\widehat{\Delta z}_1(x_2,y_2) = z_1[x_1(x_2,y_2), y_1(x_2,y_2)] + \big(x_1(a,b) + iy_1(a,b) - z_1[x_1(x_2,y_2), y_1(x_2,y_2)]\big) \quad .$$

Example 2.4.6

Take note of

$$\widehat{\Delta z}^-(x,y) = x(x,y) - iy^-(x,y) + \big[x(a,b) - iy^-(a,b) - \big(x(x,y) - iy^-(x,y)\big)\big] \quad .$$

Since z^- is a non-trivial representation, we may not directly decompose the $z_1[x_1,y_1]$ of Definition 2.4.4 into a general form $x_1(x_2,y_2) + iy_1(x_2,y_2)$. For this reason, Definition 2.4.4 specifies h in the Cartesian representation. There are other cases in which h will not have the form $a + ib$.

Definition 2.4.6

The infinite continuation of $\widehat{\Delta z}^{\gamma}$ is

$$\widehat{\Delta z}^{+}(x,y) \quad \equiv \quad \lim_{\gamma \to \widehat{\infty}} \widehat{\Delta z}^{\gamma}(x,y) \quad .$$

Example 2.4.7

The continuation of $\widehat{\Delta z}^{\gamma}$ to minus infinity, as defined, cannot be used to directly generate $\widehat{\Delta z}^{-}$. Instead we need to consider two conversion functions

$$x(x, y^{\gamma'}) = x \quad , \qquad \text{and} \qquad y(x, y^{\gamma'}) = y^{\gamma'} - \widehat{\infty} \quad ,$$

In this case $y^{\gamma'}$ has the same form as $y^{-} = \widehat{\infty} + y$. The transformation law is

$$\widehat{\Delta z}^{\gamma'}(x,y) = z^{\gamma'}[x(x,y), y^{\gamma'}(x,y)] + \big(x(a,b) + iy^{\gamma'}(a,b) - z^{\gamma'}[x(x,y), y^{\gamma'}(x,y)]\big) \quad .$$

To mimic the form of z^{-}, we will choose for this example

$$z^{\gamma'}(x, y^{\gamma'}) = x - iy^{\gamma'} \quad .$$

Evaluation yields

$$\begin{aligned}
\widehat{\Delta z}^{\gamma'}(x,y) &= x - iy^{\gamma'} + \big[a - ib^{\gamma'} - \big(x - iy^{\gamma'}\big)\big] \\
&= x - i\big(\gamma' + y\big) + \big\{a - i\big(\gamma' + b\big) - \big[x - i\big(\gamma' + y\big)\big]\big\} \\
&= \big(x + a - x\big) + i\big(y - b - y\big) - i\gamma' \\
&= \Delta x - i\big(\gamma' + \Delta y\big) \quad .
\end{aligned}$$

Definition 2.4.8

The infinite continuation of $\widehat{\Delta z}^{\gamma'}$ to $\widehat{\Delta z}^{-}$ is

$$\widehat{\Delta z}^{+}(x,y) \quad \equiv \quad \lim_{\gamma' \to \widehat{\infty}} \widehat{\Delta z}^{\gamma'}(x,y) \quad .$$

Remarks 2.4.9

If we transform the modified variation directly with the $y^{\pm}(y)$ conversion functions of the $\hat{\mathbb{C}}$ representation then we will get an undefined expression. For this reason, the infinite continuation is defined as the limit of the finite behavior rather than the infinite behavior of the transformation law for the modified variation.

Theorem 2.4.10

The infinite continuation of the transformation of the modified variation is undefined.

Proof:

Consider the conversion functions

$$x(x,y^+) = x \quad , \qquad \text{and} \qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the modified variation is

$$\widehat{\Delta z}^{+}(x,y) = z^+[x(x,y),y^+(x,y)] + \big(x(a,b) + iy^+(a,b) - z^+[x(x,y),y^+(x,y)]\big)$$

$$= x + iy + \big[x(a,b) + iy^+(a,b) - \big(x+iy\big)\big]$$

$$= x + i\big(\widehat{\infty} - y\big) + \big\{a + i\big(\widehat{\infty} - b\big) - \big[x + i\big(\widehat{\infty} - y\big)\big]\big\} \quad .$$

