# A covariant formulation of the Ashtekar-Kodama quantum gravity and its solutions

# Jan Helm

Institute of Physics, London Email: janhelm@snafu.de

### Abstract

This article consists of two parts.

In the first part A we present in a concise form the present approaches to the quantum gravity, with the ADM formulation of GR, the Ashtekar and the Kodama ansatz at the center, and we also derive the 3-dimensional Ashtekar-Kodama constraints.

In the second part B, we introduce a 4-dimensional covariant version of the 3-dimensional (spatial) Hamiltionian, Gaussian and diffeomorphism constraints of the Kodama state with positive cosmological constant  $\Lambda$  in the Ashtekar formulation of quantum gravity.

In chapter B1-4 we present the equations and their solutions.

We get 32 partial differential equations for the 16 variables  $E^{\mu\nu}$  (inverse densitized tetrad of the metric  $g_{\mu\nu}$ ) and 16 variables  $A_{\mu}^{\nu}$  (gravitational wave tensor). We impose the boundary condition: for  $r \rightarrow \infty g(E^{\mu\nu}) \rightarrow g_{\mu\nu}$  i.e. in the classical limit of large *r* the Kodama state generates the given asymptotic spacetime (normally Schwarzschild-spacetime).

For A->0 in the static (time independent) the tetrad decouples from the wave tensor and the 24 Hamiltonian equations yield for  $A_{\mu}^{\nu}$  the *constant background* solution. The diffeomorphism becomes identically zero, and the tetrad can satisfy the Schwarzschild spacetime and the Gaussian equations for all {r, $\theta$ }, i.e. it the Einstein equations *are valid everywhere* outside the horizon.

In the time-dependent case with a  $\Lambda$ - scaled wave ansatz for  $A_{\mu}^{\nu}$  and  $E^{\mu\nu}$  we get a gravitational wave equations, which yields appropriate solutions only for angular momentum  $lx \ge 2$  (quadrupole wave) the tetrad

is locally damped with  $\exp\left(-4\sqrt{\frac{r}{3}}\right)$ , only the wave tensor  $A_{\mu}^{\nu}$  carries the energy.

In chapter B5 and 6 numerical solutions for special cases of the time-independent and of time-dependent equations are discussed.

In chapter B7 the energy tensor of the Ashtekar-Kodama gravity is introduced.

In chapter B8 we present the quantum field version of the Ashtekar-Kodama gravity and demonstrate the calculation of cross-sections.

Finally, in chapter B9 we give as an outlook an overview of the QFT including the Ashtekar-Kodama gravity.

All derivations and calculations were carried out in Mathematica-programs, so the results can be considered with high probability as error-free, the programs are cited in the literature index.

In the chapters B1-4, which deal with the solutions, every subchapter consists of a flow-diagram, which gives the overview and a text part, which describes the corresponding program in detail and can be skipped at first reading.

# Part A Quantum gravity

- 1. Motivation and problems
- 2. Classical mechanics and GR fundamentals
- 2.1. Lagrangian mechanics
- 2.2. Hamiltonian mechanics
- 2.3. General relativity
- 2.4. The concept of a graviton in GR and weak gravitational waves
- 3. Quantum field theory fundamentals
- 3.1. GR-Dirac formalism
- 3.2. The gauge group inQFT
- 3.3. Gravitational scale
- 4. Semiclassical quantum gravity
- 5. Supersymmetry:quantum supergravity
- 6. The ADM-formulation (3+1 decomposition)
- 6.1. Hamiltonian form of the Einstein-Hilbert action
- 7. Canonical gravity with connections and loops (LQG)
- 7.1. Ashtekar variables
- 7.2. Discussion of the constraints
- 7.3. 3-dimensional Ashtekar-Kodama constraints
- 8. 4-dimensional Ashtekar-Kodama constraints
- 9. BF-theory
- 9.1. Palatini action as BF-theory
- 9.2. Plebanski action as BF-theory
- 9.3. From the Plebanski action to the Einstein-Hilbert action

# Part B Ashtekar-Kodama gravity

- 1. 4-dimensional Ashtekar-Kodama equations and their properties
- 1.1. AK covariant derivative and its gauge group
- 1.2. Renormalizable Einstein-Hilbert action with the Ashtekar momentum  $A_{\mu}^{\nu}$
- 2. The basic equations
- 2.1 The integrability condition
- 2.2. Solvability of static and time-dependent equations eqtoiv, eqtoiev
- 3. Solutions of static equations
- 3.1. Solution limit  $\Lambda \rightarrow 0$
- 3.1.1. The Gauss-Schwarzschild tetrad
- 3.2. Solution  $\Lambda \neq 0$  with the half-logarithmic ansatz
- 3.2.2. Solvability of the metric condition for the half-logarithmic solution
- 3.3. Behavior at Schwarzschild horizon
- 4. Solutions of time-dependent equations
- 4.1. The  $\Lambda$ -scaled wave ansatz for the A-tensor
- 4.2. Special wave solution  $\Lambda \neq 0$
- 4.3. Wave equation in Schwarzschild spacetime
- 4.3.1. Solutions of the gravitational wave equation
- 4.4. Wave equation in binary rotator spacetime
- 4.4.1. Wave equations for the binary gravitational rotator
- 4.4.2. Solution as a series in r-powers by comparison of coefficients
- 4.4.3. Solution of  $coef(1/r^4)$  as a series in  $r_0$
- 4.4.4. Complete solution of the *r*-powers-series ansatz for  $r_0=1$
- 5. Numeric solutions of time-independent equations with coupling  $\Lambda$ =1
- 5.1. The metric in AK-gravity with coupling: no horizon and no singularity
- 6. Numeric solutions of time-independent equations with weak coupling and binary gravitational rotator
- 7. Energy tensor of the gravitational wave
- 8. Quantum AK-gravitation
- 8.1. Lagrangian of Hamiltonian equations
- 8.2. Lagrangian of remaining equations
- 8.3. Dirac lagrangian for the graviton
- 8.4. The graviton wave function and cross-sections
- 8.5. The graviton propagator
- 8.6. The gravitational Compton cross section
- 9. Unified QFT:range and gauge group

# 4 Part A Quantum gravity



# A1. Motivation and problems

1. unification (successful StdModel)

-classical and quantum concepts (phase space versus Hilbert space, etc.) are most likely incompatible. -semiclassical theory, where gravity stays classical but all other fields are quantum, has failed up to now 2. cosmology and black-holes

-initial big-bang state is a quantum state

- Hawking-Penrose black-holes are quantum objects

3. problem of time

-in quantum theory, time is an external (absolute) element, not described by an operator (in special relativistic quantum field theory, the role of time is played by the external Minkowski space–time).

-in GR, space-time is a dynamical (non-absolute) object

4. superposition principle

- in QM the fundamental equations are linear in the wave function and the operators, the solutions can be combined additively(superposition principle)

- in GR the Einstein equations the Ricci-tensor is explicitly of order 2 in the metric  $g_{\mu\nu}$  and of additional

order 2 in its inverse  $g^{\mu\nu}$ , the solutions cannot be combined linearly.

5. action and renormalization

The Einstein-Hilbert action has a dimensional interaction constant  $\frac{1}{2\kappa}$ , and therefore the action is

fundamentally non-renormalizable

6. In GR, there is no adequate description of gravitational waves: a spherical gravitational wave is a metric oscillation, and satisfies the Einstein equation only for small amplitudes

In our opinion, there are five requirements, which a successful quantum gravity has to fulfill :

-it must have a dimensionally renormalizable lagrangian, i.e. the lagrangian must have the correct dimension without dimensional constants, and a covariant derivative with a gauge-group

-the static version of the theory must deliver the exact GR, except at singularities

-the static theory should remove the singularities of GR

-the time-dependent version of the theory must give a mathematically consistent classical description of gravitational waves (i.e. a graviton wave-tensor)

-the quantum version of the graviton wave-tensor must work in quantum field theory within Feynman diagrams and yield finite cross-sections in analogy to quantum electrodynamics

The Ashtekar-Kodama gravity, which we present in Part B, satisfies all five requirements, therefore it is a good candidate for the correct quantum gravity theory.

#### 6 A2. Classical mechanics and GR fundamentals

#### 2.1. Lagrangian mechanics

The non-relativistic Lagrangian for a system of particles can be defined by

$$L=T-V$$
  $T=rac{1}{2}\sum_{k=1}^{N}m_{k}v_{k}^{2}$ 

Euler-Lagrange equations

$$\delta L = \sum_{j=1}^{n} \left( \frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right)$$
, variational principle  $\delta L = 0$ 

general formulation with parameters  $\lambda_1, ..., \lambda_p$  instead of time t

$$\delta L = \sum_{j=1}^{n} \left( \frac{\partial L}{\partial q_j} \delta q_j + \sum_{k=1}^{p} \frac{\partial L}{\partial \frac{\partial q_j}{\partial \lambda_k}} \delta \frac{\partial q_j}{\partial \lambda_k} \right)$$

by partial integration and total differentiation for time:

$$\int_{t_1}^{t_2} \delta L \, \mathrm{d}t = \sum_{j=1}^n \left[ \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{j=1}^n \left( \frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j \, \mathrm{d}t \, .$$

Lagrangian with constraints

$$L' = L(\mathbf{r}_1, \mathbf{r}_2, \dots, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dots, t) + \sum_{i=1}^C \lambda_i(t) f_i(\mathbf{r}_k, t) 
onumber \ \int^{t_2} \delta L \, \mathrm{d}t = 0$$

Hamilton principle  $J_{t_1}$ 

$$\int_{t_1}^{t_2} \delta L' \mathrm{d}t = \int_{t_1}^{t_2} \sum_{k=1}^N \left( \frac{\partial L}{\partial \mathbf{r}_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\mathbf{r}}_k} + \sum_{i=1}^C \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k} \right) \cdot \delta \mathbf{r}_k \, \mathrm{d}t = 0$$

**Euler-Lagrange equations** 

$$rac{\partial L}{\partial \mathbf{r}_k} - rac{\mathrm{d}}{\mathrm{d}t}rac{\partial L}{\partial \dot{\mathbf{r}}_k} + \sum_{i=1}^C \lambda_i rac{\partial f_i}{\partial \mathbf{r}_k} = 0\,,$$

#### 2.2. Hamiltonian mechanics

Hamiltonian

$${\cal H}=T+V \quad , \quad T=rac{p^2}{2m} \quad , \quad V=V(q)$$

Hamiltonian from Lagrangian

$$\mathcal{H} = \sum_i \dot{q}^i rac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} = \sum_i \dot{q}^i p_i - \mathcal{L}$$

Hamiltonian equations

$$rac{\partial \mathcal{H}}{\partial q^j} = -\dot{p}_j \quad , \quad rac{\partial \mathcal{H}}{\partial p_j} = \dot{q}^j \quad , \quad rac{\partial \mathcal{H}}{\partial t} = -rac{\partial \mathcal{L}}{\partial t}$$

#### 7 2.3. General relativity Equations

The Einstein field equations are Minkowski metric  $\eta = diag(-1, 1, 1, 1)$  predominantly used in GR

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

where  $R_{\mu\nu}$  is the Ricci tensor, *R* the Ricci curvature,  $\kappa = \frac{8\pi G}{c^4}$ ,  $T_{\mu\nu}$  is the energy-momentum tensor,  $\Lambda$  is the

cosmological constant

with the Christoffel symbols (second kind)

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left( \frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} \right)$$

and the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho}$$

The orbit equations in vacuum ( $T_{\mu\nu} = 0$ ) are:

$$\frac{d^{2}x^{\kappa}}{d\lambda^{2}} + \Gamma^{\kappa}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0 \quad \kappa = 0...3$$

with the usual setting  $\lambda = \tau$  = proper time For  $\lambda = \tau$  we get for the line-element  $ds = c \ d\lambda = d\lambda$  and therefore trivially:

$$g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} - 1 = 0$$

The Kerr line element reads

$$-ds^{2} = \left(1 - \frac{rr_{s}}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)(dt)^{2} + \left(\frac{2rr_{s}\alpha\sin^{2}\theta}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)dt \,d\varphi$$
$$-\left(\frac{r^{2} + \alpha^{2}\cos^{2}\theta}{r^{2} - rr_{s} + \alpha^{2}}\right)dr^{2} - \left(r^{2} + \alpha^{2} + \frac{rr_{s}\alpha^{2}\sin^{2}\theta}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)\sin^{2}\theta \,d\varphi^{2} - \left(r^{2} + \alpha^{2}\cos^{2}\theta\right)(d\theta^{2})$$

where  $r_s = \frac{2GM}{c^2}$  is the Schwarzschild radius, and  $\alpha = \frac{J}{Mc}$  is the angular momentum radius (amr),  $\alpha$  has the dimension of a distance:  $[\alpha] = [r]$ , and J is the angular momentum.

