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Cross Entropy Measures of Bipolar and Interval Bipolar Neutrosophic Sets and Their Application for Multi-Attribute Decision-Making

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Received: 7 January 2018; Accepted: 22 March 2018; Published: 24 March 2018

Abstract: The bipolar neutrosophic set is an important extension of the bipolar fuzzy set. The bipolar neutrosophic set is a hybridization of the bipolar fuzzy set and neutrosophic set. Every element of a bipolar neutrosophic set consists of three independent positive membership functions and three independent negative membership functions. In this paper, we develop cross entropy measures of bipolar neutrosophic sets and prove their basic properties. We also define cross entropy measures of interval bipolar neutrosophic sets and prove their basic properties. Thereafter, we develop two novel multi-attribute decision-making strategies based on the proposed cross entropy measures. In the decision-making framework, we calculate the weighted cross entropy measures between each alternative and the ideal alternative to rank the alternatives and choose the best one. We solve two illustrative examples of multi-attribute decision-making problems and compare the obtained result with the results of other existing strategies to show the applicability and effectiveness of the developed strategies. At the end, the main conclusion and future scope of research are summarized.

Keywords: neutrosophic set; bipolar neutrosophic set; interval bipolar neutrosophic set; multi-attribute decision-making; cross entropy measure

1. Introduction

Shannon and Weaver [1] and Shannon [2] proposed the entropy measure which dealt formally with communication systems at its inception. According to Shannon and Weaver [1] and Shannon [2], the entropy measure is an important decision-making apparatus for computing uncertain information. Shannon [2] introduced the concept of the cross entropy strategy in information theory.

The measure of a quantity of fuzzy information obtained from a fuzzy set or fuzzy system is termed fuzzy entropy. However, the meaning of fuzzy entropy is quite different from the classical Shannon entropy because it is defined based on a nonprobabilistic concept [3–5], while Shannon entropy is defined based on a randomness (probabilistic) concept. In 1968, Zadeh [6] extended the Shannon entropy to fuzzy entropy on a fuzzy subset with respect to the concerned probability distribution. In 1972, De Luca and Termini [7] proposed fuzzy entropy based on Shannon's function and introduced the axioms with which the fuzzy entropy should comply. Sander [8] presented Shannon fuzzy entropy and proved that the properties sharpness, valuation, and general additivity have to be imposed on fuzzy entropy. Xie and Bedrosian [9] proposed another form of total fuzzy entropy. To overcome the drawbacks of total entropy [8,9], Pal and Pal [10] introduced hybrid entropy that can be used as an objective measure for a proper defuzzification of a certain fuzzy set. Hybrid entropy [10] considers both probabilistic entropies in the absence of fuzziness. In the same study, Pal and Pal [10] defined higher-order entropy. Kaufmann and Gupta [11] studied the degree of fuzziness of a fuzzy set by a metric distance between its membership function and the membership function (characteristic function) of its nearest crisp set. Yager [12,13] introduced a fuzzy entropy card as a fuzziness measure by observing that the intersection of a fuzzy set and its complement is not the void set. Kosko [14,15] studied the fuzzy entropy of a fuzzy set based on the fuzzy set geometry and distances between them. Parkash et al. [16] proposed two new measures of weighted fuzzy entropy.

Burillo and Bustince [17] presented an axiomatic definition of an intuitionistic fuzzy entropy measure. Szmidt and Kacprzyk [18] developed a new entropy measure based on a geometric interpretation of the intuitionistic fuzzy set (IFS). Wei et al. [19] proposed an entropy measure for interval-valued intuitionistic fuzzy sets (IVIFSs) and employed it in pattern recognition and multi criteria decision-making (MCDM). Li [20] presented a new multi-attribute decision-making (MADM) strategy combining entropy and technique for order of preference by similarity to ideal solution (TOPSIS) in the IVIFS environment.

Shang and Jiang [21] developed cross entropy in the fuzzy environment. Vlachos and Sergiadis [22] presented intuitionistic fuzzy cross entropy by extending fuzzy cross entropy [21]. Ye [23] proposed a new cross entropy in the IVIFS environment and developed an optimal decision-making strategy. Xia and Xu [24] defined a new entropy and a cross entropy and presented multi-attribute group decision-making (MAGDM) strategy in the IFS environment. Tong and Yu [25] defined cross entropy in the IVIFS environment and employed it to solve MADM problems.

Smarandache [26] introduced the neutrosophic set, which is a generalization of the fuzzy set [27] and intuitionistic fuzzy set [28]. The single-valued neutrosophic set (SVNS) [29], an instance of the neutrosophic set, has caught the attention of researchers due to its applicability in decision-making [30–61], conflict resolution [62], educational problems [63,64], image processing [65–67], cluster analysis [68,69], social problems [70,71], etc.

Majumdar and Samanta [72] proposed an entropy measure and presented an MCDM strategy in the SVNS environment. Ye [73] defined cross entropy for SVNS and proposed an MCDM strategy which bears undefined phenomena. To overcome the undefined phenomena, Ye [74] defined improved cross entropy measures for SVNSs and interval neutrosophic sets (INSs) [75], which are straightforward symmetric, and employed them to solve MADM problems. Since MADM strategies [73,74] are suitable for single-decision-maker-oriented problems, Pramanik et al. [76] defined NS-cross entropy and developed an MAGDM strategy which is straightforward symmetric and free from undefined phenomena and suitable for group decision making problem. Şahin [77] proposed two techniques to convert the interval neutrosophic information to single-valued neutrosophic information and fuzzy information. In the same study, Şahin [77] defined an interval neutrosophic cross entropy measure by utilizing two reduction methods and an MCDM strategy. Tian et al. [78] developed a transformation operator to convert interval neutrosophic numbers to single-valued neutrosophic numbers and defined cross entropy measures for two SVNSs. In the same study, Tian et al. [78] developed an MCDM strategy based on cross entropy and TOPSIS [79] where the weight of the criterion is incomplete. Tian et al. [78] defined a cross entropy for INSs and developed an MCDM strategy based on the cross entropy and TOPSIS. The MCDM strategies proposed by Sahin [77] and Tian et al. [78] are applicable for a single decision maker only. Therefore, multiple decision-makers cannot participate in the strategies in [77,78]. To tackle the problem, Dalapati et al. [80] proposed IN-cross entropy and weighted IN-cross entropy and developed an MAGDM strategy.

Deli et al. [81] proposed bipolar neutrosophic set (BNS) by hybridizing the concept of bipolar fuzzy sets [82,83] and neutrosophic sets [26]. A BNS has two fully independent parts, which are positive membership degree $T^+ \rightarrow [0, 1]$, $I^+ \rightarrow [0, 1]$, $F^+ \rightarrow [0, 1]$, and negative membership degree $T^- \rightarrow [-1, 0]$,

$I^- \rightarrow [-1, 0]$, $F^- \rightarrow [-1, 0]$, where the positive membership degrees T^+, I^+, F^+ represent truth membership degree, indeterminacy membership degree, and false membership degree, respectively, of an element and the negative membership degrees T^-, I^-, F^- represent truth membership degree, indeterminacy membership degree, and false membership degree, respectively, of an element to some implicit counter property corresponding to a BNS. Deli et al. [81] defined some operations, namely, score, accuracy, and certainty functions, to compare BNSs and provided some operators in order to aggregate BNSs. Deli and Subas [84] defined a correlation coefficient similarity measure for dealing with MCDM problems in a single-valued bipolar neutrosophic setting. Şahin et al. [85] proposed a Jaccard vector similarity measure for MCDM problems with single-valued neutrosophic information. Uluçay et al. [86] introduced a Dice similarity measure, weighted Dice similarity measure, hybrid vector similarity measure, and weighted hybrid vector similarity measure for BNSs and established an MCDM strategy. Dey et al. [87] investigated a TOPSIS strategy for solving multi-attribute decision-making (MADM) problems with bipolar neutrosophic information where the weights of the attributes are completely unknown to the decision-maker. Pramanik et al. [88] defined projection, bidirectional projection, and hybrid projection measures for BNSs and proved their basic properties. In the same study, Pramanik et al. [88] developed three new MADM strategies based on the proposed projection, bidirectional projection, and hybrid projection measures with bipolar neutrosophic information. Wang et al. [89] defined Frank operations of bipolar neutrosophic numbers (BNNs) and proposed Frank bipolar neutrosophic Choquet Bonferroni mean operators by combining Choquet integral operators and Bonferroni mean operators based on Frank operations of BNNs. In the same study, Wang et al. [89] established an MCDM strategy based on Frank Choquet Bonferroni operators of BNNs in a bipolar neutrosophic environment. Pramanik et al. [90] developed a Tomada de decisao interativa e multicritévio (TODIM) strategy for MAGDM in a bipolar neutrosophic environment. An MADM strategy based on cross entropy for BNSs is yet to appear in the literature.

Mahmood et al. [91] and Deli et al. [92] introduced the hybridized structure called interval bipolar neutrosophic sets (IBNSs) by combining BNSs and INSs and defined some operations and operators for IBNSs. An MADM strategy based on cross entropy for IBNSs is yet to appear in the literature.

Research gap:

An MADM strategy based on cross entropy for BNSs and an MADM strategy based on cross entropy for IBNSs.

This paper answers the following research questions:

i. Is it possible to define a new cross entropy measure for BNSs?
ii. Is it possible to define a new weighted cross entropy measure for BNSs?
iii. Is it possible to develop a new MADM strategy based on the proposed cross entropy measure in a BNS environment?
iv. Is it possible to develop a new MADM strategy based on the proposed weighted cross entropy measure in a BNS environment?
v. Is it possible to define a new cross entropy measure for IBNSs?
vi. Is it possible to define a new weighted cross entropy measure for IBNSs?
vii. Is it possible to develop a new MADM strategy based on the proposed cross entropy measure in an IBNS environment?
viii. Is it possible to develop a new MADM strategy based on the proposed weighted cross entropy measure in an IBNS environment?

Motivation:

The above-mentioned analysis presents the motivation behind proposing a cross-entropy-based strategy for tackling MADM in BNS and IBNS environments. This study develops two novel cross-entropy-based MADM strategies.

The objectives of the paper are:

1. To define a new cross entropy measure and prove its basic properties.
2. To define a new weighted cross measure and prove its basic properties.
3. To develop a new MADM strategy based on the weighted cross entropy measure in a BNS environment.
4. To develop a new MADM strategy based on the weighted cross entropy measure in an IBNS environment.

To fill the research gap, we propose a cross-entropy-based MADM strategy in the BNS environment and the IBNS environment.

The main contributions of this paper are summarized below:

1. We propose a new cross entropy measure in the BNS environment and prove its basic properties.
2. We propose a new weighted cross entropy measure in the IBNS environment and prove its basic properties.
3. We develop a new MADM strategy based on weighted cross entropy to solve MADM problems in a BNS environment.
4. We develop a new MADM strategy based on weighted cross entropy to solve MADM problems in an IBNS environment.
5. Two illustrative numerical examples are solved and a comparison analysis is provided.

The rest of the paper is organized as follows. In Section 2, we present some concepts regarding SVNSs, INSs, BNSs, and IBNSs. Section 3 proposes cross entropy and weighted cross entropy measures for BNSs and investigates their properties. In Section 4, we extend the cross entropy measures for BNSs to cross entropy measures for IBNSs and discuss their basic properties. Two novel MADM strategies based on the proposed cross entropy measures in bipolar and interval bipolar neutrosophic settings are presented in Section 5. In Section 6, two numerical examples are solved and a comparison with other existing methods is provided. In Section 7, conclusions and the scope of future work are provided.

2. Preliminary

In this section, we provide some basic definitions regarding SVNSs, INSs, BNSs, and IBNSs.

2.1. Single-Valued Neutrosophic Sets

An SVNS [29] S in U is characterized by a truth membership function $T_S(x)$, an indeterminate membership function $I_S(x)$, and a falsity membership function $F_S(x)$. An SVNS S over U is defined by

$$S = \{x, \langle T_S(x), I_S(x), F_S(x)\rangle | x \in U\}$$

where, $T_S(x), I_S(x), F_S(x)$: $U \to [0, 1]$ and $0 \leqslant T_S(x) + I_S(x) + F_S(x) \leqslant 3$ for each point $x \in U$.

2.2. Interval Neutrosophic Set

An interval neutrosophic set [75] P in U is expressed as given below:

$$\begin{aligned} P &= \{x, \langle T_P(x), I_P(x), F_P(x)\rangle | x \in U\} \\ &= \{x, [\inf T_p(x), \sup T_p(x)]; [\inf I_p(x), \sup I_p(x)]; [\inf F_p(x) \sup F_p(x)] | x \in U\} \end{aligned}$$

where $T_P(x), I_P(x), F_P(x)$ are the truth membership function, indeterminacy membership function, and falsity membership function, respectively. For each point x in U, $T_P(x), I_P(x), F_P(x) \subseteq [0, 1]$ satisfying the condition $0 \leqslant \sup T_P(x) + \sup I_P(x) + \sup F_P(x) \leqslant 3$.

2.3. Bipolar Neutrosophic Set

A BNS [81] E in U is presented as given below:

$$E = \{x, \langle T_E^+(x), I_E^+(x), F_E^+(x), T_E^-(x), I_E^-(x), F_E^-(x)\rangle | x \in U\}$$

where $T_E^+(x)$, $I_E^+(x)$, $F_E^+(x)$: $U \to [0, 1]$ and $T_E^-(x)$, $I_E^-(x)$, $F_E^-(x)$: $U \to [-1, 0]$. Here, $T_E^+(x)$, $I_E^+(x)$, $F_E^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions corresponding to BNS E on an element $x \in U$, and $T_E^-(x)$, $I_E^-(x)$, $F_E^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in U$ to some implicit counter property corresponding to E.