This expression is defined by the brackets but we have to remove the brackets to rearrange for the transformed variation. Consider

$$\widehat{\Delta z}^{+}(x,y) = x + i\,\widehat{\infty} - iy + a + i\,\widehat{\infty} - ib - x - i\,\widehat{\infty} + iy$$

$$= \Delta x + i\Delta y + i\,\widehat{\infty} + i\,\widehat{\infty} - i\,\widehat{\infty} \quad .$$

By Remarks 1.3.12, this expression is undefined. We can also obtain a contradiction directly from the definition of the variation. Consider two equivalent expressions such that

$$x + \big(h - x\big) = \big(x + h\big) - x$$

$$\big(\widehat{\infty} - x\big) + \big[\big(\widehat{\infty} - h\big) - \big(\widehat{\infty} - x\big)\big] = \big[\big(\widehat{\infty} - x\big) + \big(\widehat{\infty} - h\big)\big] - \big(\widehat{\infty} - x\big)$$

$$\big(\widehat{\infty} - x\big) + \big(x - h\big) = \big[\widehat{\infty} - \big(x + h\big)\big] - \big(\widehat{\infty} - x\big)$$

$$\big(\widehat{\infty} - h\big) = -h$$

Remarks 2.4.12

Although $\Delta z_1(x_2, y_2)$ and $\widehat{\Delta z}_1(x_2, y_2)$ are not always identically equal, there are cases in which they do produce the same derivatives. We will show cases in which the derivatives computed from each variation agree and disagree. Then we will show that $\widehat{\Delta z}_1$ is always valid for the case of $z_1 \in \hat{\mathbb{C}}$.

Definition 2.4.13

In the case of $\hat{\mathbb{C}}$ representations, the rule for taking the derivative with the modified variation is to compute the derivative with γ and then let $\gamma \to \widehat{\infty}$ once the expression is simplified. Division by γ shall always be avoided through L'Hôpital's rule.

Theorem 2.4.14

The modified variation, as defined, always produces the correct derivative for transformations between the Cartesian and $\hat{\mathbb{C}}$ representations of $\mathbb{C}$.

Proof:

Proof follows from Example 2.4.15, Example 2.4.16, Example 2.4.17, Example 3.2.7, and Example 3.2.8.

Example 2.4.15

In this example we will compute the derivative of $f(z) = z$ for the case of

$$\frac{d}{dz^+} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z + \widehat{\Delta z}) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x, y) = x + iy \quad ,$$

with two conversion functions

$$x(x, y^\gamma) = x \quad , \qquad\qquad \text{and} \qquad\qquad y(x, y^\gamma) = \gamma - y^\gamma \quad .$$

(The conversion functions use γ instead of $\widehat{\infty}$ because we will make the substitution only after we have evaluated the definition of the derivative.) The transformation law for the modified variation is

$$\widehat{\Delta z}(x, y^\gamma) = z[x(x, y^\gamma), y(x, y^\gamma)] + \big(x(a, b) + iy(a, b) - z[x(x, y), y^\gamma(x, y)]\big)$$

$$= x + iy + \big[x(a, b) + iy(a, b) - \big(x + iy\big)\big]$$

$$= \Delta x - i\Delta y + i\gamma \quad .$$

Evaluation yields

$$\frac{d}{dz^\gamma} z = \lim_{\substack{\Delta x \to 0 \\ \Delta y^\gamma \to 0}} \frac{f(x + iy + \widehat{\Delta z}) - f(x + iy)}{\widehat{\Delta z}}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + i\big(\gamma - y^\gamma\big) + \Delta x - i\Delta y + i\gamma) - f(x + i\big(\gamma - y^\gamma\big))}{\Delta x - i\Delta y + i\gamma}$$

$$= \lim_{\Delta x \to 0} \frac{x + i\big(\gamma - y^\gamma\big) + \Delta x + i\gamma - x - i\big(\gamma - y^\gamma\big)}{\Delta x + i\gamma}$$

Although γ is taken as a finite number, the rule is to avoid division by γ through application of L'Hôpital's rule. Since γ appears in numerator and the denominator, and since L'Hôpital's rule is typically used for limits of the form ∞/∞, we are well motivated to use this rule in the derivative for the modified variation as if γ was an infinite number. L'Hôpital's rule yields

$$\frac{d}{dz^\gamma} z \stackrel{*}{=} \lim_{\Delta x \to 0} 1 = 1 \quad .$$

This is the correct answer but this example was trivial.

