Exploring the Barnes G-Function

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I derive a integral representation for the Barnes G-function among other things.

1. INTRODUCTION

In complex analysis, the K-function is defined by

$$K(z):=(2\pi)^{\frac{-z+1}{2}}\exp\left[\binom{z}{2}+\int_0^{z-1}\ln(t!)\,\mathrm{d}t\right].$$

This function has a simple and fascinating closed form

$$K(z)=\exp[\zeta'(-1,z)-\zeta'(-1)],$$

where $\zeta'(a,\,z)$ denotes the derivative of the Hurwitz zeta function and $\zeta'(a)$ denotes the derivative of the Euler-Riemann zeta function.

Another expression for K-function, derived from the Victor S. Adamchik work [1], is given by the polygamma function of negative order:

$$K(z)=\exp\left[\psi^{(-2)}(z)+\frac{z^2-z}{2}-\frac{z}{2}\log(2\pi)\right].$$

Thus, I prove the following integral representations for Barnes G-function:

$$\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1}{4}=\int_0^1[\psi(zx+1)-x^2\,\psi(zx^2+1)]x\,\mathrm{d}x$$

and

$$\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1+\gamma}{4}=\int_0^1\int_0^1\frac{(1-y^{zx})x-\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}x\,\mathrm{d}y.$$

In addition, I propose the infinite series:

$$\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1}{4}=$$
$$=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{2k-z+2}{8z}+\frac{1}{8z^2}\log\left\{\frac{(k+1)^{2(k+1)^2}}{(k+z+1)^{2[(k+1)^2-z^2]}}\right\}\right].$$

2. THE BARNES G-FUNCTION

Theorem 2.1. *For* $0<z<1$*, then*

$$\frac{1}{2z^2}\log\Gamma(z)+\frac{1}{2z^2}\log G(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1}{4}=\int_0^1[\psi(zx+1)-x^2\,\psi(zx^2+1)]x\,\mathrm{d}x, \qquad (2.1)$$

where $G(z)$ denotes the Barnes G-function, $\Gamma(z)$ denotes the gamma function and $\psi^{(-n)}(z)$ denotes the polygamma function of negative order.

Proof. In [2, page 3], I encounter the following power series involving the Riemann zeta function:

$$2\sum_{k=2}^{\infty}(-z)^k\frac{\zeta(k)}{k+1}=\frac{2}{z}\log G(z+1)+z(\gamma+1)-\log(2\pi)+1, \tag{2.2}$$

valid for $|z|<1$ and γ denotes the Euler-Mascheroni constant.

On the other hand, in [3, page 1], I prove the integral representation

$$\frac{1}{k}=2\int_0^1 x^{k-1}(1-x^k)\mathrm{d}x, \tag{2.3}$$

therefrom, replace k by $k+1$ in (2.3)

$$\frac{1}{k+1}=2\int_0^1 x^{k}(1-x^{k+1})\mathrm{d}x, \tag{2.4}$$

Taking the right hand side of (2.4) in the left hand side of (2.2), I obtain

$$\begin{aligned}
\frac{2}{z}\log G(z+1)+z(\gamma+1)-\log(2\pi)+1 &= 4\sum_{k=2}^{\infty}(-z)^k\zeta(k)\int_0^1 x^k(1-x^{k+1})\mathrm{d}x\\
&= 4\int_0^1\left[\sum_{k=2}^{\infty}(-z)^k\zeta(k)x^k(1-x^{k+1})\right]\mathrm{d}x\\
&= 4\int_0^1[-\gamma z x^3+\gamma z x+zx\,\psi(zx+1)-zx^3\,\psi(zx^2+1)]\mathrm{d}x\\
&= -4\int_0^1[\gamma z x^3-\gamma z x+zx^3\,\psi(zx^2+1)-zx\,\psi(zx+1)]\mathrm{d}x\\
&= -4z\gamma\int_0^1(x^2-1)x\mathrm{d}x-4z\int_0^1[x^2\,\psi(zx^2+1)-\psi(zx+1)]x\,\mathrm{d}x\\
&= -4z\gamma\left(-\frac{1}{4}\right)-4z\int_0^1[x^2\,\psi(zx^2+1)-\psi(zx+1)]x\,\mathrm{d}x\\
&= z\gamma+4z\int_0^1[\psi(zx+1)-x^2\,\psi(zx^2+1)]x\,\mathrm{d}x\\
\Rightarrow\frac{1}{2z^2}\log G(z+1)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1}{4} &= \int_0^1[\psi(zx+1)-x^2\,\psi(zx^2+1)]x\,\mathrm{d}x.
\end{aligned} \tag{2.5}$$

On the other hand, the Barnes G-function satisfies the functional equation

$$G(z+1)=\Gamma(z)\,G(z), \tag{2.6}$$

provided for $z\in\mathbb{C}$.