Definition 1. Ref. [81]: *Let,* $E_1 = \{x, \left\langle T_{E_1}^+(x), I_{E_1}^+(x), F_{E_1}^+(x), T_{E_1}^-(x), I_{E_1}^-(x), F_{E_1}^-(x)\right\rangle \mid x \in U\}$ *and* $E_2 = \{x, \left\langle T_{E_2}^+(x), I_{E_2}^+(x), F_{E_2}^+(x), T_{E_2}^-(x), I_{E_2}^-(x), F_{E_2}^-(x)\right\rangle \mid x \in X\}$ *be any two BNSs. Then*

- $E_1 \subseteq E_2$ *if, and only if,*

$T_{E_1}^+(x) \leqslant T_{E_2}^+(x)$, $I_{E_1}^+(x) \leqslant I_{E_2}^+(x)$, $F_{E_1}^+(x) \geqslant F_{E_2}^+(x)$; $T_{E_1}^-(x) \geqslant T_{E_2}^-(x)$, $I_{E_1}^-(x) \geqslant I_{E_2}^-(x)$, $F_{E_1}^-(x) \leqslant F_{E_2}^-(x)$ *for all* $x \in U$.

- $E_1 = E_2$ *if, and only if,*

$T_{E_1}^+(x) = T_{E_2}^+(x)$, $I_{E_1}^+(x) = I_{E_2}^+(x)$, $F_{E_1}^+(x) = F_{E_2}^+(x)$; $T_{E_1}^-(x) = T_{E_2}^-(x)$, $I_{E_1}^-(x) = I_{E_2}^-(x)$, $F_{E_1}^-(x) = F_{E_2}^-(x)$ *for all* $x \in U$.

- *The complement of E is* $E^c = \{x, \langle T_{E^c}^+(x), I_{E^c}^+(x), F_{E^c}^+(x), T_{E^c}^-(x), I_{E^c}^-(x), F_{E^c}^-(x)\rangle \mid x \in U\}$

where

$$T_{E^c}^+(x) = F_E^+(x),\ I_{E^c}^+(x) = 1 - I_E^+(x),\ F_{E^c}^+(x) = T_E^+(x);$$
$$T_{E^c}^-(x) = F_E^-(x),\ I_{E^c}^-(x) = -1 - I_E^-(x),\ F_{E^c}^-(x) = T_E^-(x).$$

- *The union* $E_1 \cup E_2$ *is defined as follows:*

$E_1 \cup E_2 = \{Max\ (T_{E_1}^+(x), T_{E_2}^+(x)), Min\ (I_{E_1}^+(x), I_{E_2}^+(x)), Min\ (F_{E_1}^+(x), F_{E_2}^+(x)), Min\ (T_{E_1}^-(x), T_{E_2}^-(x)), Max\ (I_{E_1}^-(x), I_{E_2}^-(x)), Max\ (F_{E_1}^-(x), F_{E_2}^-(x))\}, \forall\ x \in U$.

- *The intersection* $E_1 \cap E_2$ *[88] is defined as follows:*

$E_1 \cap E_2 = \{Min\ (T_{E_1}^+(x), T_{E_2}^+(x)), Max\ (I_{E_1}^+(x), I_{E_2}^+(x)), Max\ (F_{E_1}^+(x), F_{E_2}^+(x)), Max\ (T_{E_1}^-(x), T_{E_2}^-(x)), Min\ (I_{E_1}^-(x), I_{E_2}^-(x)), Min\ (F_{E_1}^-(x), F_{E_2}^-(x))\}, \forall\ x \in U$.

2.4. Interval Bipolar Neutrosophic Sets

An IBNS [91,92] $R = \{x, <[\inf T_R^+(x), \sup T_R^+(x)]; [\inf I_R^+(x), \sup I_R^+(x)]; [\inf F_R^+(x), \sup F_R^+(x)]; [\inf T_R^-(x), \sup T_R^-(x)]; [\inf I_R^-(x), \sup I_R^-(x)]; [\inf F_R^-(x), \sup F_R^-(x)]> \mid x \in U\}$ is characterized by positive and negative truth membership functions $T_R^+(x)$, $T_R^-(x)$, respectively; positive and negative indeterminacy membership functions $I_R^+(x)$, $I_R^-(x)$, respectively; and positive and negative falsity membership functions $F_R^+(x)$, $F_R^-(x)$, respectively. Here, for any $x \in U$, $T_R^+(x)$, $I_R^+(x)$, $F_R^+(x) \subseteq [0, 1]$ and $T_R^-(x)$, $I_R^-(x)$, $F_R^-(x) \subseteq [-1, 0]$ with the conditions $0 \leqslant \sup T_R^+(x) + \sup I_R^+(x) + \sup F_R^+(x) \leqslant 3$ and $-3 \leqslant \sup T_R^-(x) + \sup I_R^-(x) + \sup F_R^-(x) \leqslant 0$.

Definition 2. Ref. [91,92]: *Let* $R = \{x, <[\inf T_R^+(x), \sup T_R^+(x)]; [\inf I_R^+(x), \sup I_R^+(x)]; [\inf F_R^+(x), \sup F_R^+(x)]; [\inf T_R^-(x), \sup T_R^-(x)]; [\inf I_R^-(x), \sup I_R^-(x)]; [\inf F_R^-(x), \sup F_R^-(x)]> \mid x \in U\}$ *and* $S = \{x, <[\inf T_S^+(x), \sup T_S^+(x)]; [\inf I_S^+(x), \sup I_S^+(x)]; [\inf F_S^+(x), \sup F_S^+(x)]; [\inf T_S^-(x), \sup T_S^-(x)]; [\inf I_S^-(x), \sup I_S^-(x)]; [\inf F_S^-(x), \sup F_S^-(x)]> \mid x \in U\}$ *be two IBNSs in U. Then*

- *$R \subseteq S$ if, and only if,*

  $inf T_R^+(x) \leqslant inf T_S^+(x)$, $sup T_R^+(x) \leqslant sup T_S^+(x)$,

  $inf I_R^+(x) \geqslant inf I_S^+(x)$, $sup I_R^+(x) \geqslant sup I_S^+(x)$,

  $inf F_R^+(x) \geqslant inf F_S^+(x)$, $sup F_R^+(x) \geqslant sup F_S^+(x)$,

  $inf T_R^-(x) \geqslant inf T_S^-(x)$, $sup T_R^-(x) \geqslant sup T_S^-(x)$,

  $inf I_R^-(x) \leqslant inf\ I_S^-(x)$, $sup I_R^-(x) \leqslant sup I_S^-(x)$,

  $inf F_R^-(x) \leqslant inf F_S^-(x)$, $sup F_R^-(x) \leqslant sup F_S^-(x)$,

  for all $x \in U$.

- *$R = S$ if, and only if,*

  $inf T_R^+(x) = inf T_S^+(x)$, $sup T_R^+(x) = sup T_S^+(x)$, $inf I_R^+(x) = inf I_S^+(x)$, $sup I_R^+(x) = sup I_S^+(x)$,

  $inf F_R^+(x) = inf F_S^+(x)$, $sup F_R^+(x) = sup F_S^+(x)$, $inf T_R^-(x) = inf T_S^-(x)$, $sup T_R^-(x) = sup T_S^-(x)$,

  $inf I_R^-(x) = inf I_S^-(x)$, $sup I_R^-(x) = sup I_S^-(x)$, $inf F_R^-(x) = inf F_S^-(x)$, $sup F_R^-(x) = sup F_S^-(x)$,

  for all $x \in U$.

- *The complement of R is defined as* The complement of R is defined as $R^C = \{x, <$ [inf $T_{R^C}^+(x)$, sup$T_{R^C}^+(x)$]; [inf$I_{R^C}^+(x)$, sup$I_{R^C}^+(x)$]; [inf$F_{R^C}^+(x)$, sup$F_{R^C}^+(x)$]; [inf$T_{R^C}^-(x)$, sup$T_{R^C}^-(x)$]; [inf$I_{R^C}^-(x)$, sup$I_{R^C}^-(x)$]; [inf $F_{R^C}^-(x)$, sup$F_{R^C}^-(x)$] $> \mid x \in U\}$ where

  inf $T_{R^C}^+(x) = \text{inf}F_R^+(x)$, sup$T_{R^C}^+(x) = \text{sup}F_R^+(x)$

  inf $I_{R^C}^+(x) = 1 - \text{sup}I_R^+(x)$, sup$I_{R^C}^+(x) = 1 - \text{inf}I_R^+(x)$

  inf$F_{R^C}^+(x) = \text{inf}T_R^+$, sup$F_{R^C}^+(x) = \text{sup}T_R^+$

  inf$T_{R^C}^-(x) = \text{inf}F_R^-$, sup$T_{R^C}^-(x) = \text{sup}F_R^-$

  inf$I_{R^C}^-(x) = -1 - \text{sup}I_R^-(x)$, sup$I_{R^C}^-(x) = -1 - \text{inf}I_R^-(x)$

  inf$F_{R^C}^-(x) = \text{inf}T_R^-\ (x)$, sup$F_{R^C}^-(x) = \text{sup}T_R^-\ (x)$

  for all $x \in U$.

3. Cross Entropy Measures of Bipolar Neutrosophic Sets

In this section we define a cross entropy measure between two BNSs and establish some of its basic properties.

Definition 3. *For any two BNSs M and N in U, the cross entropy measure can be defined as follows.*

$$C_B(M,N) = \sum_{i=1}^{n}\left[\begin{array}{l} \sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-I_M^+(x_i)\right)}+\sqrt{\left(1-I_N^+(x_i)\right)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(I_M^-(x_i)+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_M^-(x_i)\right)}+\sqrt{\left(-I_N^-(x_i)\right)}}{2}\right) + \\ \sqrt{\frac{\left(1+I_M^-(x_i)\right)+\left(1+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2}\right) \end{array}\right] \quad (1)$$

Theorem 1. *If* $M = <T_M^+(x_i),\ I_M^+(x_i),\ F_M^+(x_i),\ T_M^-(x_i),\ I_M^-(x_i),\ F_M^-(x_i)>$ *and* $N <T_N^+(x_i),\ I_N^+(x_i),\ F_N^+(x_i),\ T_N^-(x_i),\ I_N^-(x_i),\ F_N^-(x_i)>$ *are two BNSs in U, then the cross entropy measure* $C_B(M, N)$ *satisfies the following properties:*

(1) $C_B(M, N) \geqslant 0$;
(2) $C_B(M, N) = 0$ *if, and only if,* $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)$, $\forall\ x \in U$;
(3) $C_B(M, N) = C_B(N, M)$;
(4) $C_B(M, N) = C_B(M^C, N^C)$.

Proof

(1) We have the inequality $\left(\frac{a+b}{2}\right)^{\frac{1}{2}} \geqslant \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}$ for all positive numbers a and b. From the inequality we can easily obtain $C_B(M, N) \geqslant 0$.
(2) The inequality $\left(\frac{a+b}{2}\right)^{\frac{1}{2}} \geqslant \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}$ becomes the equality $\left(\frac{a+b}{2}\right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}$ if, and only if, $a = b$ and therefore $C_B(M, N) = 0$ if, and only if, $M = N$, i.e., $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)\ \forall\ x \in U$.
(3)

$$C_B(M,N) = \sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-I_M^+(x_i)\right)}+\sqrt{\left(1-I_N^+(x_i)\right)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\
\sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(I_M^-(x_i)+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_M^-(x_i)\right)}+\sqrt{\left(-I_N^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{\left(1+I_M^-(x_i)\right)+\left(1+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2}\right)
\end{array}\right]$$

$$= \sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{T_N^+(x_i)+T_M^+(x_i)}{2}} - \left(\frac{\sqrt{T_N^+(x_i)}+\sqrt{T_M^+(x_i)}}{2}\right) + \sqrt{\frac{I_N^+(x_i)+I_M^+(x_i)}{2}} - \left(\frac{\sqrt{I_N^+(x_i)}+\sqrt{I_M^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-I_N^+(x_i)\right)+\left(1-I_M^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-I_N^+(x_i)\right)}+\sqrt{\left(1-I_M^+(x_i)\right)}}{2}\right) + \sqrt{\frac{F_N^+(x_i)+F_M^+(x_i)}{2}} - \left(\frac{\sqrt{F_N^+(x_i)}+\sqrt{F_M^+(x_i)}}{2}\right) + \\
\sqrt{\frac{-\left(T_N^-(x_i)+T_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_N^-(x_i)\right)}+\sqrt{\left(-T_M^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(I_N^-(x_i)+I_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_N^-(x_i)\right)}+\sqrt{\left(-I_M^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{\left(1+I_N^-(x_i)\right)+\left(1+I_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_N^-(x_i)}+\sqrt{1+I_M^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(F_N^-(x_i)+F_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_N^-(x_i)\right)}+\sqrt{\left(-F_M^-(x_i)\right)}}{2}\right)
\end{array}\right] = C_B(N, M).$$

(4) $C_B(M^C, N^C)$

$$= \sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-I_M^+(x_i)\right)}+\sqrt{1-I_N^+(x_i))}}{2}\right) + \\
\sqrt{\frac{1-\left(1-I_M^+(x_i)\right)+1-\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-\left(1-I_M^+(x_i)\right)}+\sqrt{1-\left(1-I_N^+(x_i)\right)}}{2}\right) + \sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \\
\sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(-1-I_M^-(x_i)\right)-\left(-1-I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{-\left(-1-I_M^-(x_i)\right)}+\sqrt{-\left(-1-I_N^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{1+\left(-1-I_M^-(x_i)\right)+1+\left(-1-I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\left(-1-I_M^-(x_i)\right)}+\sqrt{1+\left(-1-I_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2}\right)
\end{array}\right]$$

$$= \sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-I_M^+(x_i)\right)}+\sqrt{\left(1-I_N^+(x_i)\right)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\
\sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(I_M^-(x_i)+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_M^-(x_i)\right)}+\sqrt{\left(-I_N^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{\left(1+I_M^-(x_i)\right)+\left(1+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2}\right)
\end{array}\right] = C_B(M, N).$$