Example 2.4.16

In this example we will compute the derivative of $f(z) = z^2$. For the case of

$$\frac{d}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

we have

$$z^\gamma(x, y^\gamma) = x + iy^\gamma \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^\gamma(x,y) = \gamma - y \quad .$$

The transformation law for the modified variation is

$$\widehat{\Delta z}^\gamma(x,y) = \Delta x - i\Delta y + i\gamma$$

Evaluation yields

$$\frac{d}{dz}\big(z^\gamma\big)^2 = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + iy^\gamma + \widehat{\Delta z}^\gamma) - f(x + iy^\gamma)}{\widehat{\Delta z}^\gamma}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + i\big(\gamma - y\big) + \Delta x - i\Delta y + i\gamma) - f(x + i\big(\gamma - y\big))}{\Delta x - i\Delta y + i\gamma}$$

$$= \lim_{\Delta y \to 0} \frac{\big[x + i\big(\gamma - y\big) - i\Delta y + i\gamma\big]^2 - \big[x + i\big(\gamma - y\big)\big]^2}{-i\Delta y + i\gamma}$$

$$\stackrel{*}{=} \lim_{\Delta y \to 0} \frac{-2i\big[x + i\big(\gamma - y\big) - i\Delta y + i\gamma\big]}{-i}$$

$$= 2\big[x + i\big(\gamma - y\big) + i\gamma\big] = 2\big[x + i\big(2\gamma - y\big)\big] \quad .$$

Having evaluated the definition of the derivative, we let $\gamma \to \widehat{\infty}$ so that

$$\frac{d}{dz}\big(z^+\big)^2 = 2\big[x + i\big(2\widehat{\infty} - y\big)\big]$$

$$= 2\big[x + i\big(\widehat{\infty} - y\big)\big]$$

$$= 2\big[x + iy^+\big]^2 = 2z^+ \quad .$$

This is the correct derivative.

Example 2.4.17

In this example we will compute the derivative of $f(z) = z^2$ for the case of

$$\frac{d}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

we have

$$z^\gamma(x, y^\gamma) = x + iy^\gamma \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad\qquad \text{and} \qquad\qquad y^\gamma(x,y) = \gamma - y \quad .$$

The transformation law for the modified variation is

$$\widehat{\Delta z}^\gamma(x,y) = \Delta x - i\Delta y + i\gamma \quad .$$

Evaluation yields

$$\frac{d}{dz}\big(z^\gamma\big)^3 = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+iy^\gamma+\widehat{\Delta z}^\gamma) - f(x+iy^\gamma)}{\widehat{\Delta z}^\gamma}$$

$$= \lim_{\Delta x\to 0} \frac{\big(x+iy^\gamma+\Delta x+i\gamma\big)^3 - \big(x+iy^\gamma\big)^3}{\Delta x + i\gamma}$$

$$\stackrel{*}{=} \lim_{\Delta x\to 0} 3\big[x+i\big(\gamma-y\big)+\Delta x+i\gamma\big]^2$$

$$= 3\big[x+i\big(\gamma-y\big)+i\gamma\big]^2$$

$$= 3\big[x+i\big(\gamma-y\big)]^2 + 2ix\gamma - 3\gamma^2 + 2\gamma y$$

$$= 3\big(z^\gamma\big)^2 + 2ix\gamma - 3\gamma^2 + 2\gamma y \quad .$$

Having evaluated the definition of the derivative, we let $\gamma \to \widehat{\infty}$ so that

$$\frac{d}{dz}\big(z^+\big)^3 = 3\big(z^\gamma\big)^2 + i\,\widehat{\infty} - \widehat{\infty}^2 + \widehat{\infty}$$

$$= 3\big[x^2 + 2ix\big(\widehat{\infty}-y\big) - \big(\widehat{\infty}^2 - 2y\widehat{\infty} + y^2\big)\big] + i\,\widehat{\infty} - \widehat{\infty}^2 + \widehat{\infty} \quad .$$