From (2.5) and (2.6), I find the desired result. □

Corollary 2.2. *For $0<z<1$, then*

$$\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1+\gamma}{4}=\int_0^1\int_0^1\frac{(1-y^{zx})x-\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}x\,\mathrm{d}y. \tag{2.7}$$

where $G(z)$ denotes the Barnes G-function, $\Gamma(z)$ denotes the gamma function, $\log(z)$ denotes the natural logarithm function.

Proof. I know the infinite sum representation for the digamma function [4]:

$$\psi(s+1)=-\gamma+\int_0^1\frac{1-y^s}{1-y}\mathrm{d}y. \tag{2.8}$$

From the right hand side of the Theorem 2.1 and (2.8), it follows that

$$\begin{aligned}
&\int_0^1 [\psi(zx+1)-x^2\psi(zx^2+1)]x\,\mathrm{d}x \\
&=\int_0^1\left[-\gamma+\int_0^1\frac{1-y^{zx}}{1-y}\mathrm{d}y-x^2\left(-\gamma+\int_0^1\frac{1-y^{zx^2}}{1-y}\mathrm{d}y\right)\right]x\,\mathrm{d}x \\
&=-\int_0^1\gamma x\,\mathrm{d}x+\int_0^1\int_0^1\frac{(1-y^{zx})x}{1-y}\mathrm{d}y\,\mathrm{d}x+\int_0^1\gamma x^3\,\mathrm{d}x-\int_0^1\int_0^1\frac{\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}y\,\mathrm{d}x \\
&=-\frac{\gamma}{2}+\frac{\gamma}{4}+\int_0^1\int_0^1\frac{(1-y^{zx})x-\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}y\,\mathrm{d}x \\
&=-\frac{\gamma}{4}+\int_0^1\int_0^1\frac{(1-y^{zx})x-\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{2.9}$$

From (2.1) and (2.9), it follows that

$$\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1+\gamma}{4}=\int_0^1\int_0^1\frac{(1-y^{zx})x-\left(1-y^{zx^2}\right)x^3}{1-y}\mathrm{d}x\,\mathrm{d}y,$$

which is the desired result. □

Theorem 2.3. *For* $0<z<1$, *then*

$$\begin{aligned}
&\frac{1}{2z^2}\log G(z)+\frac{1}{2z^2}\log\Gamma(z)-\frac{\log(2\pi)}{4z}+\frac{1}{4z}+\frac{1}{4}= \\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{2k-z+2}{8z}+\frac{1}{8z^2}\log\left\{\frac{(k+1)^{2(k+1)^2}}{(k+z+1)^{2[(k+1)^2-z^2]}}\right\}\right],
\end{aligned} \tag{2.10}$$

where $G(z)$ *denotes the Barnes G-function,* $\Gamma(z)$ *denotes the gamma function,* $\log(z)$ *denotes the natural logarithm function.*

Proof. In [5, p.18, (55), Theorem 5.1], I have the infinite series representation for the digamma function

$$\psi(u)=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(u+k), \tag{2.11}$$

valid for $u>0$.

From (2.11) and the right hand side of the Theorem 2.1, I obtain

$$\begin{aligned}
&\int_0^1[\psi(zx+1)-x^2\psi(zx^2+1)]x\,\mathrm{d}x \\
&=\int_0^1 x\,\psi(zx+1)\mathrm{d}x-\int_0^1 x^3\psi(zx^2+1)\,\mathrm{d}x \\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 x\log(zx+k+1)\mathrm{d}x \\
&-\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 x^3\log(zx^2+k+1)\,\mathrm{d}x \\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\int_0^1 x\log(zx+k+1)\mathrm{d}x-\int_0^1 x^3\log(zx^2+k+1)\,\mathrm{d}x\right] \\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k} \\
&\left[\frac{z(2k-z+2)-2(k-z+1)(k+z+1)\log(k+z+1)+2(k+1)^2\log(k+1)}{8z^2}\right] \\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{2k-z+2}{8z}+\frac{1}{8z^2}\log\left\{\frac{(k+1)^{2(k+1)^2}}{(k+z+1)^{2[(k+1)^2-z^2]}}\right\}\right].
\end{aligned} \tag{2.12}$$

From Theorem 2.1 and (2.12), I find the desired result. □

REFERENCES

[1] Victor S. Adamchik, Polygamma Functions of Negative Order, *Journal of Computational and Applied Mathematics*, **100** (1998),191-199.

[2] Victor S. Adamchik, Multiple Gamma Function and its Application to Computation of Series, arXiv.math/0308074v1[math.CA], 7 Aug 2003.

[3] Edigles B. Guedes, On the Integral Representations for Harmonic Number and Digamma Function, viXra:1503.0047, 8 March 2015.

[4] https://en.wikipedia.org/wiki/Digamma_function.