The proof is completed. □

Example 1. *Suppose that M = <0.7, 0.3, 0.4, −0.3, −0.5, −0.1> and N = <0.5, 0.2, 0.5, −0.3, −0.3, −0.2> are two BNSs; then the cross entropy between M and N is calculated as follows:*

$$C_B(M,N) = \left[\begin{array}{l} \sqrt{\frac{0.7+0.5}{2}} - \left(\frac{\sqrt{0.7}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{0.3+0.2}{2}} - \left(\frac{\sqrt{0.3}+\sqrt{0.2}}{2}\right) + \sqrt{\frac{(1-0.3)+(1-0.2)}{2}} - \left(\frac{\sqrt{1-0.3}+\sqrt{1-0.2}}{2}\right) + \sqrt{\frac{0.4+0.5}{2}} - \\ \left(\frac{\sqrt{0.4}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{-(-0.3-0.3)}{2}} - \left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.3)}}{2}\right) + \sqrt{\frac{-(-0.5-0.3)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.3)}}{2}\right) \\ +\sqrt{\frac{(1-0.5)+(1-0.3)}{2}} - \left(\frac{\sqrt{1-0.5}+\sqrt{[1-0.3]}}{2}\right) + \sqrt{\frac{-(-0.1-0.2)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.2)}}{2}\right) \end{array}\right] = 0.01738474.$$

Definition 4. *Suppose that w_i is the weight of each element x_i, i = 1, 2, ..., n, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$; then the weighted cross entropy measure between any two BNSs M and N in U can be defined as follows.*

$$C_B(M,N)_w = \sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-I_M^+(x_i)}+\sqrt{1-I_N^+(x_i)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(I_M^-(x_i)+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_M^-(x_i)\right)}+\sqrt{\left(-I_N^-(x_i)\right)}}{2}\right) + \\ \sqrt{\frac{\left(1+I_M^-(x_i)\right)+\left(1+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2}\right) \end{array}\right] \quad (2)$$

Theorem 2. *If M = <$T_M^+(x_i)$, $I_M^+(x_i)$, $F_M^+(x_i)$, $T_M^-(x_i)$, $I_M^-(x_i)$, $F_M^-(x_i)$> and N <$T_N^+(x_i)$, $I_N^+(x_i)$, $F_N^+(x_i)$, $T_N^-(x_i)$, $I_N^-(x_i)$, $F_N^-(x_i)$> are two BNSs in U, then the weighted cross entropy measure $C_B(M, N)_w$ satisfies the following properties:*

(1) $C_B(M, N)_w \geqslant 0$;
(2) $C_B(M, N)_w = 0$ *if, and only if,* $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)$, $\forall\, x \in U$;
(3) $C_B(M, N)_w = C_B(N, M)_w$;
(4) $C_B(M^C, N^C)_w = C_B(M, N)_w$.

Proof is given in Appendix A.

Example 2. *Suppose that M = <0.7, 0.3, 0.4, −0.3, −0.5, −0.1> and N = <0.5, 0.2, 0.5, −0.3, −0.3, −0.2> are two BNSs and w = 0.4; then the weighted cross entropy between M and N is calculated as given below.*

$$C_B(M,N)_w = 0.4 \times \left[\begin{array}{l} \sqrt{\frac{0.7+0.5}{2}} - \left(\frac{\sqrt{0.7}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{0.3+0.2}{2}} - \left(\frac{\sqrt{0.3}+\sqrt{0.2}}{2}\right) + \sqrt{\frac{(1-0.3)+(1-0.2)}{2}} - \left(\frac{\sqrt{1-0.3}+\sqrt{1-0.2}}{2}\right) \\ +\sqrt{\frac{0.4+0.5}{2}} - \left(\frac{\sqrt{0.4}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{-(-0.3-0.3)}{2}} - \left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.3)}}{2}\right) + \sqrt{\frac{-(-0.5-0.3)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.3)}}{2}\right) \\ +\sqrt{\frac{(1-0.5+(1-0.3)}{2}} - \left(\frac{\sqrt{1-0.5}+\sqrt{1-0.3}}{2}\right) + \sqrt{\frac{-(-0.1-0.2)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.2)}}{2}\right) \end{array}\right] = 0.006953896.$$

4. Cross Entropy Measure of IBNSs

This section extends the concepts of cross entropy and weighted cross entropy measures of BNSs to IBNSs.

Definition 5. *The cross entropy measure between any two IBNSs R = <[$infT_R^+(x_i)$, $supT_R^+(x_i)$], [$infI_R^+(x_i)$, $supI_R^+(x_i)$], [$infF_R^+(x_i)$, $supF_R^+(x_i)$], [$infT_R^-(x_i)$, $supT_R^-(x_i)$], [$infI_R^-(x_i)$, $supI_R^-(x_i)$], [$infF_R^-(x_i)$,*

$supF_R^-(x_i)$]> and S = <[$infT_S^+(x_i)$, $supT_S^+(x_i)$], [$infI_S^+(x_i)$, $supI_S^+(x_i)$], [$infF_S^+(x_i)$, $supF_S^+(x_i)$], [$infT_S^-(x_i)$, $supT_S^-(x_i)$], [$infI_S^-(x_i)$, sup $I_S^-(x_i)$], [$infF_S^-(x_i)$, $supF_S^-(x_i)$]> in U can be defined as follows.

$$
C_{IB}(R,S)=\frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{\mathrm{inf}T_R^+(x_i)+\mathrm{inf}T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}T_R^+(x_i)}+\sqrt{\mathrm{inf}T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}T_R^+(x_i)+\mathrm{sup}T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}T_R^+(x_i)}+\sqrt{\mathrm{sup}T_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{inf}I_R^+(x_i)+\mathrm{inf}I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}I_R^+(x_i)}+\sqrt{\mathrm{inf}I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}I_R^+(x_i)+\mathrm{sup}I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}I_R^+(x_i)}+\sqrt{\mathrm{sup}I_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1-\mathrm{inf}I_R^+(x_i)\right)+\left(1-\mathrm{inf}I_S^+(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\mathrm{inf}I_R^+(x_i)}+\sqrt{[1-\mathrm{inf}I_S^+(x_i)]}}{2}\right)+\sqrt{\frac{\left(1-\mathrm{sup}I_R^+(x_i)\right)+\left(1-\mathrm{sup}I_S^+(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{1-\mathrm{sup}I_R^+(x_i)}+\sqrt{1-\mathrm{sup}I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{inf}F_R^+(x_i)+\mathrm{inf}F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}F_R^+(x_i)}+\sqrt{\mathrm{inf}F_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{sup}F_R^+(x_i)+\mathrm{sup}F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}F_R^+(x_i)}+\sqrt{\mathrm{sup}F_S^+(x_i)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}T_R^-(x_i)+\mathrm{inf}T_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}T_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}T_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}T_R^-(x_i)+\mathrm{sup}T_S^+(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}T_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}T_S^-(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{-\left(\mathrm{inf}I_R^-(x_i)+\mathrm{inf}I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{inf}I_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}I_R^-(x_i)+\mathrm{sup}I_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{sup}I_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1+\mathrm{inf}I_R^-(x_i)\right)+\left(1+\mathrm{inf}I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{inf}I_R^-(x_i)}+\sqrt{1+\mathrm{inf}I_S^-(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1+\mathrm{sup}I_R^-(x_i)\right)+\left(1+\mathrm{sup}I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{sup}I_R^-(x_i)}+\sqrt{1+\mathrm{sup}I_S^-(x_i)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}F_R^-(x_i)+\mathrm{inf}F_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}F_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}F_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}F_R^-(x_i)+\mathrm{sup}F_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}F_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}F_S^-(x_i)\right)}}{2}\right)
\end{array}\right], \tag{3}
$$

Theorem 3. *If R = <[$infT_R^+(x_i)$, $supT_R^+(x_i)$], [inf, $supI_R^+(x_i)$], [$infF_R^+(x_i)$, $supF_R^+(x_i)$], [$infT_R^-(x_i)$, $supT_R^-(x_i)$], [$infI_R^-(x_i)$, $supI_R^-(x_i)$], [$infF_R^-(x_i)$, $supF_R^-(x_i)$]> and S = <[$infT_S^+(x_i)$, $supT_S^+(x_i)$], [$infI_S^+(x_i)$, $supI_S^+(x_i)$], [$infF_S^+(x_i)$, $supF_S^+(x_i)$], [$infT_S^-(x_i)$, $supT_S^-(x_i)$], [$infI_S^-(x_i)$, $supI_S^-(x_i)$], [$infF_S^-(x_i)$, $supF_S^-(x_i)$]> are two IBNSs in U, then the cross entropy measure $C_{IB}(R, S)$ satisfies the following properties:*

(1) *$C_{IB}(R, S) \geqslant 0$;*

(2) *$C_{IB}(R, S) = 0$ for R = S i.e., inf $T_R^+(x_i) = infT_S^+(x_i)$, $supT_R^+(x_i) = supT_S^+(x_i)$, $infI_R^+(x_i) = infI_S^+(x_i)$, $supI_R^+(x_i) = supI_S^+(x_i)$, $infF_R^+(x_i) = infF_S^+(x_i)$, $supF_R^+(x_i) = supF_S^+(x_i)$, $infT_R^-(x_i) = infT_S^-(x_i)$, $supT_R^-(x_i) = supT_S^-(x_i)$, $infI_R^-(x_i) = infI_S^-(x_i)$, $supI_R^-(x_i) = supI_S^-(x_i)$, $infF_R^-(x_i) = infF_S^-(x_i)$, $supF_R^-(x_i) = supF_S^-(x_i)$ $\forall$ $x \in U$;*

(3) *$C_{IB}(R, S) = C_{IB}(S, R)$;*

(4) *$C_{IB}(R^C, S^C) = C_{IB}(R, S)$.*

Proof

(1) From the inequality stated in Theorem 1, we can easily get $C_{IB}(R, S) \geqslant 0$.

(2) Since $\mathrm{inf}T_R^+(x_i) = \mathrm{inf}T_S^+(x_i)$, $\mathrm{sup}T_R^+(x_i) = \mathrm{sup}T_S^+(x_i)$, $\mathrm{inf}I_R^+(x_i) = \mathrm{inf}I_S^+(x_i)$, $\mathrm{sup}I_R^+(x_i) = \mathrm{sup}I_S^+(x_i)$, $\mathrm{inf}F_R^+(x_i) = \mathrm{inf}F_S^+(x_i)$, $\mathrm{sup}F_R^+(x_i) = \mathrm{sup}F_S^+(x_i)$, $\mathrm{inf}T_R^-(x_i) = \mathrm{inf}T_S^-(x_i)$, $\mathrm{sup}T_R^-(x_i) = \mathrm{sup}T_S^-(x_i)$, $\mathrm{inf}I_R^-(x_i) = \mathrm{inf}I_S^-(x_i)$, $\mathrm{sup}I_R^-(x_i) = \mathrm{sup}I_S^-(x_i)$, $\mathrm{inf}F_R^-(x_i) = \mathrm{inf}F_S^-(x_i)$, $\mathrm{sup}F_R^-(x_i) = \mathrm{sup}F_S^-(x_i)$ $\forall$ $x \in U$, we have $C_{IB}(R, S) = 0$.

$$
\begin{aligned}
(3)\quad C_{IB}(R,\ S) &= \frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{\mathrm{inf}T_R^{+}(x_i)+\mathrm{inf}T_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}T_R^{+}(x_i)}+\sqrt{\mathrm{inf}T_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}T_R^{+}(x_i)+\mathrm{sup}T_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}T_R^{+}(x_i)}+\sqrt{\mathrm{sup}T_S^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{inf}I_R^{+}(x_i)+\mathrm{inf}I_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}I_R^{+}(x_i)}+\sqrt{\mathrm{inf}I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}I_R^{+}(x_i)+\mathrm{sup}I_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}I_R^{+}(x_i)}+\sqrt{\mathrm{sup}I_S^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\left([1-\mathrm{inf}I_R^{+}(x_i)\right)+\left(1-\mathrm{inf}I_S^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\mathrm{inf}I_R^{+}(x_i)}+\sqrt{1-\mathrm{inf}I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\left(1-\mathrm{sup}I_R^{+}(x_i)\right)+\left(1-\mathrm{sup}I_S^{+}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{1-\mathrm{sup}I_R^{+}(x_i)}+\sqrt{1-\mathrm{sup}I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{inf}F_R^{+}(x_i)+\mathrm{inf}F_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}F_R^{+}(x_i)}+\sqrt{\mathrm{inf}F_S^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{sup}F_R^{+}(x_i)+\mathrm{sup}F_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}F_R^{+}(x_i)}+\sqrt{\mathrm{sup}F_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}T_R^{-}(x_i)+\mathrm{inf}T_S^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}T_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}T_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}T_R^{-}(x_i)+\mathrm{sup}T_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}T_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}T_S^{-}(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{-\left(\mathrm{inf}I_R^{-}(x_i)+\mathrm{inf}I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{inf}I_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}I_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}I_R^{-}(x_i)+\mathrm{sup}I_S^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{sup}I_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}I_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1+\mathrm{inf}I_R^{-}(x_i)\right)+\left(1+\mathrm{inf}I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{inf}I_R^{-}(x_i)}+\sqrt{1+\mathrm{inf}I_S^{-}(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1+\mathrm{sup}I_R^{-}(x_i)\right)+\left(1+\mathrm{sup}I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{sup}I_R^{-}(x_i)}+\sqrt{1+\mathrm{sup}I_S^{-}(x_i)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}F_R^{-}(x_i)+\mathrm{inf}F_S^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}F_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}F_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}F_R^{-}(x_i)+\mathrm{sup}F_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}F_R^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}F_S^{-}(x_i)\right)}}{2}\right)
\end{array}\right]\\
&= \frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{\mathrm{inf}T_S^{+}(x_i)+\mathrm{inf}T_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}T_S^{+}(x_i)}+\sqrt{\mathrm{inf}T_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}T_S^{+}(x_i)+\mathrm{sup}T_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}T_S^{+}(x_i)}+\sqrt{\mathrm{sup}T_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{inf}I_S^{+}(x_i)+\mathrm{inf}I_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}I_S^{+}(x_i)}+\sqrt{\mathrm{inf}I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{sup}I_S^{+}(x_i)+\mathrm{sup}I_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}I_S^{+}(x_i)}+\sqrt{\mathrm{sup}I_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1-\mathrm{inf}I_S^{+}(x_i)\right)+\left(1-\mathrm{inf}I_R^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\mathrm{inf}I_S^{+}(x_i)}+\sqrt{1-\mathrm{inf}I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\left(1-\mathrm{sup}I_S^{+}(x_i)\right)+\left(1-\mathrm{sup}I_R^{+}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{1-\mathrm{sup}I_S^{+}(x_i)}+\sqrt{1-\mathrm{sup}I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\mathrm{inf}F_S^{+}(x_i)+\mathrm{inf}F_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{inf}F_S^{+}(x_i)}+\sqrt{\mathrm{inf}F_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\mathrm{sup}F_S^{+}(x_i)+\mathrm{sup}F_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\mathrm{sup}F_S^{+}(x_i)}+\sqrt{\mathrm{sup}F_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}T_S^{-}(x_i)+\mathrm{inf}T_R^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}T_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}T_R^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}T_S^{-}(x_i)+\mathrm{sup}T_R^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}T_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}T_R^{-}(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{-\left(\mathrm{inf}I_S^{-}(x_i)+\mathrm{inf}I_R^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{inf}I_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}I_R^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}I_S^{-}(x_i)+\mathrm{sup}I_R^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{sup}I_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}I_R^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1+\mathrm{inf}I_S^{-}(x_i)\right)+\left(1+\mathrm{inf}I_R^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{inf}I_S^{-}(x_i)}+\sqrt{1+\mathrm{inf}I_R^{-}(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1+\mathrm{sup}I_S^{-}(x_i)\right)+\left(1+\mathrm{sup}I_R^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\mathrm{sup}I_S^{-}(x_i)}+\sqrt{[1+\mathrm{sup}I_R^{-}(x_i)]}}{2}\right)+\sqrt{\frac{-\left(\mathrm{inf}F_S^{-}(x_i)+\mathrm{inf}F_R^{-}(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\mathrm{inf}F_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{inf}F_R^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\mathrm{sup}F_S^{-}(x_i)+\mathrm{sup}F_R^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\mathrm{sup}F_S^{-}(x_i)\right)}+\sqrt{\left(-\mathrm{sup}F_R^{-}(x_i)\right)}}{2}\right)
\end{array}\right]\\
&= C_{IB}(S,\ R).
\end{aligned}
$$