This expression is well defined because the three infinities at end are distinct. Continued evaluation yields

$$\frac{d}{dz}\left(z^+\right)^3 = 3\Big\{x^2 + 2ix\big[(\widehat{\infty} - y) + \widehat{\infty}\big] + \left[\left(-\widehat{\infty}^2 - \frac{y^2}{2}\right) - \widehat{\infty}^2\right] + 2y\left[\left(\widehat{\infty} - \frac{y^2}{2}\right) + \widehat{\infty}\right]\Big\}$$

$$= 3\Big\{x^2 + 2ix\big(\widehat{\infty} - y\big) + \left(-\widehat{\infty}^2 - \frac{y^2}{2}\right) + 2y\left(\widehat{\infty} - \frac{y}{4}\right)\Big\}$$

$$= 3\big[x^2 + 2ix\big(\widehat{\infty} - y\big) + \big(-\widehat{\infty}^2 + 2y\widehat{\infty} - y^2\big)\big] = 3\big(z^+\big)^2 \quad .$$

This is the correct derivative. Notice that we have used the composition law defined (Definition 1.3.18) such that

$$\big(-\widehat{\infty}^2 - y^2\big) - \widehat{\infty}^2 = -\widehat{\infty}^2 - y^2 \quad .$$

3 Proof of limits of sine and cosine at infinity

3.1 Refutation of proof of nonexistence of limits at infinity

Definition 3.1.1

We say that the limit of a sequence exists if and only if all of its subsequences converge to the same value.

Theorem 3.1.2

It is impossible to compute the limits

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(x) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \cos(x) \quad .$$

Proof (Refuted):

The definition of the limit requires that for a limit

$$\lim_{x\to\infty} f(x) = l \quad ,$$

to exist, the function f must converge to l in all of its subsequences. For proof by contradiction, consider two subsequences of x

$$x_n = 2n\pi + \frac{\pi}{2} \quad , \qquad \text{and} \qquad x_m = 2m\pi.$$

For any n, m we have

$$\sin(x_n) = 1 \quad , \qquad \text{and} \qquad \sin(x_m) = 0 \quad .$$

Therefore, it is impossible for all subsequences of x to converge to some constant l.

Refutation:

The convergence of the sequences are determined by the final n points, not the first n points. Since the points in x_n and x_m are evenly spaced by 2π and the sequences both terminate at infinity, we can write the final n points of each sequence as

$$x_{\infty-n} = \widehat{\infty} - 2n\pi \quad , \qquad \text{and} \qquad x_{\infty-m} = \widehat{\infty} - 2m\pi \quad .$$

Since $\widehat{\infty} - n \neq \widehat{\infty}$, all of these points are distinct. It is obvious that both sequences converge to the same value.

Remarks 3.1.3

Expressions like

$$f(x_n) = \sin(\widehat{\infty} - 2n\pi) \quad ,$$

can be evaluated easily with the difference formulae once we get values for $\sin(\widehat{\infty})$ and $\cos(\widehat{\infty})$. Note the property of $\widehat{\mathbb{R}}$ that for any $a, b \in \mathbb{R}$

$$\widehat{\infty} - \big(b + a\big) > 0 \qquad \Longleftrightarrow \qquad \widehat{\infty} - b > a \quad .$$

This tells us that every $\widehat{\mathbb{R}}$ number is greater than every $\mathbb{R}$ number. In general, we say that if a number is greater than every real number then it is equal to infinity but Theorem 1.5.2 states that

$$0 < \widehat{\infty} - b < \infty \quad .$$

Therefore, make a definition that if x is larger than every $b \in \mathbb{R}$ and is less than infinity then

$$x = \widehat{\infty} - \varepsilon \quad .$$

Since all numbers of the form $\widehat{\infty} - 2n\pi$ can be expressed as $\widehat{\infty} - \varepsilon$ we may reexamine our sequences. With these definitions we have

$$\sin(x_n) = \sin(\widehat{\infty} - \varepsilon) \qquad \text{and} \qquad \sin(x'_n) = \sin(\widehat{\infty} - \varepsilon) \quad .$$

Therefore, the two sequences do converge to the same value. We have refuted the proof by examining the final n points of x_n and x'_n whereas the refuted proof has only examined the first n points which have no bearing on the convergence of the final n points. Since the absorptive property of ∞ is such that

$$\infty - \varepsilon = \infty \quad ,$$

we should use the difference formula to write

$$\sin(\infty - \varepsilon) = \sin(\infty)\cos(\varepsilon) - \cos(\infty)\sin(\varepsilon) \quad .$$

This cancels the absorption of ∞. For any $0 < \varepsilon < 1$

$$\sin(\infty)\cos(\varepsilon) - \cos(\infty)\sin(\varepsilon) \neq \sin(\infty) \quad .$$

Since sine and cosine are generally functions $\mathbb{R} \to \mathbb{R}$, we need a way to express $\widehat{\mathbb{R}}$ numbers as $\mathbb{R}$ numbers. With a caveat about absorption, $\infty - \varepsilon$ satisfies the main requirement

$$0 < \widehat{\infty} - \varepsilon < \infty \quad .$$

We might say that "$\infty - \varepsilon$" is a compound label referring to real numbers in the neighborhood of infinity.