In the limit  $\alpha \rightarrow 0$  the Kerr line element becomes the standard Schwarzschild line element

$$-ds^{2} = \left(1 - \frac{r_{s}}{r}\right)c^{2}dt^{2} - \frac{dr^{2}}{\left(1 - \frac{r_{s}}{r}\right)} - r^{2}\left(d\theta^{2} + \sin^{2}\theta \,d\varphi\right)$$

#### **Einstein-Hilbert action**

Einstein-Hilbert action with boundary term and external curvature K

$$S_{\rm EH} = \frac{c^4}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4 x \; \sqrt{-g} \left(R - 2\Lambda\right) - \frac{c^4}{8\pi G} \int_{\partial \mathcal{M}} \mathrm{d}^3 x \; \sqrt{h} K.$$

the Einstein field equations are obtained,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu}$$

#### 8 2.4. The concept of a graviton in GR and weak gravitational waves

 $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ [13]

the Einstein equations in linear approximation yield the gravitational wave equation for  $f_{\mu\nu}$ 

$$\Box f_{\mu\nu} = -16\pi G \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right), \quad T := \eta^{\mu\nu} T_{\mu\nu} \text{ using the gauge condition} \qquad f_{\mu\nu}{}^{\nu} = \frac{1}{2} f^{\nu}{}_{\nu,\mu}$$

analogous to the electrodynamics wave equation  $\Box A^{\mu} = -4\pi j^{\mu} \text{ with the Lorentz gauge } \partial_{\nu}A^{\nu} = 0$ 

let us consider a plane wave, purely spatial and transverse (TT-gauge  $\begin{pmatrix} e_{\mu\nu}k^{\nu} = 0 \end{pmatrix}$ ,  $e^{\nu}{}_{\nu} = 0$ ) moving in the  $x^1 \equiv x$  direction  $x^0 = t$ ,  $k^0 = k^1 = \omega > 0$ ,  $k^2 = k^3 = 0$ .

$$x^{\circ} \equiv t, \, k^{\circ} = k^{x} \equiv \omega > 0, \, k^{z} = k^{\circ} =$$
$$f_{\mu\nu} = 2 \operatorname{Re}\left(e_{\mu\nu} \mathrm{e}^{-\mathrm{i}\omega(t-x)}\right)$$

there are 2 basic polarizations (not one, as for a spin=1 wave)



with circular right and left polarized states

$$e_{R} = \frac{1}{\sqrt{2}}(e_{+} + ie_{\times}), \quad e_{L} = \frac{1}{\sqrt{2}}(e_{+} - ie_{\times})$$

Under counterclockwise rotation by an angle  $\theta$ , the circular polarization states transform according to

$$e_{\rm R}^\prime = e^{-2i\theta} e_{\rm R} \ , \quad e_{\rm L}^\prime = e^{2i\theta} e_{\rm L}. \label{eq:error}$$

, that is a rotation by 
$$2\theta$$
 with helicity  $-2$  and  $+2$ 

The corresponding left and right circularly polarized electromagnetic waves have helicity 1 and -1, respectively.

The linearized gravity lagrangian for the above gravitational wave equation is

$$\mathcal{L} = \frac{1}{64\pi G} \left( f^{\mu\nu,\sigma} f_{\mu\nu,\sigma} - f^{\mu\nu,\sigma} f_{\sigma\nu,\mu} - f^{\nu\mu,\sigma} f_{\sigma\mu,\nu} - f^{\mu}_{\ \ \mu,\nu} f^{\rho}_{\ \ \rho,\nu} + 2f^{\rho\nu}_{\ \ \nu,\nu} f^{\sigma}_{\ \ \sigma,\rho} \right) - \frac{1}{2} T_{\mu\nu} f^{\mu\nu}.$$

the energy-momentum-tensor is

0.0

$$t_{\mu\nu} := \frac{\partial \mathcal{L}}{\partial f_{\alpha\beta,\nu}} f_{\alpha\beta,\mu} - \eta_{\mu\nu} \mathcal{L}.$$
  
in TT-gauge  $t_{\mu\nu} = \frac{\frac{1}{32\pi G} f_{\alpha\beta,\mu} f^{\alpha\beta}{}_{,\nu}}{}_{,\nu}$  with the mean value  $\bar{t}_{\mu\nu} = \frac{k_{\mu}k_{\nu}}{16\pi G} e^{\alpha\beta*} e_{\alpha\beta}.$ 

# A3. Quantum field theory fundamentals

# 3.1. GR-Dirac formalism

GR-Dirac formalism ( $\hbar = c = 1$ ) [22]

$$\nabla_{\mu}A_{\nu} \equiv \partial_{\mu}A_{\nu} + \Gamma^{\lambda}_{\mu\nu}A_{\lambda}$$
$$\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} - \Gamma^{\nu}_{\mu\lambda}A^{\lambda}$$
CP covariant derivat

GR covariant derivative

its commutator is the Riemann tensor

$$[\nabla_{\mu}, \nabla_{\nu}]A_{\lambda} = R^{\rho}_{\mu\nu\lambda}A_{\rho}$$

$$R^{\rho}_{\mu\nu\lambda} = \partial_{\mu}\Gamma^{\rho}_{\nu\lambda} - \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} + \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}$$

$$e^{a}_{\mu}e^{a}_{\nu} = g_{\mu\nu}$$
the tetrad

the tetrad-Dirac matrices  $\gamma^a e^{a\mu} = \gamma^{\mu}(x)$  with the anti-commutator  $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}(x)$  the tetrad covariant derivative becomes

$$\nabla_{\mu}\psi = (\partial_{\mu} - \frac{i}{4}\omega_{\mu}^{ab}\sigma_{ab})\psi$$
  
where  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu},\gamma_{\nu}]$  are the Dirac  $\sigma$ -matrices

and  $\boldsymbol{\omega}$  the GR connection field in tetrad-expression

$$\omega_{\mu}^{ab} = \frac{1}{2}e^{a\nu}(\partial_{\mu}e_{\nu}^{b} - \partial_{\nu}e_{\mu}^{b}) + \frac{1}{4}e^{a\rho}e^{b\sigma}(\partial_{\sigma}e_{\rho}^{c} - \partial_{\rho}e_{\sigma}^{c})e_{\mu}^{c} - (a \leftrightarrow b)$$

with these denominations the GR-Dirac equation becomes

 $(i\hbar\gamma^{\mu}(x)\nabla_{\mu}-mc)\psi(x) = 0$  and the GR-Dirac Lagrangian

$$L_{GRD} = -\frac{\sqrt{-g}}{2\kappa} (R - 2\Lambda) + \sqrt{-g} \overline{\psi} (i\hbar c \gamma^{\mu}(x) \nabla_{\mu} - mc^{2}) \psi$$

#### 10 3.2. The gauge group inQFT

structure constants and the generator algebra of the gauge Lie group [22] with  $\eta = diag(1, -1, -1, -1)$ 

$$[\tau^a,\tau^b]=if^{abc}\tau^c$$

we introduce the covariant derivative with the connection  $A_{\mu}$ :

$$D_{\mu} \equiv \partial_{\mu} - igA_{\mu} \quad A_{\mu}(x) \equiv A^{a}_{\mu}(x)\tau^{a}$$

The fermion field  $\psi_i$  transforms under the Lie algebra

$$\psi_i(x) \to \Omega_{ij}(x)\psi_j(x)$$
  $\Omega_{ij}(x) = \left(e^{-i\theta^a(x)\tau^a}\right)_{ij}$ 

then the covariant derivative transforms like  $\psi_i$  under  $\Omega$  (is gauge-covariant):

$$(D_{\mu}\psi)' = \partial_{\mu}\psi' - igA'_{\mu}\psi'$$
$$= \Omega\partial_{\mu}\psi + (\partial_{\mu}\Omega)\psi - igA'_{\mu}\Omega\psi$$
$$= \Omega D_{\mu}\psi$$

 $(D_{\mu}\psi)' \ \ \Omega D_{\mu}\psi$ 

in order to achieve this,  $\psi_i$  and  $A_{\mu}$  infinitesimally transform like  $\delta A^a_{\mu} = -\frac{1}{g} \partial_{\mu} \theta^a + f^{abc} \theta^b A^c_{\mu}$ 

$$\delta\psi=-ig\theta^a\tau^a\psi$$

, which results from the ansatz above for the covariant derivative We define the field tensor  $F_{\mu\nu}{}^a$  from the commutator

$$F_{\mu\nu} = \frac{i}{g} [D_{\mu}, D_{\nu}]$$
  
=  $\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig[A_{\mu}, A_{\nu}]$   
=  $(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + gf^{abc}A_{\mu}^{b}A_{\nu}^{c})\tau^{a}$ 

 $F_{\mu\nu}^{\ \ a} = \partial_{\mu}A_{\nu}^{\ \ a} - \partial_{\nu}A_{\mu}^{\ \ a} + gf^{\ \ abc}A_{\mu}^{\ \ b}A_{\nu}^{\ \ c}$ The gauge field action becomes

$$S = \int d^4x \,\left(-\frac{1}{2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}\right) = \int d^4x \,\left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}\right)$$

and the fermion action coupled to the field is

$$S = \int d^4x \, \bar{\psi}(i \, \mathcal{D} - m) \psi$$
, where  $\mathcal{D} = \gamma^{\mu} D_{\mu}$  is the covariant "Dirac dagger"

the lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}^{\ a} F^{a\mu\nu} + \sum_{k} \left( \overline{\psi}_{k} \left( i \gamma^{\mu} D_{\mu} - m_{k} \right) \psi_{k} \right)$$

#### 11 3.3. Gravitational scale universal scale: Planck-scale

$$l_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} \text{ cm},$$

$$t_{\rm P} = \frac{l_{\rm P}}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44} \text{ s},$$

$$m_{\rm P} = \frac{\hbar}{l_{\rm P}c} = \sqrt{\frac{\hbar c}{G}} \approx 2.18 \times 10^{-5} \text{ g} \approx 1.22 \times 10^{19} \text{ GeV}/c^2.$$

$$T_{\rm P} = \frac{m_{\rm P}c^2}{k_{\rm B}} \approx 1.41 \times 10^{32} \text{ K},$$

$$\rho_{\rm P} = \frac{m_{\rm P}}{l_{\rm P}^3} \approx 5 \times 10^{93} \frac{\text{g}}{\text{cm}^3}.$$

$$Q_{\rm P} = \sqrt{m_{\rm P}l_{\rm P}} \frac{l_{\rm P}}{l_{\rm P}} = \sqrt{G}m_{\rm P} = \sqrt{\hbar c}, \quad e = \sqrt{\alpha}Q_{\rm P} = \frac{0.085}{2}Q_{\rm P},$$

$$\alpha_{\rm g} = \frac{Gm_{\rm Pr}^2}{\hbar c} = \left(\frac{m_{\rm Pr}}{m_{\rm P}}\right)^2 \approx 5.91 \times 10^{-39},$$
ne structure constant of gravity

fi

mean gravity scale  $r_{gr} = \sqrt{l_P} \sqrt{\frac{1}{\Lambda}} = 3.1 \times 10^{-5} m = 31 \mu m$ 

mean gravity scale  $r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1 \times 10^{-5} m = 31 \mu m$  with the corresponding energy scale

$$E_{gr} = c^2 M_{gr} = \frac{\hbar c}{r_{gr}} = \frac{1.05 \times 10^{-34} Js \times 3 \times 10^8 m/s}{3.1 \times 10^{-5} m} = 1.016 \times 10^{-21} J = 6.34 \times 10^{-3} eV$$