[5] Guillera, Jesús and Sondow, Jonathan, Double Integrals and Infinite Products for Some Classical Constants via Analytic Continuations of Lerch's Transcendet, arXiv:math/0506319v3 [math.NT] 5 Aug 2006.

[6] V. H. Kinkelin, Über eine mit der Gamma Funktion verwandte Transcendente und deren Anwendung auf die Integralrechnung, *J. Reine Angew. Math.* **57** (1860), 122–158.

[7] Junesang Choi and H. M. Srivastava, Certain Classes of Series Involving the Zeta Function, *Journal of Mathematical Analysis and Applications* **231**, 91-117 (1999).

[8] Joseph Edwards, A Treatise on the Integral Calculus, with Applications, Examples and Problems, Volume II, MacMillan and Co. Limited, St. Martin's Street, London, 1922.

3. APPENDIX

Corollary 3.1. *For* $0<z<1$, *then*

$$\frac{(\Gamma(z)G(z))^{1/2z^2}e^{(z+1)/4z}}{(2\pi)^{1/4z}}=\prod_{n=0}^{\infty}\left[\prod_{k=0}^{n}\left\{e^{\frac{2k-z+2}{8z}}\left[\frac{(k+1)^{2(k+1)^2}}{(k+z+1)^{2[(k+1)^2-z^2]}}\right]^{1/8z^2}\right\}^{(-1)^k\binom{n}{k}}\right]^{1/(n+1)}, \tag{3.1}$$

where $G(z)$ *denotes the Barnes G-function,* $\Gamma(z)$ *denotes the gamma function,* $\log(z)$ *denotes the natural logarithm function.*

Proof. The exponentiation of both members of (2.10) give me the desired result. □

Theorem 3.2. *For* $0<z<1$, *then*

$$z\log(2\pi)+\log\frac{G(1-z)}{G(1+z)}=\int_0^\infty\frac{1+\sinh(zt)\left[zt+\coth\left(\frac{t}{2}\right)\right]-\cosh(zt)\left[zt\coth\left(\frac{t}{2}\right)+1\right]}{t^2}\,\mathrm{d}t, \tag{3.2}$$

where $\log(z)$ *denotes the natural logarithm,* $G(z)$ *denotes the Barnes G-function,* $\sinh(z)$ *denotes the sine hyperbolic function,* $\cosh(z)$ *denotes the cosine hyperbolic function and* $\coth(z)$ *denotes the cotangent hyperbolic function.*

Proof. The following integral representation is due originally by Kinkelin [6] and Choi and Srivistava [7]:

$$\int_0^z \pi t\cot\pi t\,\mathrm{d}t=z\log(2\pi)+\log\frac{G(1-z)}{G(1+z)}. \tag{3.3}$$

In the Treatise on the Integral Calculus [8, p. 273], the author Joseph Edward give the integral representation for cotangent function as follows

$$\int_0^1\frac{x^{t-1}-x^{-t}}{1-x}\mathrm{d}x=\pi\cot\pi t, \tag{3.4}$$

provided for $0<t<1$.

Multiply (3.4) by t, integrate from 0 at z with respect to t and encounter

$$\begin{aligned}\int_0^z \pi t\cot\pi t\,\mathrm{d}t &= \int_0^z t\int_0^1 \frac{x^{t-1}-x^{-t}}{1-x}\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^1 \frac{1}{1-x}\int_0^z t(x^{t-1}-x^{-t})\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_0^1 \frac{z(x^{2z}+x)\log x-(x^z-1)(x^z+x)}{(1-x)x^{z+1}\log^2 x}\,\mathrm{d}x.\end{aligned} \tag{3.5}$$

So, from (3.3) and (3.5), I conclude that

$$\int_0^1 \frac{z(x^{2z}+x)\log x-(x^z-1)(x^z+x)}{(1-x)x^{z+1}\log^2 x}\,\mathrm{d}x = z\log(2\pi)+\log\frac{G(1-z)}{G(1+z)}.$$

With the substitution $x=e^{-t}$ and $\mathrm{d}x=-e^{-t}\mathrm{d}t$ in previous equation, I obtain

$$\begin{aligned}z\log(2\pi)+\log\frac{G(1-z)}{G(1+z)} &= -\int_0^\infty \frac{z(e^{-2zt}+e^{-t})t+(e^{-zt}-1)(e^{-zt}+e^{-t})}{(1-e^{-t})e^{-zt}t^2}\,\mathrm{d}t\\ &= \int_0^\infty \frac{1+\sinh(zt)\left[zt+\coth\left(\frac{t}{2}\right)\right]-\cosh(zt)\left[zt\coth\left(\frac{t}{2}\right)+1\right]}{t^2}\,\mathrm{d}t,\end{aligned}$$

which is the desired result. □