Corollary 3.1.4

We have shown that $\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on $\mathbb{S}^2$ in the limit where $\pm\widehat{\infty}, \pm i\widehat{\infty} \to \infty$ (Corollary 1.4.16.) Now we have reason to consider another complementary arrangement on $\mathbb{S}^2$ and we will consider a great circle $\mathbb{S}^1$ to simplify the statements. Since there are exactly as many $\widehat{\mathbb{R}}$ numbers of the form $\widehat{\infty}+b$ as there are non-zero $\mathbb{R}$ numbers, and every $\widehat{\mathbb{R}}$ number is greater than every $\mathbb{R}$ number, we should set the infinity that $\mathbb{R}$ tends toward on the equator of the sphere when $\widehat{0}$ and $\widehat{\infty}$ are the two poles. Since there are as many points in the interval $[\infty - \varepsilon, \infty]$ as there are in $[0, \infty - \varepsilon]$, one would favor the representation in which the area around the pole at infinity is stretched over an entire hemisphere because the density of numbers on the surface of the sphere is uniform when $\mathbb{R}$ tends toward a value on the equator. $\widehat{\mathbb{R}}$ numbers of the form $\widehat{\infty} - b$ will also tend toward that same value for increasing $b \in \mathbb{R}$. When $\mathbb{R}$ tends toward infinity at the opposite pole from its origin, then every $\widehat{\mathbb{R}}$ number is squeezed to one side of the sphere. Regarding the refutation of Theorem 3.1.2, all the points in the $\widehat{\mathbb{R}}$ hemisphere can take the same value $\infty - \varepsilon$ because the equator is constrained to be adjacent to the pole. We might call the equatorial infinity that $\mathbb{R}$ and $\widehat{\mathbb{R}}$ tend toward $\infty - \epsilon$ to distinguish it from the polar infinity $\widehat{\infty}$, or we might even use the label ∞. While that would bear a lot of further analysis, there are some immediate features of interest in expanding the neighborhood of polar infinity to cover an entire hemisphere. By the symmetry of the sphere, and by the symmetry of there being exactly as many $\widehat{\mathbb{R}}$ numbers less than infinity as there are $\mathbb{R}$ numbers greater than zero, we can deduce that the limits of sine and cosine at $\widehat{\infty}$ should be the same as what they are at $\widehat{0}$: the sphere has mirror symmetry about it equator. Furthermore, since ε is vanishingly small, equatorial infinity is separated from polar infinity by a vanishingly small distance. We may deduce the behavior at the equator from the behavior at the pole because there is a representation in which equatorial infinity is adjacent to polar infinity (Corollary 1.4.16.) In the next section, we will use a totally different method to derive the behavior of sine and cosine at infinity but we will find that it is exactly like the behavior at zero.

3.2 Proof of limits of sine and cosine at infinity

Theorem 3.2.1

The values of sine and cosine at infinity are

$$\sin(\infty) = 0 \quad , \qquad \text{and} \qquad \cos(\infty) = 1 \quad .$$

Proof:

We have proven in Theorem 2.2.3 that

$$\frac{d}{dz_1}\, e^{z_2} = e^{z_2} \quad .$$

For $f(z) = e^z$ in the case of

$$\frac{d}{dz}\, f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

we have

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the modified variation is

$$\widehat{\Delta z}^+(x,y) = \Delta x - i\Delta y + i\,\widehat{\infty} \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz}\, e^{z^+} &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \\
&= e^{x+iy^+} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{\Delta x - i\Delta y + i\widehat{\infty}} - 1}{\Delta x - i\Delta y + i\,\widehat{\infty}} \\
&= e^{x+iy^+} \lim_{\Delta x \to 0} \frac{e^{\Delta x + i\widehat{\infty}} - 1}{\Delta x + i\,\widehat{\infty}} \\
&\overset{*}{=} e^{x+iy^+} \lim_{\Delta x \to 0} e^{\Delta x + i\widehat{\infty}} = e^{x+iy^+} e^{i\widehat{\infty}} \quad .
\end{aligned}$$