 $\Lambda$ - energy scale

$$E_{\Lambda} = \left(\frac{\hbar^2 \Lambda^{1/2} c^6}{G}\right)^{1/3} = \left(\frac{\hbar^3 \Lambda^{1/2} c^3}{l_p^2}\right)^{1/3} \approx 15 MeV$$

with the corresponding length scale

$$l_{\Lambda} = \frac{\hbar c}{E_{\Lambda}} = \frac{1.05 * 10^{-34} * 6.24 * 10^{18} eV * 3 * 10^{8} m}{15 * 10^{6} eV} = 1.31 * 10^{-14} m$$

#### 12 A4. Semiclassical quantum gravity

Dirac equation  $\begin{aligned} \left(i\gamma^{\mu}\partial_{\mu} + \frac{mc}{\hbar}\right)\psi(x) &= 0 \\ (i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi(x) &= 0 \end{aligned} \text{ with } \hbar = 1, c = 1: (i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \end{aligned} \text{ with } \eta = diag(1, -1, -1, -1) \end{aligned} \text{ used in QFT}$ 

with commutation relations for  $\gamma^{\mu}$   $[\gamma^{\mu}, \gamma^{\nu}]_{+} := \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$ 

$$e^{a}_{\mu}e^{a}_{\nu} = g_{\mu\nu}$$
$$e^{a\mu} = g^{\mu\nu}e^{a}_{\nu}$$
$$e^{a}_{\mu}e^{b\mu} = \delta^{ab}$$

now we introduce the tetrad (vierbein)

 $\gamma^{a}e^{a\mu} = \gamma^{\mu}(x)$  with commutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}(x)$$
 from original

and local (x-dependent)  $\gamma^{\mu}(x)$ Dirac-matrices  $\gamma^{a}$ 

with this local  $\gamma^{\mu}$  we formulate the covariant derivative

$$\nabla_{\mu}\psi = (\partial_{\mu} - \frac{i}{4}\omega_{\mu}^{ab}\sigma_{ab})\psi$$
  
where  $\sigma_{ab} = \frac{i}{2}[\gamma_{a}, \gamma_{b}]$ 

and the connection

$$\omega^{ab}_{\mu} = \frac{1}{2} e^{a\nu} (\partial_{\mu} e^{b}_{\nu} - \partial_{\nu} e^{b}_{\mu}) + \frac{1}{4} e^{a\rho} e^{b\sigma} (\partial_{\sigma} e^{c}_{\rho} - \partial_{\rho} e^{c}_{\sigma}) e^{c}_{\mu} - (a \leftrightarrow b)$$

$$(i\gamma^{\mu}\nabla_{\mu}-m)\psi=0$$

the GR-Dirac equation is now and the lagrangian

$$\mathscr{L} = -\frac{1}{2\kappa^2}\sqrt{-g}R + e\bar{\psi}(i\gamma^{\mu}\nabla_{\mu} \cdot m)\psi$$
  
with  $e \equiv \det e^a_{\mu} = \sqrt{-g}$ .

$$L_{grD} = \frac{1}{2\kappa} \sqrt{-g} \left( R - 2\Lambda \right) + e \overline{\psi} \left( i\hbar c \gamma^{\mu} \nabla_{\mu} - mc^{2} \right) \psi$$

For an observer, with linear acceleration  $\boldsymbol{a}$  and angular velocity  $\boldsymbol{\omega}$  :

a non-relativistic approximation with relativistic corrections is then obtained by the standard Foldy– Wouthuysen transformation, decoupling the positive- and negative energy states. This leads to (writing  $\beta \equiv \gamma^0$ ) the Schrödinger equation

$$\begin{split} &i\hbar\frac{\partial\psi}{\partial t} = H_{\rm FW}\psi \\ &H_{\rm FW} = -\left(\begin{array}{c} \beta mc^2 + \frac{\beta}{2m}\mathbf{p}^2 - \frac{\beta}{8m^3c^2}\mathbf{p}^4 + \beta m(\mathbf{a} \ \mathbf{x}) \\ &-\omega(\mathbf{L} + \mathbf{S}) - \left(\frac{\beta}{2m}\mathbf{p}\frac{\mathbf{a} \ \mathbf{x}}{c^2}\mathbf{p} + \frac{\beta\hbar}{4mc^2}\vec{\Sigma}(\mathbf{a} \times \mathbf{p})\right) + \mathcal{O}\left(\frac{1}{c^3}\right) \end{split}$$

#### **Semiclassical Einstein equations**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle.$$

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - m\Phi\psi + \nabla^2\Phi = 4\pi Gm|\psi|^2,$$

'Schrödinger- Newton equation'

$$\mathrm{i}\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) - Gm^2\int\mathrm{d}^3y\;\frac{|\psi(\mathbf{y},t)|^2}{|\mathbf{x}-\mathbf{y}|}\psi(\mathbf{x},t)$$

# A5. Supersymmetry: quantum supergravity

Supergravity (SUGRA) is a supersymmetric theory of gravity encompassing GR. [13] Supersymmetry (SUSY) is a symmetry which mediates between bosons and fermions via N generators. the (N=1) simple SUGRA action is the sum of the Einstein-Hilbert action and the Rarita-Schwinger action for the gravitino (spin 3/2),

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \, \left(\det e^n_\mu\right) R + \frac{1}{2} \int \mathrm{d}^4 x \, \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma$$

with the tetrad  $e^n{}_\mu$  ,  $\det e^n_\mu = \sqrt{-g}, \ \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ 

$$D_{\mu} = \partial_{\mu} - \frac{1}{2} \omega_{\mu}^{nm} \sigma_{nm}$$
 and  $\Lambda = 0$ 

GR covariant derivative extended action

$$S^{sugra} = \frac{c^4}{64\pi G} \int \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} \left( R_{\mu\nu}^{\ ab} e_{\rho}^{\ c} e_{\sigma}^{\ d} - \frac{\Lambda}{3} e_{\mu}^{\ a} e_{\nu}^{\ b} e_{\rho}^{\ c} e_{\sigma}^{\ d} \right) d^4 x$$
$$- \frac{c^4}{64\pi G} \int \varepsilon^{\mu\nu\rho\sigma} \left( \frac{1}{2} \overline{\psi}_{\mu} \gamma_5 \gamma_a e_{\nu}^{\ a} D_{\rho} \psi_{\sigma} - \frac{i}{4l_p} \overline{\psi}_{\mu} \gamma_5 \gamma_a \gamma_b e_{\nu}^{\ a} e_{\rho}^{\ b} \psi_{\sigma} \right) d^4 x$$

action S is general-covariant, Poincare-invariant and also SUSY-invariant under SUSY-transformations

$$\delta e^m_\mu = \frac{1}{2} \sqrt{8\pi G} \bar{\epsilon}^\alpha \gamma^m_{\alpha\beta} \psi^\beta_\mu,$$
  
$$\delta \psi^\alpha_\mu = \frac{1}{\sqrt{8\pi G}} D_\mu \epsilon^\alpha,$$

which transform fermions into bosons and vice-versa

A special role is played by N = 8 SUGRA. As mentioned above, N = 8 is the maximal number of SUSY generators.

The theory contains an irreducible multiplet that consists of massless states including the spin-2 graviton, eight spin-3/2 gravitinos, 28 spin-1 states, 56 spin-1/2 states, and 70 spin-0 states.

The complete four-loop four-particle amplitude of N = 8 SUGRA. is ultraviolet finite.

This allows the speculation that the theory is finite at all orders. If this were true, N = 8 SUGRA would be a perturbatively consistent theory of quantum gravity.

#### 13

#### 14 A6. The ADM-formulation (3+1 decomposition)

Arnowitt, Deser, Misner 1962

# Foliation $M = R(t) \times \Sigma(x_3)$

space–time metric  $g_{\mu\nu}$  induces a three-dimensional metric on each  $\Sigma_t$  according to

$$h_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu},$$
 (4.39)

where  $n_{\mu}$  denotes again the unit normal to  $\Sigma_t$ , with  $n^{\mu}n_{\mu} = -1$ .

one can decompose t<sup> $\mu$ </sup> into its components normal and tangential to  $\Sigma_t$ 

 $t^{\mu} = N n^{\mu} + N^{\mu}$ , N is the lapse and  $N^{\mu}$  the shift vector

$$N = \frac{1}{n^{\mu} \nabla_{\mu} t}$$

we can write  $N = -t^{\mu}n_{\mu}$ , so

the four-metric can be decomposed into spatial and temporal components,

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab} \end{pmatrix}.$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^c}{N^2} & h^{ab} - \frac{N^a N^b}{N^2} \end{pmatrix}$$

the inverse is

 $\mathbf{h}^{ab} \text{ is the inverse of the three-metric} \quad h^{ab}h_{bc} = \delta^a_c \quad \text{and} \quad n^\mu = g^{\mu\nu}n_\nu = \left(\frac{1}{N}, -\frac{\mathbf{N}}{N}\right), \text{ where } N = \sqrt{N_c N^a}$ 

 $K_{\mu\nu} = h_{\mu}^{\ \rho} \nabla_{\rho} n_{\nu}$  is the embedding (external) curvature of  $\Sigma(x_3)$ 

its spatial version  $K_{ab}$  can be interpreted as the velocity' associated with  $h_{ab}$ .

$$K := K_a{}^a = h^{ab}K_{ab} =: \theta_a$$

its trace

$$K_{ab} = \frac{1}{2N} \left( \dot{h}_{ab} - D_a N_b - D_b N_a \right)$$

 $K_{ab}$  can be written as

#### 6.1. Hamiltonian form of the Einstein-Hilbert action

The 'space-time component' Gi0 of the Einstein equations reads expressed in embedding curvature

$$K^2 - K_{ab}K^{ab} + {}^{(3)}R = 0.$$

$$D_b K^b_{\ a} - D_a K = 0.$$

with the covariant derivative  $D_{\mu} = \partial_{\mu} - \frac{1}{2} \omega_{\mu}^{nm} \sigma_{nm} \quad \omega_{a}^{\ i}{}_{j} = \Gamma_{kj}^{i} e_{a}^{k}$ , connection

these constraints for  $(h_{ab}, K_{cd})$  on a boundary  $\Sigma$  determine uniquely the solutions of the Einstein equations

interconnection theorems (Kucha'r 1981):

1. If the constraints are valid on an initial hypersurface and if the dynamical evolution equations  $G_{ab} = 0$  (pure spatial components of the vacuum Einstein equations) on space-time hold, the constraints hold on every hypersurface. Together, one then has all ten Einstein equations.