(4) $C_{IB}(R^C, S^C) =$

$$\frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1-\inf I_R^+(x_i)\right)+\left(1-\inf I_S^+(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(1-\inf I_R^+(x_i)\right)}+\sqrt{\left(1-\inf I_S^+(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1-\sup I_R^+(x_i)\right)+\left(1-\sup I_S^+(x)_i\right)}{2}}-\left(\frac{\sqrt{\left(1-\sup I_R^+(x_i)\right)}+\sqrt{\left(1-\sup I_S^+(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{1-\left(1-\inf I_R^+(x_i)\right)+1-\left(1-\inf I_S^+(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\left(1-\inf I_R^+(x_i)\right)}+\sqrt{1-\left(1-\inf I_S^+(x_i)\right)}}{2}\right)+\sqrt{\frac{1-\left(1-\sup I_R^+(x_i)\right)+1-\left(1-\sup I_S^+(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{1-\left(1-\sup I_R^+(x_i)\right)}+\sqrt{1-\left(1-\sup I_S^+(x_i)\right)}}{2}\right)+\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right)\\
+\sqrt{\frac{-\left(\inf F_R^-(x_i)+\inf F_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\inf F_R^-(x_i)\right)}+\sqrt{\left(-\inf F_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup F_R^-(x_i)+\sup F_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup F_R^-(x_i)\right)}+\sqrt{\left(-\sup F_S^-(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{-\left(\left(-1-\inf I_R^-(x_i)\right)+\left(-1-\inf I_S^-(x_i)\right)\right)}{2}}-\left(\frac{\sqrt{-\left(-1-\inf I_R^-(x_i)\right)}+\sqrt{-\left(-1-\inf I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\left(-1-\sup I_R^-(x_i)\right)+\left(-1-\sup I_S^-(x_i)\right)\right)}{2}}-\\
\left(\frac{\sqrt{-\left(-1-\sup I_R^-(x_i)\right)}+\sqrt{-\left(-1-\sup I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{1+\left(-1-\inf I_R^-(x_i)\right)+1+\left(-1-\inf I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\left(-1-\inf I_R^-(x_i)\right)}+\sqrt{1+\left(-1-\inf I_S^-(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{1+\left(-1-\sup I_R^-(x_i)\right)+1+\left(-1-\sup I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\left(-1-\sup I_R^-(x_i)\right)}+\sqrt{1+\left(-1-\sup I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\inf T_R^-(x_i)+\inf T_S(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\inf T_R^-(x_i)\right)}+\sqrt{\left(-\inf T_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup T_R^-(x_i)+\sup T_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup T_R^-(x_i)\right)}+\sqrt{\left(-\sup T_S^-(x_i)\right)}}{2}\right)
\end{array}\right]$$

$$=\frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}
\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\inf I_R^+(x_i)+\inf I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf I_R^+(x_i)}+\sqrt{\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup I_R^+(x_i)+\sup I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup I_R^+(x_i)}+\sqrt{\sup I_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{[1-\inf I_R^+(x_i))+\left(1-\inf I_S^+(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\inf I_R^+(x_i)}+\sqrt{1-\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\left(1-\sup I_R^+(x_i)\right)+\left(1-\sup I_S^+(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{1-\sup I_R^+(x_i)}+\sqrt{1-\sup I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right)+\sqrt{\frac{-\left(\inf T_R^-(x_i)+\inf T_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\inf T_R^-(x_i)\right)}+\sqrt{\left(-\inf T_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup T_R^-(x_i)+\sup T_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup T_R^-(x_i)\right)}+\sqrt{\left(-\sup T_S^-(x_i)\right)}}{2}\right)+\\
\sqrt{\frac{-\left(\inf I_R^-(x_i)+\inf I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\inf I_R^-(x_i)\right)}+\sqrt{\left(-\inf I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup I_R^-(x_i)+\sup I_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\sup I_R^-(x_i)\right)}+\sqrt{\left(-\sup I_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1+\inf I_R^-(x_i)\right)+\left(1+\inf I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\inf I_R^-(x_i)}+\sqrt{1+\inf I_S^-(x_i)}}{2}\right)+\\
\sqrt{\frac{\left(1+\sup I_R^-(x_i)\right)+\left(1+\sup I_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\sup I_R^-(x_i)}+\sqrt{1+\sup I_S^-(x_i)}}{2}\right)+\sqrt{\frac{-\left(\inf F_R^-(x_i)+\inf F_S^-(x_i)\right)}{2}}-\\
\left(\frac{\sqrt{\left(-\inf F_R^-(x_i)\right)}+\sqrt{\left(-\inf F_S^-(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup F_R^-(x_i)+\sup F_S^-(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup F_R^-(x_i)\right)}+\sqrt{\left(-\sup F_S^-(x_i)\right)}}{2}\right)
\end{array}\right]$$

$= C_{IB}(R, S)$. □

Example 3. *Suppose that R = <[0.5, 0.8], [0.4, 0.6], [0.2, 0.6], [−0.3, −0.1], [−0.5, −0.1], [−0.5, −0.2]> and S = <[0.5, 0.9], [0.4, 0.5], [0.1, 0.4], [−0.5, −0.3], [−0.7, −0.3], [−0.6, −0.3]> are two IBNSs; the cross entropy between R and S is computed as follows:*

$$C_{IB}(R,S)=\frac{1}{2}\left[\begin{array}{l}
\sqrt{\frac{0.5+0.5}{2}}-\left(\frac{\sqrt{0.5}+\sqrt{0.5}}{2}\right)+\sqrt{\frac{0.8+0.9}{2}}-\left(\frac{\sqrt{0.8}+\sqrt{0.9}}{2}\right)+\sqrt{\frac{0.4+0.4}{2}}-\left(\frac{\sqrt{0.4}+\sqrt{0.4}}{2}\right)+\sqrt{\frac{0.6+0.5}{2}}-\left(\frac{\sqrt{0.6}+\sqrt{0.5}}{2}\right)+\\
\sqrt{\frac{[1-0.4]+[1-0.4]}{2}}-\left(\frac{\sqrt{1-0.4}+\sqrt{[1-0.4]}}{2}\right)+\sqrt{\frac{[1-0.6]+[1-0.5]}{2}}-\left(\frac{\sqrt{1-0.6}+\sqrt{[1-0.5]}}{2}\right)+\sqrt{\frac{0.2+0.1}{2}}-\left(\frac{\sqrt{0.2}+\sqrt{0.1}}{2}\right)+\\
\sqrt{\frac{0.6+0.4}{2}}-\left(\frac{\sqrt{0.6}+\sqrt{0.4}}{2}\right)+\sqrt{\frac{-(-0.3-0.5)}{2}}-\left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.5)}}{2}\right)+\sqrt{\frac{-(-0.1-0.3)}{2}}-\left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right)+\\
\sqrt{\frac{-(-0.5-0.7)}{2}}-\left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.7)}}{2}\right)+\sqrt{\frac{-(-0.1-0.3)}{2}}-\left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right)+\sqrt{\frac{[1-0.5]+[1-0.7]}{2}}-\\
\left(\frac{\sqrt{1-0.5}+\sqrt{[1-0.7]}}{2}\right)+\sqrt{\frac{[1-0.1]+[1-0.3]}{2}}-\left(\frac{\sqrt{1-0.1}+\sqrt{[1-0.3]}}{2}\right)+\sqrt{\frac{-(-0.5-0.6)}{2}}-\left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.6)}}{2}\right)+\\
\sqrt{\frac{-(-0.2-0.3)}{2}}-\left(\frac{\sqrt{-(-0.2)}+\sqrt{-(-0.3)}}{2}\right)
\end{array}\right]=0.02984616.$$

Definition 6. *Let w_i be the weight of each element x_i, i = 1, 2, ..., n, and $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$; then the weighted cross entropy measure between any two IBNSs R = <[$infT_R^+(x_i)$, $supT_R^+(x_i)$], [$infI_R^+(x_i)$, $supI_R^+(x_i)$], [$infF_R^+(x_i)$, $supF_R^+(x_i)$], [$infT_R^-(x_i)$, $supT_R^-(x_i)$], [$infI_R^-(x_i)$, $supI_R^-(x_i)$], [$infF_R^-(x_i)$, $supF_R^-(x_i)$]> and S = <[$infT_S^+(x_i)$, $supT_S^+(x_i)$], [$infI_S^+(x_i)$, $supI_S^+(x_i)$], [$infF_S^+(x_i)$, $supF_S^+(x_i)$], [$infT_S^-(x_i)$, $supT_S^-(x_i)$], [$infI_S^-(x_i)$, $supI_S^-(x_i)$], [$infF_S^-(x_i)$, $supF_S^-(x_i)$]> in U can be defined as follows.*

$$
C_{IB}(R,S)_w = \frac{1}{2}\sum_{i=1}^{n} w_i
\left[\begin{array}{l}
\sqrt{\frac{\mathrm{inf}T_R^+(x_i)+\mathrm{inf}T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{inf}T_R^+(x_i)}+\sqrt{\mathrm{inf}T_S^+(x_i)}}{2}\right) + \sqrt{\frac{\mathrm{sup}T_R^+(x_i)+\mathrm{sup}T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{sup}T_R^+(x_i)}+\sqrt{\mathrm{sup}T_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\mathrm{inf}I_R^+(x_i)+\mathrm{inf}I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{inf}I_R^+(x_i)}+\sqrt{\mathrm{inf}I_S^+(x_i)}}{2}\right) + \sqrt{\frac{\mathrm{sup}I_R^+(x_i)+\mathrm{sup}I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{sup}I_R^+(x_i)}+\sqrt{\mathrm{sup}I_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-\mathrm{inf}I_R^+(x_i)\right)+\left(1-\mathrm{inf}I_S^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-\mathrm{inf}I_R^+(x_i)}+\sqrt{1-\mathrm{inf}I_S^+(x_i)}}{2}\right) + \sqrt{\frac{\left(1-\mathrm{sup}I_R^+(x_i)\right)+\left(1-\mathrm{sup}I_S^+(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{1-\mathrm{sup}I_R^+(x_i)}+\sqrt{\left(1-\mathrm{sup}I_S^+(x_i)\right)}}{2}\right) + \sqrt{\frac{\mathrm{inf}F_R^+(x_i)+\mathrm{inf}F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{inf}F_R^+(x_i)}+\sqrt{\mathrm{inf}F_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\mathrm{sup}F_R^+(x_i)+\mathrm{sup}F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\mathrm{sup}F_R^+(x_i)}+\sqrt{\mathrm{sup}F_S^+(x_i)}}{2}\right) + \sqrt{\frac{-\left(\mathrm{inf}T_R^-(x_i)+\mathrm{inf}T_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\mathrm{inf}T_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}T_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\mathrm{sup}T_R^-(x_i)+\mathrm{sup}T_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\mathrm{sup}T_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}T_S^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{-\left(\mathrm{inf}I_R^-(x_i)+\mathrm{inf}I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\mathrm{inf}I_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\mathrm{sup}I_R^-(x_i)+\mathrm{sup}I_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\mathrm{sup}I_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{\left(1+\mathrm{inf}I_R^-(x_i)\right)+\left(1+\mathrm{inf}I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\mathrm{inf}I_R^-(x_i)}+\sqrt{1+\mathrm{inf}I_S^-(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1+\mathrm{sup}I_R^-(x_i)\right)+\left(1+\mathrm{sup}I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\mathrm{sup}I_R^-(x_i)}+\sqrt{[1+\mathrm{sup}I_S^-(x_i)]}}{2}\right) + \sqrt{\frac{-\left(\mathrm{inf}F_R^-(x_i)+\mathrm{inf}F_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\mathrm{inf}F_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{inf}F_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\mathrm{sup}F_R^-(x_i)+\mathrm{sup}F_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\mathrm{sup}F_R^-(x_i)\right)}+\sqrt{\left(-\mathrm{sup}F_S^-(x_i)\right)}}{2}\right)
\end{array}\right] \tag{4}
$$