The exponential is an eigenfunction of the derivative with eigenvalue 1 so

$$1 = e^{i\widehat{\infty}} = \cos(\widehat{\infty}) + i\sin(\widehat{\infty}) \quad .$$

Equating the real and imaginary parts gives

$$\sin(\widehat{\infty}) = 0 \quad , \qquad \text{and} \qquad \cos(\widehat{\infty}) = 1 \quad .$$

Theorem is proven with

$$\sin(\widehat{\infty}) = \sin(\infty) \quad , \qquad \text{and} \qquad \cos(\widehat{\infty}) = \cos(\infty) \quad .$$

Theorem 3.2.3

The limits of sine and cosine at infinity are

$$\lim_{x\to\infty} \sin(x) = 0 \quad , \qquad \text{and} \qquad \lim_{x\to\infty} \cos(x) = 1 \quad .$$

Proof:

A function has a limit l if and only if the function converges to l in any subsequence. We have shown in the refutation of of Theorem 3.1.2 that the final n points of any sequence $\sin(x_n)$ have the form $\sin(\infty - \varepsilon)$. In the limit $\varepsilon \to 0$ we find that for any $x_n \in \mathbb{R}$

$$\lim_{n\to\infty} \sin(x_n) = \sin(\infty) \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \cos(x_n) = \cos(\infty) \quad .$$

Theorem is proven with $\sin(\infty) = 0$, $\cos(\infty) = 1$, and $x_n \to x$.

Theorem 3.2.4

Sine and cosine are continuous at infinity.

Proof:

We say that a function is continuous at a point if

$$\lim_{x\to x_0} f(x) = f(x_0) \quad .$$

Sine and cosine are such that

$$\lim_{x\to\infty} \sin(x) = \sin(\infty) \quad , \qquad\qquad \text{and} \qquad\qquad \lim_{x\to\infty} \cos(x) = \cos(\infty) \quad .$$

Both functions are continuous at infinity.

<u>Theorem 3.2.5</u>

The values of sine and cosine at ∞ preserve the odd- and evenness of sine and cosine respectively.

<u>Proof:</u>

In Example 2.4.7 we found that $\widehat{\Delta z}^{\gamma'}(x,y)$ which is continued to $\widehat{\Delta z}^{-}(x,y)$ as

$$\widehat{\Delta z}^{-}(x,y) = \Delta x - i\big(\widehat{\infty} + \Delta y\big) \quad .$$

We can plug this into

$$\frac{d}{dz} f(z^-) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^- + \widehat{\Delta z}^{-}) - f(z^-)}{\widehat{\Delta z}^{-}} \quad ,$$

to obtain

$$\begin{aligned}
\frac{d}{dz} e^{z^-} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^- + \widehat{\Delta z}^{-}) - f(z^-)}{\widehat{\Delta z}^{-}} \\
&= e^{x-iy^-} \lim_{\Delta y\to 0} \frac{e^{-i\Delta y - i\widehat{\infty}} - 1}{-\,i\Delta y - i\,\widehat{\infty}} \\
&\overset{*}{=} e^{x-iy^-} \lim_{\Delta x\to 0} e^{-i\Delta y - i\widehat{\infty}} = e^{x-iy^-} e^{-i\widehat{\infty}} \quad .
\end{aligned}$$

It follows that

$$\cos(-\widehat{\infty}) = 1 \quad , \qquad\qquad \text{and} \qquad\qquad \sin(-\widehat{\infty}) = 0 \quad .$$

Therefore,

$$\cos(-\widehat{\infty}) = \cos(\widehat{\infty}) \quad , \qquad\qquad \text{and} \qquad\qquad \sin(-\widehat{\infty}) = -\sin(\widehat{\infty}) \quad .$$

Sine is an odd function and cosine is an even function.

Theorem 3.2.6

Sine and cosine satisfy the double angle identities at infinity.

Proof:

The relevant identities are

$$\sin(2x) = 2\sin(x)\cos(x) \quad , \qquad\qquad \text{and} \qquad\qquad \cos(2x) = 1 - \sin^2(x) \quad .$$

These identities are satisfied trivially for $x = \widehat{\infty}$.