2. If the constraints hold on every hypersurface, the equations  $G_{ab} = 0$  hold on space-time.

In electrodynamics, for comparison, one has to specify A and E on  $\Sigma$  satisfying the constraint Gauss's law

 $\nabla E = 0.$ 

For the volume element we get

One then gets in space-time a solution of Maxwell's equations that is unique up to gauge transformation.

$$\sqrt{-g} = N\sqrt{h}.$$

The reformulated Einstein-Hilbert action becomes the ADM action after Arnowitt, Deser, Misner 1962:

$$16\pi G \ S_{\rm EH} = \int_{\mathcal{M}} dt \, d^3x \ N\sqrt{h} (K_{ab}K^{ab} - K^2 + {}^{(3)}R - 2\Lambda)$$
  

$$\equiv \int_{\mathcal{M}} dt \, d^3x \ N \left( G^{abcd}K_{ab}K_{cd} + \sqrt{h} [{}^{(3)}R - 2\Lambda] \right),$$
  
here  $\kappa = \frac{8\pi l_p{}^2}{\hbar c} = \frac{8\pi G}{c^4}$ ,  $16\pi G = 2\kappa c^4$ , where  $G^{abcd}K_{ab}K_{cd} = K_{ab}K^{ab} - K^2$   
 $G^{abcd} = \frac{\sqrt{h}}{2} (h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd})$  (DeWitt-metric)

We get for the spatial metric  $h_{ab}$  and the canonical spatial momenta  $p^{ab}$ 

$$p^{ab} := \frac{\partial \mathcal{L}^{g}}{\partial \dot{h}_{ab}} = \frac{1}{16\pi G} G^{abcd} K_{cd} = \frac{\sqrt{h}}{16\pi G} \left( K^{ab} - Kh^{ab} \right)$$

and for the action

 $\mathcal{H}_{a}^{a} = -2D_{b}p_{a}^{a}$  diffeomorphism constraint Variation with respect to the Lagrange multipliers N and N<sup>a</sup> yields the constraints

$$\begin{aligned} \mathcal{H}_{\perp}^{\mathbf{g}} &= 16\pi G \, G_{abcd} p^{ab} p^{cd} - \frac{\sqrt{h}}{16\pi G} (\,^{(3)}R - 2\Lambda) \approx 0 \\ \mathcal{H}_{a}^{\mathbf{g}} &= -2D_{b} p_{a}^{\ b} \approx 0. \end{aligned}$$

4 pdeqs 2. order in r,  $\theta$  for 6 symmetric  $h_a^{\ b}$  and 6 symmetric  $p_a^{\ b}$ If non-gravitational fields are coupled, the constraints acquire extra terms.

$$2G_{\mu\nu}n^{\mu}n^{\nu} = 16\pi G T_{\mu\nu}n^{\mu}n^{\nu} =: 16\pi G\rho.$$
 with the energy density  $\rho = T_{\mu\nu}n^{\mu}n^{\nu}$ 

$$\mathcal{H}_{\perp} = 16\pi G \, G_{abcd} p^{ab} p^{cd} - \frac{\sqrt{h}}{16\pi G} ({}^{(3)}R - 2\Lambda) + \sqrt{h}\rho \approx 0$$

Hamiltonian constraint

 $\mathcal{H}_a = -2D_b p_a^{\ b} + \sqrt{h} J_a \approx 0,$  diffeomorphism constraint with external current

where  $J_a := h_a^{\ \mu} T_{\mu\nu} n^{\nu}$  is the gravitational Poynting vector

#### 16 A7. Canonical gravity with connections and loops (LQG)

#### 7.1. Ashtekar variables

definition of the (inverse) triad  $e_i^a$ 

$$h_{ab}e^a_i e^b_j = \delta_{ij},$$

$$h^{ab} = \delta^{ij} e^a_i e^b_j \equiv e^a_i e^b_i. \quad e^0_a = -n_a = Nt_{,a}$$

 $E^{\,a}_i \text{ is the inverse densitized triad } E^a_i(x) := \sqrt{h}(x)e^a_i(x), \ \sqrt{h} = |\det(e^i_a)|.$ 

the extrinsic curvature  $K_a^i(x) := K_{ab}(x)e^{bi}(x)$ , is the canonical conjugate to  $E_i^a$ 

$$\begin{split} K^i_a \delta E^{ia} &= \frac{K_{ab}}{2\sqrt{h}} \delta \left( E^{ia} E^{ib} \right) = \frac{K_{ab}}{2\sqrt{h}} \left( h \delta h^{ab} + h^{ab} \delta h \right) \\ &= -\frac{\sqrt{h}}{2} \left( K^{ab} - K h^{ab} \right) \delta h_{ab} = -8\pi G p^{ab} \delta h_{ab} \\ &\text{ with } 8\pi G = \kappa c^4 \end{split}$$

$$\mathcal{G}_i(x) := \epsilon_{ijk} K_a^j(x) E^{ka}(x) \approx 0,$$

resulting Gauss constraint

with the GR connection

$$v^a = v^i e^a_i$$
.  $D_a v^i = \partial_a v^i + \omega^{\ i}_a{}_j v^j$ ,

the covariant derivative for a vector field

$$\omega_{a\ j}^{\ i} = \Gamma_{kj}^{i} e_{a}^{k}, \qquad \Gamma_{kj}^{i} = e_{k}^{d} e_{j}^{f} e_{c}^{i} \Gamma_{df}^{c} - e_{k}^{d} e_{j}^{f} \partial_{d} e_{f}^{i}.$$
 are the Christoffel

symbols (Levi-Civita connection)

$$D_a e_b^i = 0$$
,  
:  $D_a h_{bc} = 0$ 

the triad is covariant consistent Parallel transport is defined by

$$\mathrm{d}v^i = -\omega_a{}^i{}_j v^j \mathrm{d}x^a, \quad \Gamma_a^i = -\frac{1}{2} \ \omega_{ajk} \epsilon^{ijk}, \quad \delta\omega^i = \Gamma_a^i \ \mathrm{d}x^a, \quad \mathrm{d}v^i = \epsilon^i_{jk} v^j \delta\omega^k.$$

the Riemann curvature components are

$$\begin{split} R^i_{ab} &= 2\partial_{[a}\Gamma^i_{b]} + \epsilon^i_{jk}\Gamma^j_a\Gamma^k_b \\ R^i_{ab}e^b_i &= 0. \\ \text{with the Riemann scalar} \\ \text{re generalized impulse was introduced by Ashtekar 1986:} \\ \end{split}$$

th

Ashtekar variables  $A_a^{i}(x) = \Gamma_a^{i}(x) + \beta K_a^{i}(x)$ , with dimension  $[A_a^{i}] = 1/cm$ ,  $\beta$  Barbero-Immirzi parameter

 $A_a^i$  and  $E_b^j$  are canonically conjugate

$$\{A_{a}^{i}(x), E_{j}^{b}(y)\} = 8\pi\beta l_{p}^{2}\delta_{a}^{b}\delta_{j}^{i}\delta(x, y)$$
$$\{A_{a}^{i}(x), A_{j}^{b}(y)\} = 0$$

i.e. we can replace  $E_a^{i}$  by the operator  $E_a^{i} \Rightarrow \frac{\beta}{\iota l_r} \frac{\delta}{\delta A_i^{a}}$ 

#### 7.2. Discussion of the constraints

Gauss constraints  $G_i = \partial_a E_i^a + \varepsilon_{ijk} A_a^j E^{ka} \equiv D_a E_i^a \approx 0$ 

field strength tensor  $F_{ab}^{i} = \partial_{a}A_{b}^{i} - \partial_{b}A_{a}^{i} + \varepsilon_{ijk}A_{a}^{j}A_{b}^{k}$ 

$$\tilde{\mathcal{H}}_{\perp} = -\frac{\sigma}{2} \frac{\epsilon^{ijk} F_{abk}}{\sqrt{|\det E_i^a|}} E_i^a E_j^b + \frac{\beta^2 \sigma - 1}{\beta^2 \sqrt{|\det E_i^a|}} E_{[i}^a E_{j]}^b \Big( A_a^i - \Gamma_a^i \Big) \Big( A_b^j - \Gamma_b^j \Big) \approx 0$$

Hamiltonian constraint  $\sigma=-1$  Lorentzian ,  $\sigma=1$  Euclidean

diffeomorphism  $\tilde{\mathcal{H}}_a = F^i_{ab} E^b_i \approx 0.$ for  $\beta = \iota = \sqrt{-1}$ 

in the Lorentzian case

the Hamiltonian constraint simplifies 
$$\tilde{\mathcal{H}}_{\perp} = \epsilon^{ijk} F_{abk} E^a_i E^b_j \approx 0.$$

5 pdeqs order 1 in r,  $\theta$  non-linear (quadratic) for 6 symmetric  $E_i^a$  and 6 symmetric  $A_i^a$ 

### 7.3. 3-dimensional Ashtekar-Kodama constraints

We construct a theory based on the densitized inverse tetrad  $E_j^b(\mathbf{y})$  and the connection  $\frac{A_a^i(\mathbf{x})}{dt}$  with the commutator

$$[A_{a}^{i}(x), E_{j}^{b}(y)] = -8\pi l_{p}^{2} \delta_{j}^{i} \delta_{a}^{b} \delta(x, y) \beta i \text{ where } \kappa = \frac{8\pi l_{p}^{2}}{\hbar c} = \frac{8\pi G}{c^{4}} , 8\pi \hbar G = 8\pi l_{p}^{2} c^{3}$$

the operators act on the wave functional  $\Psi[A]$ 

$$\hat{A}_{a}^{i}(\mathbf{x})\Psi[A] = A_{a}^{i}(\mathbf{x})\Psi[A],$$

$$\hat{E}_{j}^{b}(y)\Psi[A] = -8\pi l_{p}^{2} \frac{\beta}{i} \frac{\delta\Psi[A]}{\delta A_{j}^{b}(y)}, \quad \hat{E}_{j}^{b}(y)\Psi[A] = -8\pi l_{p}^{2} \frac{3}{\lambda} \frac{\beta}{i} \varepsilon^{bcd} F_{cdj}$$

where  $\lambda = 8\pi l_P^2 \Lambda$ 

$$D_a \frac{\delta \Psi}{\delta A_a^i} = 0.$$

the Gauss constraint becomes

$$F^i_{ab} \frac{\delta \Psi}{\delta A^i_b} = 0.$$

the diffeomorphism constraint becomes

and the Hamiltonian constraint with  $\Lambda=0$  and  $\beta=\iota=\sqrt{-1}$ 

$$\epsilon^{ijk}F_{kab}\frac{\delta^2\Psi}{\delta A_a^i\delta A_b^j}=0$$

In the case of vacuum gravity with  $\Lambda \neq 0$ , an exact formal solution in the connection representation was found by Kodama 1990.

The Hamiltonian constraint becomes for  $\beta = \iota = \sqrt{-1}$ 

$$\varepsilon^{ijk} \frac{\delta}{\delta A_a^{i}} \frac{\delta}{\delta A_b^{j}} \left( F_{abk} - \frac{\Lambda}{3l_P} \varepsilon_{abc} \frac{\delta}{\delta A_c^{k}} \right) \Psi[A] = 0$$

with the global wave function

$$\Psi[A] = N \exp\left(\frac{3}{\lambda} \int_{\Sigma} d^{3}x \,\varepsilon^{abc} \, tr\left(A_{a} \partial_{a} A_{c} + \frac{1}{3} A_{a} A_{b} A_{c}\right)\right)$$

$$\frac{c^3}{G\hbar\Lambda} = \frac{1}{8\pi l_P{}^2\Lambda} = \frac{1}{\lambda}$$

derived from the Chern-Simons action

$$S_{CS}[A] = \int_{\Sigma} d^{3}x \,\varepsilon^{abc} \, tr \left( A_{a} \partial_{b} A_{c} + \frac{1}{3} A_{a} A_{b} A_{c} \right)$$
$$\varepsilon_{abc} \, \frac{\delta \Psi}{\delta A_{c}^{k}} = \frac{3}{\Lambda} F_{kab}$$

results from the variation of the Chern-Simons covariant Lagrangian

$$L_{CS} = \varepsilon^{\mu\nu\lambda} \left( A^{\kappa}_{\mu} \partial_{\nu} A_{\lambda\kappa} + \frac{1}{3} \varepsilon_{\kappa_{1}\kappa_{2}\kappa_{3}} A^{\kappa_{1}}_{\mu} A^{\kappa_{2}}_{\mu} A^{\kappa_{3}}_{\mu} \right)$$
$$\frac{\delta L_{CS}}{\delta A^{\sigma}_{\rho}} = \varepsilon^{\rho\nu\lambda} F_{\nu\lambda\sigma}$$

The resulting constraints are [4] [5] 3 Gauss constraints  $G_i = \partial_a E^a{}_i + \varepsilon_{ij}{}^k A_a{}^j E^a{}_k$ 3 diffeomorphism constraints  $D_a = E^b{}_i F_{ab}{}^i$ 3\*3=9 Hamiltonian constraint  $H_{(a,b)}{}^k = F_{ab}{}^k + \frac{\Lambda}{3} \varepsilon_{ab}{}^c E_c{}^k$ 

alltogether 15 pdeqs order 1 in r,  $\theta$  nonlinear (quadratic in  $E^a_i$  and  $A^a_i$ , cubic in both), for 9  $E^a_i$  and 9  $A^a_i$ 