Theorem 4. *For any two IBNSs R = <[$infT_R^+(x_i)$, sup $T_R^+(x_i)$], [$infI_R^+(x_i)$, $supI_R^+(x_i)$], [$infF_R^+(x_i)$, $supF_R^+(x_i)$], [$infT_R^-(x_i)$, $supT_R^-(x_i)$], [$infI_R^-(x_i)$, $supI_R^-(x_i)$], [$infF_R^-(x_i)$, $supF_R^-(x_i)$]> and S = <[$infT_S^+(x_i)$, $supT_S^+(x_i)$], [$infI_S^+(x_i)$, $supI_S^+(x_i)$], [$infF_S^+(x_i)$, $supF_S^+(x_i)$], [$infT_S^-(x_i)$, $supT_S^-(x_i)$], [$infI_S^-(x_i)$, $supI_S^-(x_i)$], [$infF_S^-(x_i)$, $supF_S^-(x_i)$]> in U, the weighted cross entropy measure $C_{IB}(R, S)_w$ also satisfies the following properties:*

(1) $C_{IB}(R, S)_w \geqslant 0$;

(2) $C_{IB}(R, S)_w = 0$ *if, and only if, R = S i.e.,* inf $T_R^+(x_i) = infT_S^+(x_i)$, $supT_R^+(x_i) = supT_S^+(x_i)$, $infI_R^+(x_i) = infI_S^+(x_i)$, $supI_R^+(x_i) = supI_S^+(x_i)$, $infF_R^+(x_i) = infF_S^+(x_i)$, $supF_R^+(x_i) = supF_S^+(x_i)$, $infT_R^-(x_i) = infT_S^-(x_i)$, $supT_R^-(x_i) = supT_S^-(x_i)$, $infI_R^-(x_i) = infI_S^-(x_i)$, $supI_R^-(x_i) = supI_S^-(x_i)$, $infF_R^-(x_i) = infF_S^-(x_i)$, $supF_R^-(x_i) = supF_S^-(x_i)$ $\forall$ *x* $\in$ *U*;

(3) $C_{IB}(R, S)_w = C_{IB}(S, R)_w$;

(4) $C_{IB}(R^C, S^C)_w = C_{IB}(R, S)_w$.

The proofs are presented in Appendix B.

Example 4. *Consider the two IBNSs R = <[0.5, 0.8], [0.4, 0.6], [0.2, 0.6], [−0.3, −0.1], [−0.5, −0.1], [−05, −0.2]> and S = <[0.5, 0.9], [0.4, 0.5], [0.1, 0.4], [−0.5, −0.3], [−0.7, −0.3], [−0.6, −0.3]>, and let w = 0.3; then the weighted cross entropy between R and S is calculated as follows:*

$$C_{IB}(R,S) = \frac{1}{2} \times 0.3 \times \left[\begin{array}{l} \sqrt{\frac{0.5+0.5}{2}} - \left(\frac{\sqrt{0.5}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{0.8+0.9}{2}} - \left(\frac{\sqrt{0.8}+\sqrt{0.9}}{2}\right) + \sqrt{\frac{0.4+0.4}{2}} - \left(\frac{\sqrt{0.4}+\sqrt{0.4}}{2}\right) + \sqrt{\frac{0.6+0.5}{2}} - \left(\frac{\sqrt{0.6}+\sqrt{0.5}}{2}\right) + \\ \sqrt{\frac{[1-0.4]+[1-0.4]}{2}} - \left(\frac{\sqrt{1-0.4}+\sqrt{[1-0.4]}}{2}\right) + \sqrt{\frac{[1-0.6]+[1-0.5]}{2}} - \left(\frac{\sqrt{1-0.6}+\sqrt{[1-0.5]}}{2}\right) + \sqrt{\frac{0.2+0.1}{2}} - \left(\frac{\sqrt{0.2}+\sqrt{0.1}}{2}\right) + \\ \sqrt{\frac{0.6+0.4}{2}} - \left(\frac{\sqrt{0.6}+\sqrt{0.4}}{2}\right) + \sqrt{\frac{-(-0.3-0.5)}{2}} - \left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.5)}}{2}\right) + \sqrt{\frac{-(-0.1-0.3)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right) + \\ \sqrt{\frac{-(-0.5-0.7)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.7)}}{2}\right) + \sqrt{\frac{-(-0.1-0.3)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right) + \sqrt{\frac{[1-0.5]+[1-0.7]}{2}} - \\ \left(\frac{\sqrt{1-0.5}+\sqrt{[1-0.7]}}{2}\right) + \sqrt{\frac{[1-0.1]+[1-0.3]}{2}} - \left(\frac{\sqrt{1-0.1}+\sqrt{[1-0.3]}}{2}\right) + \sqrt{\frac{-(-0.5-0.6)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.6)}}{2}\right) + \\ \sqrt{\frac{-(-0.2-0.3)}{2}} - \left(\frac{\sqrt{-(-0.2)}+\sqrt{-(-0.3)}}{2}\right) \end{array} \right] = 0.00895385.$$

5. MADM Strategies Based on Cross Entropy Measures

In this section, we propose two new MADM strategies based on weighted cross entropy measures in bipolar neutrosophic and interval bipolar neutrosophic environments. Let $B = \{B_1, B_2, \ldots, B_m\}$ $(m \geqslant 2)$ be a discrete set of m feasible alternatives which are to be evaluated based on n attributes $C = \{C_1, C_2, \ldots, C_n\}$ $(n \geqslant 2)$ and let w_j be the weight vector of the attributes such that $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$.

5.1. MADM Strategy Based on Weighted Cross Entropy Measures of BNS

The procedure for solving MADM problems in a bipolar neutrosophic environment is presented in the following steps:

Step 1. The rating of the performance value of alternative B_i $(i = 1, 2, \ldots, m)$ with respect to the predefined attribute C_j $(j = 1, 2, \ldots, n)$ can be expressed in terms of bipolar neutrosophic information as follows:

$$B_i = \{C_j, < T^+_{B_i}(C_j), I^+_{B_i}(C_j), F^+_{B_i}(C_j), T^-_{B_i}(C_j), I^-_{B_i}(C_j), F^-_{B_i}(C_j) > |C_j \in C_j, j = 1, 2, \ldots, n\},$$

where $0 \leqslant T^+_{B_i}(C_j) + I^+_{B_i}(C_j) + F^+_{B_i}(C_j) \leqslant 3$ and $-3 \leqslant T^-_{B_i}(C_j) + I^-_{B_i}(C_j) + F^-_{B_i}(C_j) \leqslant 0$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Assume that $\widetilde{d}_{ij} = <T^+_{ij}, I^+_{ij}, F^+_{ij}, T^-_{ij}, I^-_{ij}, F^-_{ij}>$ is the bipolar neutrosophic decision matrix whose entries are the rating values of the alternatives with respect to the attributes provided by the expert or decision-maker. The bipolar neutrosophic decision matrix $[\widetilde{d}_{ij}]_{m\times n}$ can be expressed as follows:

$$[\widetilde{d}_{ij}]_{m\times n} = \begin{array}{c} \\ B_1 \\ B_2 \\ . \\ . \\ B_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{pmatrix} \end{array}.$$

Step 2. The positive ideal solution (PIS) $<p^* = (d^*_1, d^*_2, \ldots, d^*_n)>$ of the bipolar neutrosophic information is obtained as follows:

$$p_j^* = \left\langle T_j^{*+}, I_j^{*+}, F_j^{*+}, T_j^{*-}, I_j^{*-}, F_j^{*-} \right\rangle = < [\{\underset{i}{Max}\,(T_{ij}^{+}) | j \in H_1\};\ \{\underset{i}{Min}\,(T_{ij}^{+}) | j \in H_2\}],$$
$$[\{\underset{i}{Min}\,(I_{ij}^{+}) | j \in H_1\};\ \{\underset{i}{Max}\,(I_{ij}^{+}) | j \in H_2\}],\ [\{\underset{i}{Min}\,(F_{ij}^{+}) | j \in H_1\};\ \{\underset{i}{Max}\,(F_{ij}^{+}) | j \in H_2\}],$$
$$[\{\underset{i}{Min}\,(T_{ij}^{-}) | j \in H_1\};\ \{\underset{i}{Max}\,(T_{ij}^{-}) | j \in H_2\}],\ [\{\underset{i}{Max}\,(I_{ij}^{-}) | j \in H_1\};\ \{\underset{i}{Min}\,(I_{ij}^{-}) | j \in H_2\}],$$
$$[\{\underset{i}{Max}\,(F_{ij}^{-}) | j \in H_1\};\ \{\underset{i}{Min}\,(F_{ij}^{-}) | j \in H_2\}] >,\ j = 1, 2, \ldots, n;$$

where H_1 and H_2 represent benefit and cost type attributes, respectively.

Step 3. The weighted cross entropy between an alternative B_i, i = 1, 2, ... , m, and the ideal alternative p^* is determined by

$$C_B(B_i, p*)_w = \sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_{ij}^{+}+T_j^{*+}}{2}} - \left(\frac{\sqrt{T_{ij}^{+}}+\sqrt{T_j^{*+}}}{2}\right) + \sqrt{\frac{I_{ij}^{+}+I_j^{*+}}{2}} - \left(\frac{\sqrt{I_{ij}^{+}}+\sqrt{I_j^{*+}}}{2}\right) + \sqrt{\frac{[1-I_{ij}^{+}]+[1-I_j^{*+}]}{2}} - \\ \left(\frac{\sqrt{1-I_j^{+}}+\sqrt{[1-I_j^{*+}]}}{2}\right) + \sqrt{\frac{F_{ij}^{+}+F_j^{*+}}{2}} - \left(\frac{\sqrt{F_{ij}^{+}}+\sqrt{F_j^{*+}}}{2}\right) + \sqrt{\frac{-\left(T_{ij}^{-}+T_j^{*-}\right)}{2}} - \\ \left(\frac{\sqrt{\left(-T_{ij}^{-}\right)}+\sqrt{\left(-T_j^{*-}\right)}}{2}\right) + \sqrt{\frac{-\left(I_{ij}^{-}+I_j^{*-}\right)}{2}} - \left(\frac{\sqrt{\left(-I_{ij}^{-}\right)}+\sqrt{\left(-I_j^{*-}\right)}}{2}\right) + \sqrt{\frac{[1+I_{ij}^{-}]+[1+I_j^{*-}]}{2}} \\ -\left(\frac{\sqrt{1+I_{ij}^{-}}+\sqrt{[1+I_j^{*-}]}}{2}\right) + \sqrt{\frac{-\left(F_{ij}^{-}+F_j^{*-}\right)}{2}} - \left(\frac{\sqrt{\left(-F_{ij}^{-}\right)}+\sqrt{\left(-F_j^{*-}\right)}}{2}\right) \end{array} \right]. \tag{5}$$

Step 4. A smaller value of $C_B(B_i, p^*)_w$, i = 1, 2, ..., m represents that an alternative B_i, i = 1, 2, ... , m is closer to the PIS p^*. Therefore, the alternative with the smallest weighted cross entropy measure is the best alternative.

5.2. MADM Strategy Based on Weighted Cross Entropy Measures of IBNSs

The steps for solving MADM problems with interval bipolar neutrosophic information are presented as follows.

Step 1. In an interval bipolar neutrosophic environment, the rating of the performance value of alternative B_i (i = 1, 2, ... , m) with respect to the predefined attribute C_j (j = 1, 2, ... , n) can be represented as follows:

$$B_i = \{C_j, < [\inf T_{B_i}^{+}(C_j),\ \sup T_{B_i}^{+}(C_j)],\ [\inf I_{B_i}^{+}(C_j),\ \sup I_{B_i}^{+}(C_j)],\ [\inf F_{B_i}^{+}(C_j),\ \sup F_{B_i}^{+}(C_j)],$$
$$[\inf T_{B_i}^{-}(C_j),\ \sup T_{B_i}^{-}(C_j)],\ [\inf I_{B_i}^{-}(C_j),\ \sup I_{B_i}^{-}(C_j)],\ [\inf F_{B_i}^{-}(C_j),\ \sup F_{B_i}^{-}(C_j)] > | C_j \in C_j,$$
$$j = 1, 2, \ldots, n\}$$

where $0 \leqslant \sup T_{B_i}^{+}(C_j) + \sup I_{B_i}^{+}(C_j) + \sup F_{B_i}^{+}(C_j) \leqslant 3$ and $-3 \leqslant \sup T_{B_i}^{-}(C_j) + \sup I_{B_i}^{-}(C_j) + \sup F_{B_i}^{-}(C_j) \leqslant 0$; j = 1, 2, ... , n. Let $\tilde{g}_{ij}$ = <$[{}^{L}T_{ij}^{+}, {}^{U}T_{ij}^{+}]$, $[{}^{L}I_{ij}^{+}, {}^{U}I_{ij}^{+}]$, $[{}^{L}F_{ij}^{+}, {}^{U}F_{ij}^{+}]$, $[{}^{L}T_{ij}^{-}, {}^{U}T_{ij}^{-}]$, $[{}^{L}I_{ij}^{-}, {}^{U}I_{ij}^{-}]$, $[{}^{L}F_{ij}^{-}, {}^{U}F_{ij}^{-}]$> be the bipolar neutrosophic decision matrix whose entries are the rating values of the alternatives with respect to the attributes provided by the expert or decision-maker. The interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{m \times n}$ can be presented as follows:

$$[\tilde{g}_{ij}]_{m \times n} = \begin{array}{c} \\ B_1 \\ B_2 \\ . \\ . \\ B_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{pmatrix} \end{array}.$$