Example 3.2.7

To further confirm Theorem 2.4.17, namely that modified variation always produces teh correct derivative and that, therefore, the derivation of limits in this paper is completely sound, we will now use the modified variation to compute a derivative which requires an application of the chain rule. We will consider $f(z)=3ze^{2z}$ in the case of

$$\frac{d}{dz}f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\Delta z^+} \quad .$$

We have

$$z^\gamma(x, y^\gamma) = x + iy^\gamma \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^\gamma(x,y) = \gamma - y \quad .$$

The transformation law for the modified variation is

$$\widehat{\Delta z}^\gamma(x,y) = \Delta x - i\Delta y + i\gamma \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz}\, 3z^\gamma e^{2z^\gamma} &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + iy^\gamma + \widehat{\Delta z}^\gamma) - f(x + iy^\gamma)}{\widehat{\Delta z}^\gamma} \\
&= \lim_{\Delta x \to 0} \frac{3\big(z^\gamma + \Delta x + i\gamma\big)e^{2z^\gamma + 2\Delta x + 2i\gamma} - 3\big(z^\gamma\big)e^{2z^\gamma}}{\Delta x + i\gamma} \\
&\overset{*}{=} \lim_{\Delta x \to 0} \big[3 + 6\big(z^\gamma + \Delta x + i\gamma\big)\big]e^{2z^\gamma + 2\Delta x + 2i\gamma} \\
&= \big[3 + 6\big(z^\gamma + i\gamma\big)\big]e^{2z^\gamma + 2i\gamma} \quad .
\end{aligned}$$

We obtain the expression for z^+ by letting $\gamma \to \widehat{\infty}$. This gives

$$\begin{aligned}
\frac{d}{dz}\, 3z^+ e^{2z^+} &= \big[3 + 6\big(z^+ + i\widehat{\infty}\big)\big]e^{2z^+ + 2i\widehat{\infty}} \\
&= \big\{3 + 6\big[x + \big(i\widehat{\infty} - iy\big) + i\widehat{\infty}\big]\big\}e^{2z^+}e^{2i\widehat{\infty}} \\
&= \big\{3 + 6\big[x + \big(i\widehat{\infty} - iy\big)\big]\big\}e^{2z^+} = \big(3 + 6z^+\big)e^{2z^+} \quad .
\end{aligned}$$

This is the correct derivative.

Example 3.2.8

To continue with the proof of Theorem 2.4.17, we will now use the modified variation to compute a derivative which requires an application of the chain rule. We will consider $f(z)=7z^2\tan(6z)$ in the

$$\frac{d}{dz}f(z^+) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^+ + \widehat{\Delta z}^{+}) - f(z^+)}{\Delta z^+} \quad ,$$

variant of the representational derivative. Evaluation yields

$$\frac{d}{dz}7\big(z^\gamma\big)^2\tan(6z^\gamma) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+iy^\gamma + \widehat{\Delta z}^{\gamma}) - f(x+iy^\gamma)}{\widehat{\Delta z}^{\gamma}}$$

$$= \lim_{\Delta x\to 0} \frac{7\big(z^\gamma + \Delta x + i\gamma\big)^2\tan(6z^\gamma + 6\Delta x + 6i\gamma) - 7\big(z^\gamma\big)^2\tan(6z^\gamma)}{\Delta x + i\gamma}$$

$$\stackrel{*}{=} \lim_{\Delta x\to 0} 14\big(z^\gamma + \Delta x + i\gamma\big)\tan(6z^\gamma + 6\Delta x + 6i\gamma) + \ldots$$

$$\ldots + 7\big(z^\gamma + \Delta x + i\gamma\big)^2 6\sec(12z^\gamma + 12\Delta x + 12i\gamma)$$

$$= 14\big(z^\gamma + i\gamma\big)\tan(6z^\gamma + 6i\gamma) + 7\big(z^\gamma + i\gamma\big)^2 6\sec(12z^\gamma + 12i\gamma)$$

We obtain the expression for z^+ by letting $\gamma \to \widehat{\infty}$. This gives

$$\frac{d}{dz}7\big(z^+\big)^2\tan(6z^+) = 14\big(z^+ + i\,\widehat{\infty}\big)\tan(6z^+ + 6i\,\widehat{\infty}) + 42\big(z^+ + i\,\widehat{\infty}\big)^2\sec(12z^+ + 12i\,\widehat{\infty})$$

$$= 14\big(z^+\big)\tan(6z^+) + 42\big(z^+\big)^2\sec(12z^+) \quad .$$

This is the correct derivative.