#### A8. 4-dimensional Ashtekar-Kodama constraints

We can transform the 3-dimensional Ashtekar-Kodama equations uniquely into the 4-dimensional relativistic form by generalizing the  $\varepsilon$ -tensor from 3 spatial indices (1,2,3) to 4 spacetime indices (0,1,2,3), which is mathematically uniquely and well-defined.

with 16 variables  $E^{\mu\nu}$ : inverse densitized triad of the metric  $g_{\mu\nu}$ 

with 16 variables  $A_{\mu}^{\nu}$  connection tensor

spatial spacetime curvature  $F_{\mu\nu}^{\ \kappa} = \partial_{\mu}A_{\nu}^{\ \kappa} - \partial_{\nu}A_{\mu}^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}A_{\nu}^{\ \kappa_{2}}$ 

4 Gauss constraints  $G^{\mu} = \partial_{\nu} E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda} A_{\nu}{}^{\kappa} E^{\nu\lambda}$  (covariant derivative of  $E^{\mu\nu}$  vanishes)

4 diffeomorphism constraints  $I_{\mu} = E^{\kappa} F_{\mu\kappa}^{\nu}$ 

24 Hamiltonian constraints  $H_{(\mu,\nu)}^{\ \kappa} = F_{\mu\nu}^{\ \kappa} + \frac{\Lambda}{3} \varepsilon_{\mu\nu\rho} E^{\rho\kappa}$ 

The expression  $(\mu, \nu)$  in the index of H means that only *pairs*  $(\mu, \nu)$  where  $\mu \neq \nu$  in the first index yield different constraints, as the right side is antisymmetric in  $(\mu, \nu)$ , that results in 6\*4=24 Hamiltonian constraints. So we have 32 partial differential equations of degree 1, nonlinear (quadratic in  $E^{\mu\nu}$  and  $A^{\mu\nu}$ , cubic in both) in  $\{t, r, \theta\}$  for 32 variables, with the  $E_g^{\mu\nu}$  =tetrad $(g_{\mu\nu})$  boundary condition  $(r \rightarrow \infty)$  for  $E^{\mu\nu}$ .

#### 20 A9. BF-theory 9.1. Palatini action as BF-theory

Palatini action (Durka) (in the following the constant factor in the action  $\frac{1}{2\kappa} = \frac{c^4}{16\pi G}$  is skipped)

$$S=rac{1}{64\pi G}\,\int d^4x\,\epsilon^{abcd}(R_{\mu
u\,\,ab}\,e_{
ho\,k}\,e_{\sigma\,d}-rac{\Lambda}{3}e_{\mu\,\,a}e_{
u\,\,b}\,e_{
ho\,c}\,e_{\sigma\,d})\epsilon^{\mu
u
ho c}$$

Riemann tensor expressed by the GR connection  $\omega_{\mu}^{ab}$ 

$$\begin{aligned} R_{\mu\nu}{}^{ab} &= \partial_{\mu}\omega_{\nu}{}^{ab} - \partial_{\nu}\omega_{\mu}{}^{ab} + \omega_{\mu}{}^{a}{}_{c}\omega_{\nu}{}^{cb} - \omega_{\nu}{}^{a}{}_{c}\omega_{\mu}{}^{cb}, \qquad R^{a}{}_{b} &= d\omega^{a}{}_{b} + \omega^{a}{}_{c}\wedge\omega^{c}{}_{b} \end{aligned}$$
and tetrad derivatives
$$\begin{aligned} T_{\mu\nu}{}^{a} &= \mathcal{D}_{\mu}^{\omega}e_{\nu}{}^{a} - \mathcal{D}_{\nu}^{\omega}e_{\mu}{}^{a}, \qquad T^{a} &= \mathcal{D}^{\omega}e^{a} &= de^{a} + \omega^{a}{}_{b}\wedge e^{b} \end{aligned}$$

$$\begin{aligned} T_{\mu\nu}{}^{a} &= \mathcal{D}_{\mu}e_{\nu}{}^{a} - \mathcal{D}_{\nu}e_{\mu}{}^{a} \qquad D_{\mu}e_{\nu}{}^{a} &= \partial_{\mu}e_{\nu}{}^{a} + \varepsilon^{a}{}_{bcd}\omega_{\mu}{}^{bc}e_{\nu}{}^{d} \end{aligned}$$
with the covariant derivative
$$\begin{aligned} \mathcal{L}_{\text{Palatini}}[e,\omega] &= \int_{\mathcal{X}} \mathfrak{U}_{\alpha\beta}\wedge \mathbb{R}^{\alpha\beta} \qquad \text{Vey 1.88} \end{aligned}$$

$$*u \text{ Hodge- transformed} \qquad \begin{bmatrix} \mathbb{I}\mathfrak{U} \end{bmatrix}_{\mu\nu}^{IJ} &= \frac{1}{2}\varepsilon^{IJ}{}_{KL}e_{\mu}^{K}e_{\nu}^{L} \qquad u = \begin{pmatrix} \mathbb{U}_{\Lambda} \end{pmatrix}_{\mu\nu}^{IJ} &= e_{\alpha}^{I}e_{\beta}^{J} \\ 1.78 \qquad I^{Palatini} &= \int \epsilon_{abcd} \left( e^{a}\wedge e^{b}\wedge F^{cd} + \frac{\Lambda}{2}e^{a}\wedge e^{b}\wedge e^{c}\wedge e^{d} \right) \end{aligned}$$

$$[5 (35)] \text{ Lorentzian case} \end{aligned}$$

$$F_{\mu\nu}(\omega) = d\omega + \omega \wedge \omega \qquad [14], \end{aligned}$$

where  $\omega = (\omega_v)^{ab}$  is a matrix-vector and  $F_{\mu\nu} = (F_{\mu\nu})^{ab}$  is a matrix-matrix or 4-degree-tensor explicitly:

$$F_{\mu\nu}^{\ ab}(\omega_{\nu}^{\ ab}) = d\omega^{ab} + \omega^{ab} \wedge \omega^{ab} = \partial_{\mu}\omega_{\nu}^{\ ab} - \partial_{\nu}\omega_{\mu}^{\ ab} + \omega_{\mu}^{\ a}{}_{c}\omega_{\nu}^{\ cb} - \omega_{\nu}^{\ a}{}_{c}\omega_{\mu}^{\ cb}$$

$$F^{ab}(\omega^{ab}) = R_{\mu\nu}^{\ ab}$$

eom's of Palatini action:

$$\frac{\delta I^{Pal}}{\delta A^{ab}} = \nabla e^a \wedge e^b \quad \nabla e^a \wedge e^b = 0 \quad \text{, solution } A^{ab} = \omega^{ab}$$

where  $\omega$  is the SO(4) spin connection

 $\frac{\delta I^{Pal}}{\delta e_{\mu}^{a}}$  corresponding derived Einstein equations

$$\epsilon_{abcd} \left( e^b \wedge F^{cd} + \Lambda e^a \wedge e^b \wedge e^c \wedge e^d \right) = 0.$$

#### 9.2. Plebanski action as BF-theory

original Lorentzian Plebanski action Smolin [5] (26)

$$I^{\text{Plebanski}} = i \int B^{i} \wedge F_{i} - \frac{1}{2} \phi_{ij} B^{i} \wedge B^{j}$$
where  $i = \sqrt{-1}$  and  $\phi$  ij generates the cosmological constant  
we add a Chern-Simons boundary term to the action  $S_{CS}[A] = \frac{3i}{\lambda} \int_{\Sigma} d^{3}x \,\varepsilon^{abc} tr \left( A_{a} \partial_{b} A_{c} + \frac{1}{3} A_{a} A_{b} A_{c} \right)$ 

to enforce on the boundary  $F^i = -\frac{\Lambda}{3}B^i$ 

explicitly:

$$I^{Pleb} = \iota \int d^4 x \, \varepsilon^{abcd} \left( B_{ab}^{\ i} F_{cdi} - \frac{1}{2} \varphi_{ij} B_{ab}^{\ i} B_{cd}^{\ j} \right) + \frac{3 \iota}{\lambda} \int_{\Sigma} d^3 x \, \varepsilon^{abc} \, tr \left( A_a \partial_b A_c + \frac{1}{3} A_a A_b A_c \right)$$
  
$$\phi_i^{\ i} = -\Lambda; \qquad \phi_{[ij]} = 0$$

the eom's are

$$\frac{\delta I^{Pleb}}{\delta A} = 2D \wedge B \qquad, \frac{\delta I^{Pleb}}{\delta A_{\rho}^{\sigma}} = 2\varepsilon^{ab\rho\sigma} \left(\partial_{\rho} B_{ab}^{\sigma} + \varepsilon_{i\rho\kappa} A_{\rho}^{\kappa} B_{ab}^{i}\right)$$
$$\frac{\delta I^{Pleb}}{\delta B^{i}} = F^{i} - \phi^{i}{}_{j}B^{j}$$

 $\Phi^i_j = -\frac{\Lambda}{3} \delta^i_j \qquad F^i = -\frac{\Lambda}{3} B^i$  and we get a solution

if we set B equal to the self-dual tetrad field, we get  $B_{ab}^{\ \ i} = \varepsilon_{abc} E^{ci}$ 

$$F_{ab}{}^{i} = -\frac{\Lambda}{3}\varepsilon_{abc}E^{ci}$$

, which gives the Hamiltonian Ashtekar-Kodama constraint

$$0 = D \wedge B = D_a E^{ai}$$

the first eom becomes for spatial indices

and by generalization for covariant 4 indices,

which gives the Gaussian constraint.

The Palatini action can be derived from the more general Plebanski action, setting

$$\phi_{ij} = -\Lambda \qquad B_{ab}^{\ i} = \varepsilon^{i}{}_{jk} e_{a}^{\ j} e_{b}^{\ k}$$
and

#### 9.3. From the Plebanski action to the Einstein-Hilbert action

the general ansatz for the Plebanski action is

$$S(B,\omega,\phi) = \int B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2}\phi_{IJKL} B^{IJ} \wedge B^{KL} - \frac{1}{6} \left(\frac{\lambda}{2}\epsilon_{IJKL} + \mu \,\delta_{IJKL}\right) B^{IJ} \wedge B^{KL}.$$

From the resulting eom's one get the new action with a parameter  $\gamma = 1/\beta$ , where  $\beta$  is the Immirzi parameter.

$$S(e,\omega) = \int (\star + \gamma) e^{I} \wedge e^{J} \wedge F_{IJ} - 2 e \left(\lambda (1 + \gamma^{2}) + 2\mu\gamma\right)$$

and \* is the Hodge-operator \*  $e_a^{\ J} = \varepsilon^{\ Ia}{}_J e_a^{\ J}$  is the "anisymmetrized tetrad" and

$$e = \frac{1}{24} \varepsilon_{abcd} \varepsilon_{\mu\nu\kappa\lambda} e_{\mu}^{\ a} e_{\nu}^{\ b} e_{\kappa}^{\ c} e_{\lambda}^{\ d}$$

with the cosmological constant  $\Lambda = \lambda (1+\gamma^2) + 2\mu\gamma.$ 

 $S(e) = \int e(R - 2\Lambda) + \gamma \epsilon \cdot R,$ and  $\epsilon \cdot R \equiv \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0$  due to the Bianchi identity we get the action

 $R_{\mu\nu\rho\sigma} = 0$ , so we get the Einstein-Hilbert action with a cosmological constant