Step 2. The PIS $<q^* = (g_1^*, g_2^*, \ldots, g_n^*)>$ of the interval bipolar neutrosophic information is obtained as follows:

$$
\begin{aligned}
q_j^* &= <[{}^{L}T_{ij}^{*+}, {}^{U}T_{ij}^{*+}], [{}^{L}I_{ij}^{*+}, {}^{U}I_{ij}^{*+}], [{}^{L}F_{ij}^{*+}, {}^{U}F_{ij}^{*+}], [{}^{L}T_{ij}^{*-}, {}^{U}T_{ij}^{*-}], [{}^{L}I_{ij}^{*-}, {}^{U}I_{ij}^{*-}], [{}^{L}F_{ij}^{*-}, {}^{U}F_{ij}^{*-}] >, \\
&= < [\{\underset{i}{Max}\,({}^{L}T_{ij}^{+})|j \in H_1\}; \{\underset{i}{Min}\,({}^{L}T_{ij}^{+})|j \in H_2\}, \{\underset{i}{Max}\,({}^{U}T_{ij}^{+})|j \in H_1\}; \{\underset{i}{Min}\,({}^{U}T_{ij}^{+})|j \in H_2\}], \\
&[\{\underset{i}{Min}\,({}^{L}I_{ij}^{+})|j \in H_1\}; \{\underset{i}{Max}\,({}^{L}I_{ij}^{+})|j \in H_2\}, \{\underset{i}{Min}\,({}^{U}I_{ij}^{+})|j \in H_1\}; \{\underset{i}{Max}\,({}^{U}I_{ij}^{+})|j \in H_2\}], \\
&[\{\underset{i}{Min}\,({}^{L}F_{ij}^{+})|j \in H_1\}; \{\underset{i}{Max}\,({}^{L}F_{ij}^{+})|j \in H_2\}, \{\underset{i}{Min}\,({}^{U}F_{ij}^{+})|j \in H_1\}; \{\underset{i}{Max}\,({}^{U}F_{ij}^{+})|j \in H_2\}], \\
&[\{\underset{i}{Min}\,({}^{L}T_{ij}^{-})|j \in H_1\}; \{\underset{i}{Max}\,({}^{L}T_{ij}^{-})|j \in H_2\}, \{\underset{i}{Min}\,({}^{U}T_{ij}^{-})|j \in H_1\}; \{\underset{i}{Max}\,({}^{U}T_{ij}^{-})|j \in H_2\}], \\
&[\{\underset{i}{Max}\,({}^{L}I_{ij}^{-})|j \in H_1\}; \{\underset{i}{Min}\,({}^{L}I_{ij}^{-})|j \in H_2\}, \{\underset{i}{Max}\,({}^{U}I_{ij}^{-})|j \in H_1\}; \{\underset{i}{Min}\,({}^{U}I_{ij}^{-})|j \in H_2\}], \\
&[\{\underset{i}{Max}\,({}^{L}F_{ij}^{-})|j \in H_1\}; \{\underset{i}{Min}\,({}^{L}F_{ij}^{-})|j \in H_2\}, \{\underset{i}{Max}\,({}^{U}F_{ij}^{-})|j \in H_1\}; \{\underset{i}{Min}\,({}^{U}F_{ij}^{-})|j \in H_2\}] >, \\
&j = 1, 2, \ldots, n
\end{aligned}
$$

where H_1 and H_2 stand for benefit and cost type attributes, respectively.

Step 3. The weighted cross entropy between an alternative B_i, $i = 1, 2, \ldots, m$, and the ideal alternative q^* under an interval bipolar neutrosophic setting is computed as follows:

$$
C_{IB}(B_i, q^*)_w = \frac{1}{2} \times \sum_{i=1}^{n} w_i \left[\begin{array}{l}
\sqrt{\frac{{}^{L}T_{ij}^{+} + {}^{L}T_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{L}T_{ij}^{+}} + \sqrt{{}^{L}T_{j}^{*+}}}{2} \right) + \sqrt{\frac{{}^{U}T_{ij}^{+} + {}^{U}T_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{U}T_{ij}^{+}} + \sqrt{{}^{U}T_{j}^{*+}}}{2} \right) + \sqrt{\frac{{}^{L}I_{ij}^{+} + {}^{L}I_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{L}I_{ij}^{+}} + \sqrt{{}^{L}I_{j}^{*+}}}{2} \right) + \\
\sqrt{\frac{[1 - {}^{L}I_{ij}^{+}] + [1 - {}^{L}I_{j}^{*+}]}{2}} - \left(\frac{\sqrt{1 - {}^{L}I_{j}^{+}} + \sqrt{[1 - {}^{L}I_{j}^{*+}]}}{2} \right) + \sqrt{\frac{{}^{U}I_{ij}^{+} + {}^{U}I_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{U}I_{ij}^{+}} + \sqrt{{}^{U}I_{j}^{*+}}}{2} \right) + \sqrt{\frac{[1 - {}^{U}I_{ij}^{+}] + [1 - {}^{U}I_{j}^{*+}]}{2}} - \\
\left(\frac{\sqrt{1 - {}^{U}I_{j}^{+}} + \sqrt{[1 - {}^{U}I_{j}^{*+}]}}{2} \right) + \sqrt{\frac{{}^{L}F_{ij}^{+} + {}^{L}F_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{L}F_{ij}^{+}} + \sqrt{{}^{L}F_{j}^{*+}}}{2} \right) + \sqrt{\frac{{}^{U}F_{ij}^{+} + {}^{U}F_{j}^{*+}}{2}} - \left(\frac{\sqrt{{}^{U}F_{ij}^{+}} + \sqrt{{}^{U}F_{j}^{*+}}}{2} \right) + \sqrt{\frac{-({}^{L}T_{ij}^{-} + {}^{L}T_{j}^{*-})}{2}} - \\
\left(\frac{\sqrt{(-{}^{L}T_{ij}^{-})} + \sqrt{(-{}^{L}T_{j}^{*-})}}{2} \right) + \sqrt{\frac{-({}^{U}T_{ij}^{-} + {}^{U}T_{j}^{*-})}{2}} - \left(\frac{\sqrt{(-{}^{U}T_{ij}^{-})} + \sqrt{(-{}^{U}T_{j}^{*-})}}{2} \right) + \sqrt{\frac{-({}^{L}I_{ij}^{-} + {}^{L}I_{j}^{*-})}{2}} - \left(\frac{\sqrt{(-{}^{L}I_{ij}^{-})} + \sqrt{(-{}^{L}I_{j}^{*-})}}{2} \right) + \\
\sqrt{\frac{-({}^{U}I_{ij}^{-} + {}^{U}I_{j}^{*-})}{2}} - \left(\frac{\sqrt{(-{}^{U}I_{ij}^{-})} + \sqrt{(-{}^{U}I_{j}^{*-})}}{2} \right) + \sqrt{\frac{[1 + {}^{L}I_{ij}^{-}] + [1 + {}^{L}I_{j}^{*-}]}{2}} - \left(\frac{\sqrt{1 + {}^{L}I_{ij}^{-}} + \sqrt{[1 + {}^{L}I_{j}^{*-}]}}{2} \right) + \sqrt{\frac{[1 + {}^{U}I_{ij}^{-}] + [1 + {}^{U}I_{j}^{*-}]}{2}} - \\
\left(\frac{\sqrt{1 + {}^{U}I_{ij}^{-}} + \sqrt{[1 + {}^{U}I_{j}^{*-}]}}{2} \right) + \sqrt{\frac{-({}^{L}F_{ij}^{-} + {}^{L}F_{j}^{*-})}{2}} - \left(\frac{\sqrt{(-{}^{L}F_{ij}^{-})} + \sqrt{(-{}^{L}F_{j}^{*-})}}{2} \right) + \sqrt{\frac{-({}^{U}F_{ij}^{-} + {}^{U}F_{j}^{*-})}{2}} - \left(\frac{\sqrt{(-{}^{U}F_{ij}^{-})} + \sqrt{(-{}^{U}F_{j}^{*-})}}{2} \right)
\end{array} \right]. \quad (6)
$$

Step 4. A smaller value of $C_{IB}(B_i, p^*)_w$, $i = 1, 2, \ldots, m$ indicates that an alternative B_i, $i = 1, 2, \ldots, m$ is closer to the PIS q^*. Hence, the alternative with the smallest weighted cross entropy measure will be identified as the best alternative.

A conceptual model of the proposed strategy is shown in Figure 1.

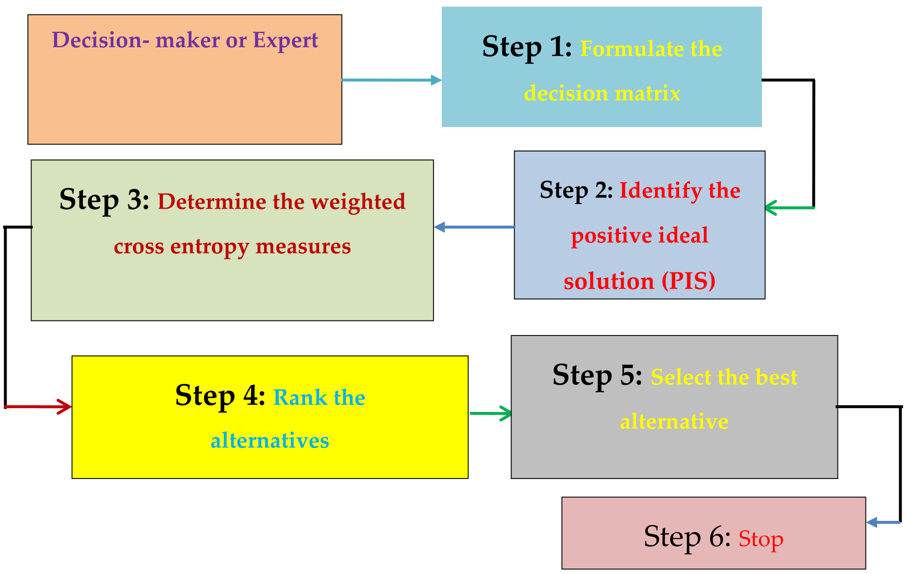

Figure 1. Conceptual model of the proposed strategy.

6. Illustrative Example

In this section we solve two numerical MADM problems and a comparison with other existing strategies is presented to verify the applicability and effectiveness of the proposed strategies in bipolar neutrosophic and interval bipolar neutrosophic environments.

6.1. Car Selection Problem with Bipolar Neutrosophic Information

Consider the problem discussed in [81,86–88] where a buyer wants to purchase a car based on some predefined attributes. Suppose that four types of cars (alternatives) B_i, (i = 1, 2, 3, 4) are available in the market. Four attributes are taken into consideration in the decision-making environment, namely, Fuel economy (C_1), Aerod (C_2), Comfort (C_3), Safety (C_4), to select the most desirable car. Assume that the weight vector for the four attributes is known and given by $w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125)$. Therefore, the bipolar neutrosophic decision matrix $\langle d_{ij} \rangle_{4\times4}$ can be obtained as given below.

The bipolar neutrosophic decision matrix $[\tilde{d}_{ij}]_{4\times4}$ =

	C_1	C_2	C_3	C_4
B_1	<0.5, 0.7, 0.2, −0.7, −0.3, −0.6>	<0.4, 0.4, 0.5, −0.7, −0.8, −0.4>	<0.7, 0.7, 0.5, −0.8, −0.7, −0.6>	<0.1, 0.5, 0.7, −0.5, −0.2, −0.8>
B_2	<0.9, 0.7, 0.5, −0.7, −0.7, −0.1>	<0.7, 0.6, 0.8, −0.7, −0.5, −0.1>	<0.9, 0.4, 0.6, −0.1, −0.7, −0.5>	<0.5, 0.2, 0.7, −0.5, −0.1, −0.9>
B_3	<0.3, 0.4, 0.2, −0.6, −0.3, −0.7>	<0.2, 0.2, 0.2, −0.4, −0.7, −0.4>	<0.9, 0.5, 0.5, −0.6, −0.5, −0.2>	<0.7, 0.5, 0.3, −0.4, −0.2, −0.2>
B_4	<0.9, 0.7, 0.2, −0.8, −0.6, −0.1>	<0.3, 0.5, 0.2, −0.5, −0.5, −0.2>	<0.5, 0.4, 0.5, −0.1, −0.7, −0.2>	<0.2, 0.4, 0.8, −0.5, −0.5, −0.6>

The positive ideal bipolar neutrosophic solutions are computed from $[\tilde{d}_{ij}]_{4\times4}$ as follows:

$$p^* = [<0.9, 0.4, 0.2, -0.8, -0.3, -0.1>, <0.7, 0.2, 0.2, -0.7, -0.5, -0.1>,$$
$$<0.9, 0.4, 0.5, -0.8, -0.5, -0.2>, <0.7, 0.2, 0.3, -0.5, -0.1, -0.2>].$$

Using Equation (5), the weighted cross entropy measure $C_B(B_i, p^*)_w$ is obtained as follows:

$$C_B(B_1, p^*)_w = 0.0734,\ C_B(B_2, p^*)_w = 0.0688,\ C_B(B_3, p^*)_w = 0.0642,\ C_B(B_4, p^*)_w = 0.0516. \quad (7)$$

According to the weighted cross entropy measure $C_B(B_i, p^*)_w$, the order of the four alternatives is $B_4 \prec B_3 \prec B_2 \prec B_1$; therefore, B_4 is the best car.

We compare our obtained result with the results of other existing strategies (see Table 1), where the known weight of the attributes is given by $w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125)$. It is to be noted that the ranking results obtained from the other existing strategies are different from the result of the proposed strategies in some cases. The reason is that the different authors adopted different decision-making strategies and thereby obtained different ranking results. However, the proposed strategies are simple and straightforward and can effectively solve decision-making problems with bipolar neutrosophic information.