21

and finally

# 22 Part B Ashtekar-Kodama gravity

# **4-dimensional Kodama-Ashtekar equations** 16 variables $E^{\mu\nu}$ : inverse densitized triad of the metric $g_{\mu\nu}$ 16 variables $A_{\mu}^{\nu}$ connection tensor spatial spacetime curvature $F_{\mu\nu}^{\ \kappa} = \partial_{\mu}A_{\nu}^{\ \kappa} - \partial_{\nu}A_{\mu}^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}A_{\nu}^{\ \kappa_{2}}$ 4 Gauss constraints $G^{\mu} = \partial_{\nu}E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda}A_{\nu}^{\ \kappa}E^{\nu\lambda}$ 4 diffeomorphism constraints $I_{\mu} = E^{\kappa}{}_{\nu}F_{\mu\kappa}^{\ \nu}$ 24 Hamiltonian constraints $H_{(\mu,\nu)}^{\ \kappa} = F_{\mu\nu}^{\ \kappa} + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa}$

# AK covariant derivative and its gauge group

$$\begin{split} D_{\mu}t_{\nu}^{\ \lambda} &= \partial_{\mu}t_{\nu}^{\ \lambda} + \varepsilon^{\lambda}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}t^{\nu\kappa_{2}} \\ \text{left spin-1/2 representation of the Lorentz-algebra with 4 generators} \\ \tau_{i} &= T_{+}^{\ i}; i = 1, 2, 3 \\ \tau_{0} &= (T_{+}^{\ 1} + T_{-}^{\ 1}) - (T_{+}^{\ 2} + T_{-}^{\ 2}) + (T_{+}^{\ 3} + T_{-}^{\ 3}) \\ 4 \text{ extended generators } \tau_{i} \text{ satisfy the extended SU(2) commutator algebra with spacetime indices } \{0, 1, 2, 3\} \\ \left[\tau_{\kappa}, \tau_{\lambda}\right] &= i\varepsilon_{\kappa\lambda\mu}\tau_{\mu} \end{split}$$

Renormalizable Einstein-Hilbert action with the Ashtekar momentum  $A_{\mu}^{\nu}$  Einstein-Hilbert action

$$S = \frac{\hbar c}{16\pi {l_P}^2} \int (R - 2\Lambda) \sqrt{-g} d^4 x \quad , \ \kappa = \frac{8\pi {l_P}^2}{\hbar c}$$

half-asymmetric background  $A_{\mu}^{\nu} = \frac{1}{l_p} \begin{bmatrix} 1, 1, -1, 1\\ 1, 1, -1, 1\\ 1, 1, -1, 1\\ 1, 1, -1, 1\\ 1, 1, -1, 1\\ 1, 1, -1, 1 \end{bmatrix} = \frac{1}{l_p} \Omega_{\mu}^{\nu}$ 

reformulated Einstein-Hilbert action with  $\Lambda \approx 0$ 

 $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} d^{4}x$  is dimensionally renormalizable

variation with respect to  $g_{\mu\nu}$  yields the Einstein equation as before

variation with respect to 
$$A_{\mu}^{\nu}$$
 gives  $\frac{\partial}{\partial A_{\mu}^{\nu}} \frac{\hbar c}{\pi} (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} = -16 \frac{l_p}{r_s} \Omega_{\mu}^{\nu} T \sqrt{-g}$   
This is  $\approx 0$  in the classical region, so the eom is satisfied.

# 23 Solutions of static equations

# **Solution limit** $\Lambda \rightarrow \theta$ :

A-tensor becomes a *constant half-antisymmetric background*  $A_{hab}$  in the form  $AO_i = AOOc \{1, 1, -1, 1\}, AI_i = AIOc \{1, 1, -1, 1\}, AZ_i = A2Oc \{1, 1, -1, 1\}, AJ_i = A3Oc \{1, 1, -1, 1\}$ E-tensor is the Gauss-Schwarzschild tetrad  $E_{GS}$ , satisfying gaussian equations  $\frac{\partial_{\theta} E^{2\nu}}{r} + \partial_r E^{1\nu} = 0$ , and the metric condition for all r > 1  $E \eta E^t = g^{-1} / (-\det(g))^{3/4}$ 

the metric generated by  $E_{GS}$  is the Schwarzschild metric and the Einstein equations are satisfied

# Solution $\Lambda \neq 0$ with the half-logarithmic ansatz

solution E in  $(r_{th}, \theta)$ ,  $r_{th} = \theta + log(r)$   $Eb_{ij}(r, \theta) = \pm Eb_{ij}(r_{th}) + \sum \frac{c_{1kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}(r_{th}) + \sum \frac{c_{4kl}}{L} \frac{Ab_{0k}'(r_{th})}{\exp(r_{th} - \theta)} + \sum \frac{c_{5kl}}{L} \frac{Ab_{3k}'(r_{th})}{\exp(r_{th} - \theta)} + \sum \frac{c_{2kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}'(r_{th})(r_{th} - \theta) + \sum \frac{c_{3kl}}{L} Ab_{0k}'(r_{th}) Ab_{3l}(r_{th})(r_{th} - \theta)$ metric condition: half-logarithmic Schwarzschild metric

# Behavior at Schwarzschild horizon

Schwarzschild tetrad diverges

 $E_{dS}^{0,0} = \frac{1}{r\sqrt{r-1}\sin^{3/4}(\theta)} \rightarrow \infty \text{, so the term } \frac{\Lambda}{3}E^{\mu\nu} \text{ becomes significant}$ at r = 1 + dr,  $dr = \sqrt{\Lambda}$ , i.e.  $E00(\theta) = \frac{1}{\Lambda^{1/4}\sin^{3/4}(\theta)}$ , the peak in the metric is  $g_{1,1} = \frac{1}{\sqrt{\Lambda}}$ gravitational limit for the quantum realm becomes  $r_{gr} = \sqrt{l_p\sqrt{\frac{1}{\Lambda}}} = 3.1*10^{-5}m = 31\mu m$ objective collapse theory links the spontaneous collapse of the wave function to quantum gravitation, this

puts the limit for quantum behavior at  $r \le r_{or}$ 

# 24 Solutions of time-dependent equations

Λ-scaled wave ansatz  $A_{\mu}^{\nu} = Ab_{\mu}^{\nu} + \Lambda \frac{As_{\mu}^{\nu}}{r} \exp(-ik(r-t))$   $E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp(-ik(r-t))$ 

# Wave equation in Schwarzschild spacetime: solutions

solution lx=0 spherical wave: incoming wave, only zero solution solution lx=1 dipole wave: divergent, only zero soltion

solution lx=2 quadrupole wave:

-the E-tensor is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

-the A-tensor components As0 and As1 are pure quadrupole waves, As2 is a linearly damped quadrupole wave,

As3 is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

#### Wave equation in binary rotator spacetime

bgr described by Kerr spacetime with  $\alpha = \frac{c_0}{r_0}$ eqtoiev  $\Lambda$ -scaled wave ansatz, backgrund equation eqtoeivnu3b=eqtoiv standard solution: Eb-tensor= the Kerr-Schwarzschild-tetrad  $E_{KS}$ : Ab-tensor  $Ab = A_{hab} + dAb$  perturbed half-antisymmetric background wave equation eqtoievnu3wdA = eqtoievnu3wdA(As, Es,  $\alpha$ , k)

solution of wave equation of bgr as a series in r-powers by comparison of coefficients

result: free parameter  $As00(r, \theta, r_0) = \frac{As00n01}{r_0} + \dots$ 

 $As1 \approx As0, \{As2, As3\} = O(1/r^2), Es2 = O(1/r^2), \{Es0, Es1, Es3\} = O(1/r)$ function(As00n01)

#### Numerical solutions

static *eqtoiv* with full coupling  $(\Lambda = I)$ : Ritz-Galerkin method with trigonometric polynomials in  $\theta$ metric in AK-gravity with coupling: no horizon and no singularity

time-dependent *eqtoiev* with weak coupling ( $\Lambda = 0.001$ ) and binary gravitational rotator (bgr) with  $r_0 = 1$  Ritz-Galerkin method with trigonometric polynomials in  $\theta$ 

#### gravitational Ashtekar-Kodama energy

AK grav. wave energy density 
$$t_{\mu\nu} = D_{\kappa}A_{\mu}^{\ \kappa}D_{\lambda}A_{\nu}^{\ \lambda}\hbar c\left(\frac{1}{l_{p}^{\ 2}\Lambda^{2}r_{s}^{\ 2}}\right)$$
  
GR  $t_{\mu\nu} = \frac{\hbar c}{16\pi l_{p}^{\ 2}}k_{\mu}k_{\nu}\left(e^{\lambda\kappa^{*}}e_{\lambda\kappa}-\frac{1}{2}\left|e^{\lambda}_{\lambda}\right|^{2}\right)$   
Einstein power formula for bgr  $P_{GR} = \frac{\hbar c^{2}}{2l_{p}^{\ 2}}\frac{r_{s}^{\ 5}}{r_{0}^{\ 5}}\left(\frac{m_{1}m_{2}}{m^{2}}\right)^{2}$   
power AK-gravity for bgr  $P_{KA} = k_{0}^{\ 2}As_{00}^{\ 2}4\pi \hbar c^{2}\left(\frac{1}{l_{p}r_{s}}\right)^{2}$   
result:  $As_{00} = \frac{f_{m}}{4\sqrt{2\pi}}\frac{r_{s}^{\ 3}}{r_{0}}$  with  $f_{m} = \frac{m_{1}m_{2}...m_{n}}{m^{n}} = \frac{m_{r}}{m}$ 

#### Lagrangian of AK-gravitation

electrodynamics: Maxwell lagrangian  $L_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ 

diffeomorph lagrangian  $L_{I} = \hbar c C_{\mu} C^{\mu} = \hbar c E^{\kappa_{1}}{}_{\nu_{1}} F_{\mu\kappa_{1}}{}^{\nu_{1}} E^{\kappa_{2}}{}_{\nu_{2}} F^{\mu}{}_{\kappa_{2}}{}^{\nu_{2}}$ hamiltonian lagrangian  $L_{H} = -\hbar c \left( \frac{1}{4} F_{\mu\nu}{}^{\kappa} F^{\mu\nu}{}_{\kappa} + \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}{}_{\mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}{}_{\kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}{}^{\lambda_{2}} A_{\mu_{1}}{}^{\mu_{2}}) \right)$ the complete AK lagrangian is then  $L_{gr} = L_{H} + L_{I} = \hbar c \left( -\frac{1}{4} F_{\mu\nu}{}^{\kappa} F^{\mu\nu}{}_{\kappa} - \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}{}_{\mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}{}_{\kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}{}^{\lambda_{2}} A_{\mu_{1}}{}^{\mu_{2}}) \right)$ where  $\Lambda$  is generated by a scalar field  $\varphi_{\Lambda}$ 

#### B1. 4-dimensional Ashtekar-Kodama equations and their properties

We can transform the 3-dimensional Ashtekar-Kodama equations *uniquely* into the 4-dimensional relativistic form by generalizing the  $\varepsilon$ -tensor from 3 spatial indices (1,2,3) to 4 spacetime indices (0,1,2,3), which is mathematically uniquely and well-defined.

with 16 variables  $E^{\mu\nu}$ : inverse densitized triad of the metric  $g_{\mu\nu}$ 

with 16 variables  $A_{\mu}^{\nu}$  connection tensor

spatial spacetime curvature  $F_{\mu\nu}^{\ \kappa} = \partial_{\mu}A_{\nu}^{\ \kappa} - \partial_{\nu}A_{\mu}^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}A_{\nu}^{\ \kappa_{2}}$ 

4 Gauss constraints  $G^{\mu} = \partial_{\nu} E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda} A_{\nu}{}^{\kappa} E^{\nu\lambda}$  (covariant derivative of  $E^{\mu\nu}$  vanishes)

4 diffeomorphism constraints  $I_{\mu} = E^{\kappa} F_{\mu\kappa}^{\nu}$ 

24 Hamiltonian constraints  $H_{(\mu,\nu)}^{\ \kappa} = F_{\mu\nu}^{\ \kappa} + \frac{\Lambda}{3} \varepsilon_{\mu\nu\rho} E^{\rho\kappa}$ 

The expression  $(\mu, \nu)$  in the index of H means that only *pairs*  $(\mu, \nu)$  where  $\mu \neq \nu$  in the first index yield different constraints, as the right side is antisymmetric in  $(\mu, \nu)$ , that results in 6\*4=24 Hamiltonian constraints. So we have 32 partial differential equations of degree 1, nonlinear (quadratic in  $E^{\mu\nu}$  and  $A^{\mu\nu}$ , cubic in both) in  $\{t, r, \theta\}$  for 32 variables, with the  $E_g^{\mu\nu}$  =tetrad $(g_{\mu\nu})$  boundary condition  $(r \rightarrow \infty)$  for  $E^{\mu\nu}$ .

 $E_g^{\mu\nu}$  is the solution of the original defining *densitized tetrad* equation  $E^{\mu\kappa}E^{\nu}{}_{\kappa} = g^{\mu\nu}/(-\det(g))^{3/4}$  or in matrix-notation for d=4:  $E \eta E^t = g^{-1}/(-\det(g))^{3/4}$  with the Lorentz signature  $\eta = -diag(1,-1,-1,-1)$ , which is generalized from the densitized triad equation for d=3:  $E \eta E^t = g^{-1}/(-\det(g))$  with the *scaling behavior* 

 $det(E) = \frac{1}{det(g)^2}$ . As is easily shown, the densitized tetrad has the same scaling behavior  $det(E) = \frac{1}{det(g)^2}$  and

for the scaling transformation with a scalar  $\alpha \ g \to \alpha \ g$  follows  $E \to \frac{E}{\alpha^2}$  for both d=3 and d=4.

For the (normalized with  $r_s=1$ ) Schwarzschild metric in spherical coordinates

$$g_{\mu\nu} = -diag\left[\left(1-\frac{1}{r}\right), -\frac{1}{\left(1-\frac{1}{r}\right)}, -r^2, -r^2\sin^2\theta\right]$$

the diagonal tetrad solution is

$$(E_{dS})^{\mu\nu} = diag\left(\frac{1}{\sqrt{r-1}r\sin(\theta)^{3/4}}, \frac{\sqrt{r-1}}{r^{2}\sin(\theta)^{3/4}}, \frac{1}{r^{5/2}\sin(\theta)^{3/4}}, \frac{1}{r^{7/2}\sin(\theta)^{3/4}}\right)$$

but, as the tetrad equation has 10 equations for 16 variables, there are 6 degrees of freedom (dof) left.

So we can enforce in addition the validity of the Gauss constraint: this can be achieved, and the solution  $(E_{GS})^{\mu\nu}$  can be calculated in a half-analytical form.

# 27**1.1. AK covariant derivative and its gauge group**

Here the covariant derivative (of the SO(3) group as gauge group) acting on a tensor  $t^{\nu\lambda}$  is  $D_{\mu}t_{\nu}^{\ \lambda} = \partial_{\mu}t_{\nu}^{\ \lambda} + \varepsilon^{\lambda}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}t^{\nu\kappa_{2}} \quad (D_{\mu})^{\lambda}{}_{\kappa} = \partial_{\mu} + \varepsilon^{\lambda}{}_{\kappa_{1}\kappa}A_{\mu}^{\ \kappa_{1}}$ where  $F_{\mu\nu}^{\ \kappa} = [D_{\mu}^{\ \lambda}, D_{\nu}^{\ \lambda}]$   $D_{\mu} = \partial_{\mu} - iA_{\mu}^{\ a}\tilde{\tau}^{a}$ , where  $\tilde{\tau}^{a} = i\varepsilon_{\nu}^{\ a}{}_{\lambda}$  satisfy the *extended SU(2) Lie-algebra*   $[\tilde{\tau}^{a}, \tilde{\tau}^{b}] = i\varepsilon^{abc}\tilde{\tau}^{c}$  $\varepsilon^{\lambda}{}_{\kappa_{1}\kappa_{2}}$  are the structure constants of the arter ded SU(2) Lie algebra

 $\mathcal{E}_{\kappa_1\kappa_2}$  are the structure constants of the *extended SU*(2) *Lie-algebra* A well-known representation of this *extended SU*(2) *Lie-algebra* are the following 4x4 martices  $\tau_i = T_+^{\ i}; i = 1,2,3$ 

$$\tau_{0} = (T_{+}^{1} + T_{-}^{1}) - (T_{+}^{2} + T_{-}^{2}) + (T_{+}^{3} + T_{-}^{3})$$

The  $T_+, T_-$  are combinations of the 6 generators of the Lorentz group:

$$T_{\pm}^{k} = \frac{1}{2} \left( J^{k} \pm K^{k} \right)$$

of the 3 spatial rotators  $J^k$  and the 3 boosts  $K^k$ , which are 4x4 matrices derived from the 4-tensor generator  $(M^{\mu\nu})^{\rho}{}_{\sigma} = -i(\eta^{\mu\nu}\delta_{\sigma}{}^{\nu} - \eta^{\nu\rho}\delta_{\sigma}{}^{\mu})$ ,  $J^k = \frac{1}{2}\varepsilon^{ijk}M^{ij}$ ,  $K^k = M^{0k}$  where  $\eta$  is the Minkowski metric, e.g.

The  $T_{+}^{i}$  are the generators of the left spin-1/2 representation of the Lorentz-algebra SO(1,3) and  $T_{-}^{i}$  are the generators of the right spin-1/2 representation of the Lorentz-algebra SO(1,3),

the 3 generators  $\tau^i$  satisfy with spatial indices i=1,2,3: the ordinary SU(2) algebra  $\left[T_{+}^{i}, T_{+}^{j}\right] = i\varepsilon_{ijk}T_{+}^{k}, \quad \left[T_{-}^{i}, T_{-}^{j}\right] = i\varepsilon_{ijk}T_{-}^{k}, \quad \left[T_{+}^{i}, T_{-}^{j}\right] = 0$ 

and the 4 extended generators  $\tau^{\mu}$  satisfy the *extended SU*(2) algebra with spacetime indices  $\mu = \{0, 1, 2, 3\}$  $[\tau_{\kappa}, \tau_{\lambda}] = i\varepsilon_{\kappa\lambda\mu}\tau_{\mu}$ 

#### 28 1.2. Renormalizable Einstein-Hilbert action with the Ashtekar momentum $A_{\mu}^{\nu}$

semiclassical Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle.$$
(Kiefer 1.37)

Einstein-Hilbert action

$$S = \frac{1}{2\kappa} \int R\sqrt{-g} \, \mathrm{d}^4 x \qquad \kappa = 8\pi G c^{-4} \quad \kappa = \frac{8\pi l_P^2}{\hbar c} \quad S = \frac{\hbar c}{16\pi l_P^2} \int R\sqrt{-g} \, \mathrm{d}^4 x \text{, with lambda:}$$

$$S = \frac{\hbar c}{16\pi l_P^2} \int (R - 2\Lambda) \sqrt{-g} \, \mathrm{d}^4 x$$

setting  $A_{\mu}^{\nu} = \frac{1}{l_{p}} \begin{pmatrix} 1, 1, -1, 1\\ 1, 1, -1, 1\\ 1, 1, -1, 1 \end{pmatrix} = \frac{1}{l_{p}} \Omega_{\mu}^{\nu}$ 

(constant background in the Ashtekar-Kodama equations) , one can reformulate the Einstein-Hilbert action with  $\Lambda{\approx}0$ 

 $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} d^4 x$ , which makes it dimensionally renormalizable, with the dimensionless

interaction constant  $g_{gr} = \frac{1}{\pi}$ . Variation with respect to  $g_{\mu\nu}$  yields then, as before, the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad \text{or equivalent} \quad R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right)$$

From this we derive with  $\Lambda \approx 0$ :

$$R = g^{\nu\mu}R_{\mu\nu} = \kappa(T - \frac{T}{2}g^{\nu\mu}g_{\mu\nu}) = -\kappa T$$

Now, variation with respect to  $A_{\mu}^{\nu}$  gives the left side of the equation-of-motion (eom)

$$\frac{\partial}{\partial A_{\mu}^{\nu}}\frac{\hbar c}{\pi}(A_{\mu}^{\nu}A_{\nu}^{\mu})R\sqrt{-g} = -\frac{\hbar c}{\pi}2A_{\mu}^{\nu}\kappa T\sqrt{-g} = -16l_{P}\Omega_{\mu}^{\nu}T\sqrt{-g}$$

The above expression is calculated, as usually, dimensionless, with correct dimension we have

$$\frac{\partial}{\partial A_{\mu}^{\nu}} \frac{\hbar c}{\pi} (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} = -16 \frac{l_{p}}{r_{s}} \Omega_{\mu}^{\nu} T \sqrt{-g}$$

This is  $\approx 0$  in the classical region, so the eom is satisfied.

It is interesting to assess the dimensionless factor  $f_{gr} = \frac{l_p}{r_s}$ :  $f_{gr} = 5.3 \times 10^{-39}$  for  $r_s = 3km$  (Schwarzschild of sun),

which is about the ratio  $f_{grem} = \frac{E_{gr}}{E_{em}} \approx 10^{-40}$  of the gravitational and electrodynamic interaction strength

# 29 **B2. The basic equations**

AK equations 24 hamiltonian scheme  $A \bullet A + \partial A + (\Lambda/3)E$ 4 gaussian scheme  $A \bullet E + \partial E$ 4 diffeomorphism scheme  $E \bullet A \bullet A + E \bullet \partial A$ coordinates {*t*, *r*, *θ*} derivatives order 1:  $\partial_t$ ,  $\partial_r$ ,  $\partial_\theta$ 





The Ashtekar-Kodama equations (AKe) consist of

24 hamiltonian equations with the expression scheme  $A \bullet A + \partial A + (\Lambda/3)E$ 

4 gaussian equations with the expression scheme A $\bullet$ E+  $\partial$ E

4 diffeomorphism equations with the expression scheme  $E \bullet A \bullet A + E \bullet \partial A$ 

where • represents multiplicative terms and  $\partial$  means derivatives for covariant coordinates, here the spherical coordinates spacetime {*t*, *r*,  $\theta$ ,  $\varphi$ }

$$\partial^{\mu} = (\partial_{t}, \partial_{r}, \frac{1}{r}\partial_{\theta}, \frac{1}{r\sin\theta}\partial_{\varphi})$$

We consider here only spacetimes with axial symmetry, i.e.  $\partial_{\varphi} = 0$  and the variables  $E^{\mu\nu}$  and  $A_{\mu}^{\nu}$  are functions of  $\{t, r, \theta\}$ 

### 2.1. The integrability conditions

In the static (time-independent) AKe equations eq1..4 and eq13..16 contain resp.  $\partial_r A0i$  and  $\partial_{\theta} A0i$  as the only derivative, also eq9..12 and eq17..20 contain resp.  $\partial_{\theta} A3i$  and  $\partial_r A3i$  as the only derivative.

Therefore we have to impose integrability conditions  $\partial_{\theta} \partial_r A 0i = \partial_r \partial_{\theta} A 0i$  and  $\partial_{\theta} \partial_r A 3i = \partial_r \partial_{\theta} A 3i$ .

This changes the expression scheme for in eq9..12, eq13..16 :  $A \bullet A + A \bullet \partial A + (\Lambda/3)(\partial E + E)$ 

Accordingly in the time-dependent AKe equations eq9..12 and eq21..24 are transformed.

Equations with integrability condition static ( $\partial_t=0$ ): *eqtoiv* 

Equations with integrability condition time-dependent: eqtoiev

```
(* eqtocv;
   rvars= A2i,E1i;
   thvars=A1i,E2i;
   rthvars=A0i,A3i;
   avars=E0i,E3i; *)
 (* eqtoiv;
 rvars= A2i,E1i;
   thvars=A1i,E2i;
   rthvars=A0i,A3i,E3i,E0i;
    avars={};
                               changed eq9...12: no single DthA3, instead Dr, Dth all;
 *)
                               changed eq13...16: no single DthA0, instead Dr, Dth all
 (* eqtocev;
 tvars= A1i,A2i,A3i;E0i;
 rvars= A2i,E1i;
   thvars=A1i,E2i;
   rthvars=A0i,A3i;
   avars=E3i; *)
(* eqtoiev;
 tvars= A1,A2,A3,E0,E2;
 rvars= A2, E1;
   thvars=A1;
   rthvars=A0,A3,E0,E2;
    avars=E3;
 *)
```

The static equations *eqtoiv* are 32 pdeq's of first order in r,  $\theta$ , quadratic in the variables  $E^{\mu\nu}$  and  $A_{\mu}^{\nu}$  in the 24 hamiltonian equations and 4 gaussian equations and cubic in the variables  $E^{\mu\nu}$  and  $A_{\mu}^{\nu}$  in the last 4 diffeomorphism equations.