Table 1. The results of the car selection problem obtained from different methods.

Methods	Ranking Results	Best Option
The proposed weighted cross entropy measure	$B_4 \prec B_3 \prec B_2 \prec B_1$	B_4
Dey et al.'s TOPSIS strategy [87]	$B_1 \prec B_3 \prec B_2 \prec B_4$	B_4
Deli et al.'s strategy [81]	$B_1 \prec B_2 \prec B_4 \prec B_3$	B_3
Projection measure [88]	$B_3 \prec B_4 \prec B_1 \prec B_2$	B_2
Bidirectional projection measure [88]	$B_2 \prec B_1 \prec B_4 \prec B_3$	B_3
Hybrid projection measure [88] with ρ = 0.25	$B_2 \prec B_1 \prec B_3 \prec B_4$	B_4
Hybrid projection measure [88] with ρ = 0.50	$B_3 \prec B_2 \prec B_1 \prec B_4$	B_4
Hybrid projection measure [88] with ρ = 0.75	$B_1 \prec B_3 \prec B_4 \prec B_2$	B_2
Hybrid projection measure [88] with ρ = 0.90	$B_3 \prec B_4 \prec B_2 \prec B_1$	B_1
Hybrid similarity measure [88] with ρ = 0.25	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3
Hybrid similarity measure [88] with ρ = 0.30	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3
Hybrid similarity measure [88] with ρ = 0.60	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3
Hybrid similarity measure [88] with ρ = 0.90	$B_2 \prec B_4 \prec B_3 \prec B_1$	B_1

6.2. Interval Bipolar Neutrosophic MADM Investment Problem

Consider an interval bipolar neutrosophic MADM problem studied in [91] with four possible alternatives with the aim to invest a sum of money in the best choice. The four alternatives are:

> a food company (B_1),
> a car company (B_2),
> an arms company (B_3), and
> a computer company (B_4).

The investment company selects the best option based on three predefined attributes, namely, growth analysis (C_1), risk analysis (C_2), and environment analysis (C_3). We consider C_1 and C_2 to be benefit type attributes and C_3 to be a cost type attribute based on Ye [93]. Assume that the weight vector [91] of C_1, C_2, and C_3 is given by $w = (w_1, w_2, w_3) = (0.35, 0.25, 0.4)$. The interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{4\times 3}$ presented by the decision-maker or expert is as follows.

Interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{4\times 3}$ =

$$C_1$$

$$\begin{pmatrix} B_1 & [[0.4, 0.5], [0.2, 0.3], [0.3, 0.4], [-0.3, -0.2], [-0.4, -0.3], [-0.5, -0.4]] \\ B_2 & [[0.6, 0.7], [0.1, 0.2], [0.2, 0.3], [-0.2, -0.1], [-0.3, -0.2], [-0.7, -0.6]] \\ B_3 & [[0.3, 0.6], [0.2, 0.3], [0.3, 0.4], [-0.3, -0.2], [-0.4, -0.3], [-0.6, -0.3]] \\ B_4 & [[0.7, 0.8], [0.0, 0.1], [0.1, 0.2], [-0.1, -0.0], [-0.2, -0.1], [-0.8, -0.7]] \end{pmatrix}$$

$$C_2$$

$$\begin{pmatrix} B_1 & [[0.4, 0.6], [0.1, 0.3], [0.2, 0.4], [-0.3, -0.1], [-0.4, -0.2], [-0.6, -0.4]] \\ B_2 & [[0.6, 0.7], [0.1, 0.2], [0.2, 0.3], [-0.2, -0.1], [-0.3, -0.2], [-0.7, -0.6]] \\ B_3 & [[0.5, 0.6], [0.2, 0.3], [0.3, 0.4], [-0.3, -0.2], [-0.4, -0.3], [-0.6, -0.5]] \\ B_4 & [[0.6, 0.7], [0.1, 0.2], [0.1, 0.3], [-0.2 - 0.1], [-0.3, -0.1], [-0.7, -0.6]] \end{pmatrix}$$

$$C_3$$

$$\begin{pmatrix} B_1 & [[0.7, 0.9], [0.2, 0.3], [0.4, 0.5], [-0.3, -0.2], [-0.5, -0.4], [-0.9, -0.7]] \\ B_2 & [[0.3, 0.6], [0.3, 0.5], [0.8, 0.9], [-0.5, -0.3], [-0.9, -0.8], [-0.6, -0.3]] \\ B_3 & [[0.4, 0.5], [0.2, 0.4], [0.7, 0.9], [-0.4, -0.2], [-0.9, -0.7], [-0.5, -0.4]] \\ B_4 & [[0.6, 0.7], [0.3, 0.4], [0.8, 0.9], [-0.4, -0.3], [-0.9, -0.8], [-0.7, -0.6]] \end{pmatrix}$$

From the matrix $[\tilde{g}_{ij}]_{4\times 3}$, we determine the positive ideal interval bipolar neutrosophic solution (q^*) by using Equation (6) as follows:

$$\begin{aligned} q^* = &<[0.7, 0.8], [0.0, 0.1], [0.1, 0.2], [-0.3, -0.2], [-0.2, -0.1], [-0.5, -0.3]>; \\ &<[0.6, 0.7], [0.1, 0.2], [0.1, 0.3], [-0.3, -0.2], [-0.3, -0.1], [-0.6, -0.4]>; \\ &<[0.3, 0.5], [0.3, 0.5], [0.8, 0.9], [-0.3, -0.2], [-0.9, -0.8], [-0.9, -0.7]>. \end{aligned}$$

The weighted cross entropy between an alternative B_i, $i = 1, 2, \ldots, m$, and the ideal alternative q^* can be obtained as given below:

$$C_{IB}(B_1, q^*)_w = 0.0606, C_{IB}(B_2, q^*)_w = 0.0286, C_{IB}(B_3, q^*)_w = 0.0426, C_{IB}(B_4, q^*)_w = 0.0423.$$

On the basis of the weighted cross entropy measure $C_{IB}(B_i, q^*)_w$, the order of the four alternatives is $B_2 \prec B_4 \prec B_3 \prec B_1$; therefore, B_2 is the best choice.

Next, the comparison of the results obtained from different methods is presented in Table 2 where the weight vector of the attribute is given by $w = (w_1, w_2, w_3) = (0.35, 0.25, 0.4)$. We observe that B_2 is the best option obtained using the proposed method and B_4 is the best option obtained using the method of Mahmood et al. [91]. The reason for this may be that we use the interval bipolar neutrosophic cross entropy method whereas Mahmood et al. [91] derived the most desirable alternative based on a weighted arithmetic average operator in an interval bipolar neutrosophic setting.

Table 2. The results of the investment problem obtained from different methods.

Methods	Ranking Results	Best Option
The proposed weighted cross entropy measure	$B_2 \prec B_4 \prec B_3 \prec B_1$	B_2
Mahmood et al.'s strategy [91]	$B_2 \prec B_3 \prec B_1 \prec B_4$	B_4

7. Conclusions

In this paper we defined cross entropy and weighted cross entropy measures for bipolar neutrosophic sets and proved their basic properties. We also extended the proposed concept to the interval bipolar neutrosophic environment and proved its basic properties. The proposed cross entropy measures were then employed to develop two new multi-attribute decision-making strategies. Two illustrative numerical examples were solved and comparisons with existing strategies were provided to demonstrate the feasibility, applicability, and efficiency of the proposed strategies. We hope that the proposed cross entropy measures can be effective in dealing with group decision-making, data analysis, medical diagnosis, selection of a suitable company to build power plants [94], teacher selection [95], quality brick selection [96], weaver selection [97,98], etc. In future, the cross entropy measures can be extended to other neutrosophic hybrid environments, such as bipolar neutrosophic soft expert sets, bipolar neutrosophic refined sets, etc.

Author Contributions: Surapati Pramanik and Partha Pratim Dey conceived and designed the experiments; Surapati Pramanik performed the experiments; Jun Ye and Florentin Smarandache analyzed the data; Surapati Pramanik and Partha Pratim Dey wrote the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof of Theorem 2

(1) From the inequality stated in Theorem 1, we can easily obtain $C_B(M, N)_w \geqslant 0$.

(2) $C_B(M, N)_w = 0$ if, and only if, $M = N$, i.e., $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)$ $\forall\, x \in U$.

(3)
$$C_B(M,N)_w = \sum_{i=1}^{n} w_{i\ i} \left[\begin{array}{l} \sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2} \right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2} \right) + \\ \sqrt{\frac{\left(1-I_M^+(x_i)\right)+\left(1-I_N^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-I_M^+(x_i)}+\sqrt{1-I_N^+(x_i)}}{2} \right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2} \right) + \\ \sqrt{\frac{-\left(T_M^-(x_i)+T_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_M^-(x_i)\right)}+\sqrt{\left(-T_N^-(x_i)\right)}}{2} \right) + \sqrt{\frac{-\left(I_M^-(x_i)+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_M^-(x_i)\right)}+\sqrt{\left(-I_N^-(x_i)\right)}}{2} \right) + \\ \sqrt{\frac{\left(1+I_M^-(x_i)\right)+\left(1+I_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2} \right) + \sqrt{\frac{-\left(F_M^-(x_i)+F_N^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_M^-(x_i)\right)}+\sqrt{\left(-F_N^-(x_i)\right)}}{2} \right) \end{array} \right]$$

$$= \sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_N^+(x_i)+T_M^+(x_i)}{2}} - \left(\frac{\sqrt{T_N^+(x_i)}+\sqrt{T_M^+(x_i)}}{2} \right) + \sqrt{\frac{I_N^+(x_i)+I_M^+(x_i)}{2}} - \left(\frac{\sqrt{I_N^+(x_i)}+\sqrt{I_M^+(x_i)}}{2} \right) + \\ \sqrt{\frac{\left(1-I_N^+(x_i)\right)+\left(1-I_M^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-I_N^+(x_i)}+\sqrt{1-I_M^+(x_i)}}{2} \right) + \sqrt{\frac{F_N^+(x_i)+F_M^+(x_i)}{2}} - \left(\frac{\sqrt{F_N^+(x_i)}+\sqrt{F_M^+(x_i)}}{2} \right) + \\ \sqrt{\frac{-\left(T_N^-(x_i)+T_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-T_N^-(x_i)\right)}+\sqrt{\left(-T_M^-(x_i)\right)}}{2} \right) + \sqrt{\frac{-\left(I_N^-(x_i)+I_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-I_N^-(x_i)\right)}+\sqrt{\left(-I_M^-(x_i)\right)}}{2} \right) + \\ \sqrt{\frac{\left(1+I_N^-(x_i)\right)+\left(1+I_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+I_N^-(x_i)}+\sqrt{1+I_M^-(x_i)}}{2} \right) + \sqrt{\frac{-\left(F_N^-(x_i)+F_M^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-F_N^-(x_i)\right)}+\sqrt{\left(-F_M^-(x_i)\right)}}{2} \right) \end{array} \right]$$

$= C_B(N, M)_w$.

(4) $C_B(M^C, N^C)_w =$

$$\sum_{i=1}^{n}\left[\begin{array}{l} \sqrt{\frac{F_M^{+}(x_i)+F_N^{+}(x_i)}{2}}-\left(\frac{\sqrt{F_M^{+}(x_i)}+\sqrt{F_N^{+}(x_i)}}{2}\right)+\sqrt{\frac{\left(1-I_M^{+}(x_i)\right)+\left(1-I_N^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(1-I_M^{+}(x_i)\right)}+\sqrt{1-I_N^{+}(x_i))}}{2}\right)+ \\ \sqrt{\frac{1-\left(1-I_M^{+}(x_i)\right)+1-\left(1-I_N^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\left(1-I_M^{+}(x_i)\right)}+\sqrt{1-\left(1-I_N^{+}(x_i)\right)}}{2}\right)+\sqrt{\frac{T_M^{+}(x_i)+T_N^{+}(x_i)}{2}}-\left(\frac{\sqrt{T_M^{+}(x_i)}+\sqrt{T_N^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{-\left(F_M^{-}(x_i)+F_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-F_M^{-}(x_i)\right)}+\sqrt{\left(-F_N^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(-1-I_M^{-}(x_i)\right)-\left(-1-I_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{-\left(-1-I_M^{-}(x_i)\right)}+\sqrt{-\left(-1-I_N^{-}(x_i)\right)}}{2}\right)+ \\ \sqrt{\frac{1+\left(-1-I_M^{-}(x_i)\right)+1+\left(-1-I_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\left(-1-I_M^{-}(x_i)\right)}+\sqrt{1+\left(-1-I_N^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(T_M^{-}(x_i)+T_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-T_M^{-}(x_i)\right)}+\sqrt{\left(-T_N^{-}(x_i)\right)}}{2}\right) \end{array}\right]$$

$$= \sum_{i=1}^{n} w_i\ i\left[\begin{array}{l} \sqrt{\frac{T_M^{+}(x_i)+T_N^{+}(x_i)}{2}}-\left(\frac{\sqrt{T_M^{+}(x_i)}+\sqrt{T_N^{+}(x_i)}}{2}\right)+\sqrt{\frac{I_M^{+}(x_i)+I_N^{+}(x_i)}{2}}-\left(\frac{\sqrt{I_M^{+}(x_i)}+\sqrt{I_N^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{\left(1-I_M^{+}(x_i)\right)+\left(1-I_N^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{1-I_M^{+}(x_i)}+\sqrt{1-I_N^{+}(x_i)}}{2}\right)+\sqrt{\frac{F_M^{+}(x_i)+F_N^{+}(x_i)}{2}}-\left(\frac{\sqrt{F_M^{+}(x_i)}+\sqrt{F_N^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{-\left(T_M^{-}(x_i)+T_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-T_M^{-}(x_i)\right)}+\sqrt{\left(-T_N^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(I_M^{-}(x_i)+I_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-I_M^{-}(x_i)\right)}+\sqrt{\left(-I_N^{-}(x_i)\right)}}{2}\right)+ \\ \sqrt{\frac{\left(1+I_M^{-}(x_i)\right)+\left(1+I_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+I_M^{-}(x_i)}+\sqrt{1+I_N^{-}(x_i)}}{2}\right)+\sqrt{\frac{-\left(F_M^{-}(x_i)+F_N^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-F_M^{-}(x_i)\right)}+\sqrt{\left(-F_N^{-}(x_i)\right)}}{2}\right) \end{array}\right]$$

$= C_B(M^C, N^C)_w$. □

Appendix B

Proof of Theorem 4

(1) Obviously, we can easily get $C_{IB}(R, S)_w \geqslant 0$.

(2) If $C_{IB}(R, S)_w = 0$ then $R = S$, and if $\inf T_R^{+}(x_i) = \inf T_S^{+}(x_i)$, $\sup T_R^{+}(x_i) = \sup T_S^{+}(x_i)$, $\inf I_R^{+}(x_i) = \inf I_S^{+}(x_i)$, $\sup I_R^{+}(x_i) = \sup I_S^{+}(x_i)$, $\inf F_R^{+}(x_i) = \inf F_S^{+}(x_i)$, $\sup F_R^{+}(x_i) = \sup F_S^{+}(x_i)$, $\inf T_R^{-}(x_i) = \inf T_S^{-}(x_i)$, $\sup T_R^{-}(x_i) = \sup T_S^{-}(x_i)$, $\inf I_R^{-}(x_i) = \inf I_S^{-}(x_i)$, $\sup I_R^{-}(x_i) = \sup I_S^{-}(x_i)$, $\inf F_R^{-}(x_i) = \inf F_S^{-}(x_i)$, $\sup F_R^{-}(x_i) = \sup F_S^{-}(x_i)$ $\forall\, x \in U$, then we obtain $C_{IB}(R, S)_w = 0$.

(3) $$C_{IB}(R, S)_w = \frac{1}{2}\sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{\inf T_R^{+}(x_i)+\inf T_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^{+}(x_i)}+\sqrt{\inf T_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^{+}(x_i)+\sup T_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^{+}(x_i)}+\sqrt{\sup T_S^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{\inf I_R^{+}(x_i)+\inf I_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf I_R^{+}(x_i)}+\sqrt{\inf I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\sup I_R^{+}(x_i)+\sup I_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup I_R^{+}(x_i)}+\sqrt{\sup I_S^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{\left(1-\inf I_R^{+}(x_i)\right)+\left(1-\inf I_S^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{1-\inf I_R^{+}(x_i)}+\sqrt{1-\inf I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\left(1-\sup I_R^{+}(x_i)\right)+\left(1-\sup I_S^{+}(x_i)\right)}{2}}- \\ \left(\frac{\sqrt{1-\sup I_R^{+}(x_i)}+\sqrt{1-\sup I_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{\inf F_R^{+}(x_i)+\inf F_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^{+}(x_i)}+\sqrt{\inf F_S^{+}(x_i)}}{2}\right)+ \\ \sqrt{\frac{\sup F_R^{+}(x_i)+\sup F_S^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^{+}(x_i)}+\sqrt{\sup F_S^{+}(x_i)}}{2}\right)+\sqrt{\frac{-\left(\inf T_R^{-}(x_i)+\inf T_S^{+}(x_i)\right)}{2}}- \\ \left(\frac{\sqrt{\left(-\inf T_R^{-}(x_i)\right)}+\sqrt{\left(-\inf T_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup T_R^{-}(x_i)+\sup T_S^{+}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup T_R^{-}(x_i)\right)}+\sqrt{\left(-\sup T_S^{-}(x_i)\right)}}{2}\right)+ \\ \sqrt{\frac{-\left(\inf I_R^{-}(x_i)+\inf I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\inf I_R^{-}(x_i)\right)}+\sqrt{\left(-\inf I_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup I_R^{-}(x_i)+\sup I_S^{-}(x_i)\right)}{2}}- \\ \left(\frac{\sqrt{\left(-\sup I_R^{-}(x_i)\right)}+\sqrt{\left(-\sup I_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{\left(1+\inf I_R^{-}(x_i)\right)+\left(1+\inf I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\inf I_R^{-}(x_i)}+\sqrt{1+\inf I_S^{-}(x_i)}}{2}\right)+ \\ \sqrt{\frac{\left(1+\sup I_R^{-}(x_i)\right)+\left(1+\sup I_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{1+\sup I_R^{-}(x_i)}+\sqrt{1+\sup I_S^{-}(x_i)}}{2}\right)+\sqrt{\frac{-\left(\inf F_R^{-}(x_i)+\inf F_S^{-}(x_i)\right)}{2}}- \\ \left(\frac{\sqrt{\left(-\inf F_R^{-}(x_i)\right)}+\sqrt{\left(-\inf F_S^{-}(x_i)\right)}}{2}\right)+\sqrt{\frac{-\left(\sup F_R^{-}(x_i)+\sup F_S^{-}(x_i)\right)}{2}}-\left(\frac{\sqrt{\left(-\sup F_R^{-}(x_i)\right)}+\sqrt{\left(-\sup F_S^{-}(x_i)\right)}}{2}\right) \end{array}\right]$$

$$
= \frac{1}{2}\sum_{i=1}^{n} w_i \left[\begin{array}{l}
\sqrt{\frac{\inf T_S^+(x_i)+\inf T_R^+(x_i)}{2}} - \left(\frac{\sqrt{\inf T_S^+(x_i)}+\sqrt{\inf T_R^+(x_i)}}{2}\right) + \sqrt{\frac{\sup T_S^+(x_i)+\sup T_R^+(x_i)}{2}} - \left(\frac{\sqrt{\sup T_S^+(x_i)}+\sqrt{\sup T_R^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\inf I_S^+(x_i)+\inf I_R^+(x_i)}{2}} - \left(\frac{\sqrt{\inf I_S^+(x_i)}+\sqrt{\inf I_R^+(x_i)}}{2}\right) + \sqrt{\frac{\sup I_S^+(x_i)+\sup I_R^+(x_i)}{2}} - \left(\frac{\sqrt{\sup I_S^+(x_i)}+\sqrt{\sup I_R^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-\inf I_S^+(x_i)\right)+\left(1-\inf I_R^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-\inf I_S^+(x_i)}+\sqrt{1-\inf I_R^+(x_i)}}{2}\right) + \sqrt{\frac{\left(1-\sup I_S^+(x_i)\right)+\left(1-\sup I_R^+(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{1-\sup I_S^+(x_i)}+\sqrt{[1-\sup I_R^+(x_i)]}}{2}\right) + \sqrt{\frac{\inf F_S^+(x_i)+\inf F_R^+(x_i)}{2}} - \left(\frac{\sqrt{\inf F_S^+(x_i)}+\sqrt{\inf F_R^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\sup F_S^+(x_i)+\sup F_R^+(x_i)}{2}} - \left(\frac{\sqrt{\sup F_S^+(x_i)}+\sqrt{\sup F_R^+(x_i)}}{2}\right) + \sqrt{\frac{-\left(\inf T_S^-(x_i)+\inf T_R^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\inf T_S^-(x_i)\right)}+\sqrt{\left(-\inf T_R^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup T_S^-(x_i)+\sup T_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup T_S^-(x_i)\right)}+\sqrt{\left(-\sup T_R^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{-\left(\inf I_S^-(x_i)+\inf I_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\inf I_S^-(x_i)\right)}+\sqrt{\left(-\inf I_R^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup I_S^-(x_i)+\sup I_R^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\sup I_S^-(x_i)\right)}+\sqrt{\left(-\sup I_R^-(x_i)\right)}}{2}\right) + \sqrt{\frac{\left(1+\inf I_S^-(x_i)\right)+\left(1+\inf I_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\inf I_S^-(x_i)}+\sqrt{1+\inf I_R^-(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1+\sup I_S^-(x_i)\right)+\left(1+\sup I_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\sup I_S^-(x_i)}+\sqrt{[1+\sup I_R^-(x_i)]}}{2}\right) + \sqrt{\frac{-\left(\inf F_S^-(x_i)+\inf F_R^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\inf F_S^-(x_i)\right)}+\sqrt{\left(-\inf F_R^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup F_S^-(x_i)+\sup F_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup F_S^-(x_i)\right)}+\sqrt{\left(-\sup F_R^-(x_i)\right)}}{2}\right)
\end{array} \right]
$$

$= C_{IB}(S, R)_w$.

(4) $C_{IB}(R^C, S^C)_w =$

$$
\frac{1}{2}\sum_{i=1}^{n} w_i \left[\begin{array}{l}
\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right) + \sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-\inf I_R^+(x_i)\right)+\left(1-\inf I_S^+(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(1-\inf I_R^+(x_i)\right)}+\sqrt{\left(1-\inf I_S^+(x_i)\right)}}{2}\right) + \sqrt{\frac{\left(1-\sup I_R^+(x_i)\right)+\left(1-\sup I_S^+(x)_i\right)}{2}} - \left(\frac{\sqrt{\left(1-\sup I_R^+(x_i)\right)}+\sqrt{\left(1-\sup I_S^+(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{1-\left(1-\inf I_R^+(x_i)\right)+1-\left(1-\inf I_S^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-\left(1-\inf I_R^+(x_i)\right)}+\sqrt{1-\left(1-\inf I_S^+(x_i)\right)}}{2}\right) + \sqrt{\frac{1-\left(1-\sup I_R^+(x_i)\right)+1-\left(1-\sup I_S^+(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{1-\left(1-\sup I_R^+(x_i)\right)}+\sqrt{1-\left(1-\sup I_S^+(x_i)\right)}}{2}\right) + \sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right) + \sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right) \\
+\sqrt{\frac{-\left(\inf F_R^-(x_i)+\inf F_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\inf F_R^-(x_i)\right)}+\sqrt{\left(-\inf F_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup F_R^-(x_i)+\sup F_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup F_R^-(x_i)\right)}+\sqrt{\left(-\sup F_S^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{-\left(\left(-1-\inf I_R^-(x_i)\right)+\left(-1-\inf I_S^-(x_i)\right)\right)}{2}} - \left(\frac{\sqrt{-\left(-1-\inf I_R^-(x_i)\right)}+\sqrt{-\left(-1-\inf I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\left(-1-\sup I_R^-(x_i)\right)+\left(-1-\sup I_S^-(x_i)\right)\right)}{2}} - \\
\left(\frac{\sqrt{-\left(-1-\sup I_R^-(x_i)\right)}+\sqrt{-\left(-1-\sup I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{1+\left(-1-\inf I_R^-(x_i)\right)+1+\left(-1-\inf I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\left(-1-\inf I_R^-(x_i)\right)}+\sqrt{1+\left(-1-\inf I_S^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{1+\left(-1-\sup I_R^-(x_i)\right)+1+\left(-1-\sup I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\left(-1-\sup I_R^-(x_i)\right)}+\sqrt{1+\left(-1-\sup I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\inf T_R^-(x_i)+\inf T_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\inf T_S^-(x_i)\right)}+\sqrt{\left(-\inf T_R^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup T_S^-(x_i)+\sup T_R^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup T_S^-(x_i)\right)}+\sqrt{\left(-\sup T_R^-(x_i)\right)}}{2}\right)
\end{array} \right]
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n} w_i \left[\begin{array}{l}
\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right) + \sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\inf I_R^+(x_i)+\inf I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf I_R^+(x_i)}+\sqrt{\inf I_S^+(x_i)}}{2}\right) + \sqrt{\frac{\sup I_R^+(x_i)+\sup I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup I_R^+(x_i)}+\sqrt{\sup I_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1-\inf I_R^+(x_i)\right)+\left(1-\inf I_S^+(x_i)\right)}{2}} - \left(\frac{\sqrt{1-\inf I_R^+(x_i)}+\sqrt{1-\inf I_S^+(x_i)}}{2}\right) + \sqrt{\frac{\left(1-\sup I_R^+(x_i)\right)+\left(1-\sup I_S^+(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{1-\sup I_R^+(x_i)}+\sqrt{1-\sup I_S^+(x_i)}}{2}\right) + \sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right) + \\
\sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right) + \sqrt{\frac{-\left(\inf T_R^-(x_i)+\inf T_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\inf T_R^-(x_i)\right)}+\sqrt{\left(-\inf T_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup T_R^-(x_i)+\sup T_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup T_R^-(x_i)\right)}+\sqrt{\left(-\sup T_S^-(x_i)\right)}}{2}\right) + \\
\sqrt{\frac{-\left(\inf I_R^-(x_i)+\inf I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\inf I_R^-(x_i)\right)}+\sqrt{\left(-\inf I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup I_R^-(x_i)+\sup I_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\sup I_R^-(x_i)\right)}+\sqrt{\left(-\sup I_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{\left(1+\inf I_R^-(x_i)\right)+\left(1+\inf I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\inf I_R^-(x_i)}+\sqrt{1+\inf I_S^-(x_i)}}{2}\right) + \\
\sqrt{\frac{\left(1+\sup I_R^-(x_i)\right)+\left(1+\sup I_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{1+\sup I_R^-(x_i)}+\sqrt{1+\sup I_S^-(x_i)}}{2}\right) + \sqrt{\frac{-\left(\inf F_R^-(x_i)+\inf F_S^-(x_i)\right)}{2}} - \\
\left(\frac{\sqrt{\left(-\inf F_R^-(x_i)\right)}+\sqrt{\left(-\inf F_S^-(x_i)\right)}}{2}\right) + \sqrt{\frac{-\left(\sup F_R^-(x_i)+\sup F_S^-(x_i)\right)}{2}} - \left(\frac{\sqrt{\left(-\sup F_R^-(x_i)\right)}+\sqrt{\left(-\sup F_S^-(x_i)\right)}}{2}\right)
\end{array}\right] \\
&= C_{IB}(R,S)_w.
\end{aligned}
$$

This completes the proof. □

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