The row-variables in the A- tensor and the E-tensor have different derivative behavior:  $A2_i$  and  $E1_i$  are pure *r*-variables (only  $\partial_r$  derivative present),  $A1_i$  and  $E2_i$  are pure  $\theta$ -variables (only  $\partial_{\theta}$ derivative present),  $(A0_i, E0_i, A3_i, E3_i)$  are *r*- $\theta$ -variables (both  $\partial_r$  derivative and  $\partial_{\theta}$  derivative present). The time-dependent equations eqtoiev are 32 pdeq's of first order in *t*, *r*,  $\theta$ , quadratic in the variables  $E^{\mu\nu}$  and  $A_{\mu}^{\nu}$  in the 24 hamiltonian equations and 4 Gaussian equations and cubic in the variables  $E^{\mu\nu}$  and  $A_{\mu}^{\nu}$  in the last 4 diffeomorphism equations.

Here  $A_{2_i}$  and  $E_{1_i}$  are *r*-variables ,  $A_{1_i}$  are  $\theta$ -variables ,  $(A_{0_i}, E_{0_i}, A_{3_i}, E_{2_i})$  are *r*- $\theta$ -variables ,  $(A_{1_i}, A_{2_i}, A_{3_i}, E_{0_i}, E_{2_i})$  are *t*-variables ( $\partial_t$  derivative present) and  $E_{3_i}$  are algebraic variables (no derivative present).

The overall scheme of the static equations *eqtoiv* becomes 24 hamiltonian  $A \bullet A + A \bullet \partial A + (\Lambda/3)(\partial E + E)$  or  $A \bullet A + \partial A + (\Lambda/3) E$ 4 gaussian  $A \bullet E + \partial E$  31 4 diffeomorphism  $A \bullet A \bullet E + E \bullet \partial A$ The overall scheme of the static equations *eqtoiev* becomes 24 hamiltonian  $A \bullet A + A \bullet \partial A + (\Lambda/3)(\partial E + E)$  or  $A \bullet A + A \bullet \partial A + (\Lambda/3)\partial E$  or  $A \bullet A + \partial A + (\Lambda/3)E$ 4 gaussian  $A \bullet E + \partial E$ 4 diffeomorphism  $A \bullet A \bullet E + E \bullet \partial A$ 

### 2.2. Solvability of static and time-dependent equations eqtoiv, eqtoiev

By setting the A-variables and E-variables with derivatives and the coordinates r,  $\theta$  to random values one can

determine the rank of the Jacobi derivative-equation matrix  $\frac{\partial eq_i}{\partial (\partial_\kappa A_\mu^{\nu}, \partial_\kappa E^{\mu\nu})}$ , i.e. the solvability of the

equations for the highest derivatives. This ensures, given appropriate boundary conditions, the solvability of the partial differential equations system (pdeq) in a vicinity of the boundary, according to the famous theorem by Kovalevskaya.

Dr1(rvars1),Dth(thvars1),Dr1(rthvars1),Dth(rthvars1), fvars elim: free A2,E0,Dr1{E2,E3}-> free 12 params 2. systematic elim, L=1: {E1,A2 free} rank 24

solved Dt(tvarsA),<u>Dth(thvarsA),Dr(rthvars),Dth(rthvars)</u>,Dt(<u>trthvars</u>),Dr(<u>trthvars</u>),<u>Dth(trthvars</u>),no free f1varse, uniquely solvable genuine 32 <u>dvars</u>={<u>Dth</u>(A1,E0,A02,A03),Dr1(A3,A23),Dt(A1,A2,A33,E0,E2) }

The result for *eqtoiv* is : the rank of Jacobi matrix is 24, there are 8 free parameters  $(A2_i, E1_i)$ The Jacobi matrix of *eqtoiev* has full rank: the equations are solvable for the 32 derivatives  $\partial_{\theta} (A1_i, E0_i, A02, A03)$ ,  $\partial_r (A3_i, E0_i, A23)$ ,  $\partial_t (A1_i, A2_i, E0_i, E2_i, A33)$ 

# 32 **B3. Solutions of static equations**

# **3.1. Solution limit** $\Lambda \rightarrow \theta$

Solution *eqtoiv*  $\Lambda \rightarrow 0$  : Einstein equations valid, Schwarzschild & Kerr-spacetime



When  $\Lambda = 0$ , the tetrad  $E^{\mu\nu}$  decouples in the hamiltonian equations from the graviton tensor  $A_{\mu}^{\nu}$ , the 24 hamiltonian equations are in general overdetermined with 16 variables of the A-tensor. By stepwise elimination we get the following solution:

A1 = {1, 1, -1, 1} A10c; A0 = {1, 1, -1, 1} A00c; A2 = {1, 1, -1, 1} A20c; A3 = {1, 1, -1, 1} A30c; eqdiff = 0, eqgauss =: Dth E2 / r1 + Dr1 E1 =: 0 The A-tensor becomes a *constant half-antisymmetric background*  $A_{hab}$  in the form  $AO_i = AOOc \{1, 1, -1, 1\}, AI_i = A10c \{1, 1, -1, 1\}, A2_i = A20c \{1, 1, -1, 1\}, A3_i = A30c \{1, 1, -1, 1\}$ The diffeomorphism equations vanish identically, we are left with the 4 gaussian equations for the E-tensor  $\partial E^{2\nu}$ 

 $\frac{\partial_{\theta} E^{2\nu}}{r} + \partial_{r} E^{1\nu} = 0$ , and the E-tensor has to satisfy the 10 equations metric condition at r  $\rightarrow$  infinity  $E \eta E^{t} = g^{-1} / (-\det(g))^{3/4}$ 

Now with 16 variables both the Gaussian equations and the metric condition *can be satisfied for all r>1*, so the Einstein equations are satisfied, and *GR is valid* not only in the limit r—infinity, but *everywhere for r>1*. The only exception arises at the horizon (Schwarzschild or Kerr), where the E-tensor diverges, and the coupling reappears in the hamiltonian equations. In this case there is *no singularity*, but only a *peak for r→1*.

#### 3.1.1. The Gauss-Schwarzschild tetrad

The metric condition for Schwarzschild spacetime has a diagonal solution, *diagonal Schwarzschild tetrad*  $E_{dS} =$ 

$$\left\{\left\{\frac{1}{\sqrt{-1+r1}},0,0,0\right\},\left[0,\frac{\sqrt{-1+r1}}{r1^{2}\sin[th]^{3/4}},0,0\right\},\left[0,0,\frac{1}{r1^{5/2}\sin[th]^{3/4}},0\right],\left[0,0,0,\frac{1}{r1^{5/2}\sin[th]^{3/4}},0\right],\left[0,0,0,\frac{1}{r1^{5/2}\sin[th]^{3/4}},0\right]\right\}$$

For the Kerr metric, there is a *semi-diagonal Kerr tetrad* solution  $E_{dK}$  with a non-zero (0,3)-element. The solution of the gaussian and Schwarzschild metric equations, the *Gauss-Schwarzschild tetrad*  $E_{GS}$ , can be calculated from the series in  $1/r^{3/2}$  for  $r \rightarrow inf$ 

 $E_{GS} = \frac{E_{GS,1}(\theta)}{r^{3/2}} + \frac{E_{GS,2}(\theta)}{r^{5/2}} + \frac{E_{GS,3}(\theta)}{r^{7/2}} + \dots \text{, the coefficients } E_{GS,i}(\theta) \text{ are calculated from the corresponding data in } \theta$ 

deq in  $\theta$ .

It has the semi-diagonal block-matrix form

$$E_{GS} = \frac{1}{r^{3/2}} \begin{pmatrix} \frac{1}{\sin(\theta)^{3/4}} & 0 & 0 & 0 \\ 0 & E_{GS,1}^{-1,1}(\theta) & E_{GS,1}^{-1,2}(\theta) & E_{GS,1}^{-1,3}(\theta) \\ 0 & E_{GS,1}^{-2,1}(\theta) & E_{GS,1}^{-2,2}(\theta) & E_{GS,1}^{-2,3}(\theta) \\ 0 & E_{GS,1}^{-3,1}(\theta) & E_{GS,1}^{-3,2}(\theta) & E_{GS,1}^{-3,3}(\theta) \end{pmatrix} + \dots$$

The first coefficient function of the Gauss-Schwarzschild tetrad can be given in closed form  $E_{GS,I}(\theta)=$ 



It contains the hypergeometric function

The coefficients  $E_{GS,2}(\theta)$  and  $E_{GS3}(\theta)$  have been calculated numerically with Ritz-Galerkin method as an power series in  $(sin(\theta)^{1/4}, cos(\theta))$  of order 8. The resulting order  $1/r^{7/2}$  for  $E_{GS}$  is sufficient to ensure the metric condition exactly at infinity.

#### 35 3.2. Solution $\Lambda \neq 0$ with the half-logarithmic ansatz

eliminated ( $E1_i$ ,  $E2_i$ ,  $E3_i$ ) all variables *half-logarithmic ansatz f*( $\theta$ +*log(r*)) new coordinate  $r_{th} = \theta$ +*log(r)* satisfies automatically gaussian equs

solution E in  $(r_{th}, \theta)$  $Eb_{ij}(r, \theta) = \pm Eb_{ij}(r_{th}) + \sum \frac{c_{1kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}(r_{th}) + \sum \frac{c_{4kl}}{L} \frac{Ab_{0k}'(r_{th})}{\exp(r_{th} - \theta)} + \sum \frac{c_{5kl}}{L} \frac{Ab_{3k}'(r_{th})}{\exp(r_{th} - \theta)} + \sum \frac{c_{2kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}'(r_{th}) (r_{th} - \theta) + \sum \frac{c_{3kl}}{L} Ab_{0k}'(r_{th}) Ab_{3l}(r_{th}) (r_{th} - \theta)$ 



We eliminate  $(E1_i, E2_i, E3_i)$  and make for the remaining variables the *half-logarithmic ansatz*  $f(\theta + log(r))$ , which satisfies the gaussian equations automatically. The results are: **Ab1== Ab0[th+Log[r1]]**, **Ab2== Ab3[th+Log[r1]]** 

Ebi=Ebi[th+Log[r1]] with free par. Ab0i[th+Log[r1]],Ab3i[th+Log[r1]],Eb0i[th+Log[r1]]

$$\begin{split} Eb_{ij}(r,\theta) &= \pm Eb_{ij}(r_{th}) + \sum \frac{c_{1kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}(r_{th}) + \sum \frac{c_{4kl}}{L} \frac{Ab_{0k}'(r_{th})}{\exp(r_{th} - \theta)} + \sum \frac{c_{5kl}}{L} \frac{Ab_{3k}'(r_{th})}{\exp(r_{th} - \theta)} \\ &+ \sum \frac{c_{2kl}}{L} Ab_{0k}(r_{th}) Ab_{3l}'(r_{th})(r_{th} - \theta) + \sum \frac{c_{3kl}}{L} Ab_{0k}'(r_{th}) Ab_{3l}(r_{th})(r_{th} - \theta) \\ \text{e.g.} \\ e.g. \\ Eb00(r,\theta) &= \frac{Eb00[rth] - \frac{3 e^{-rth+th} Ab30'[rth]}{L}}{Eb01[rth] - \frac{6 rth Ab32[rth] Ab00'[rth]}{L} + \frac{6 th Ab32[rth] Ab00'[rth]}{L} - \frac{6 rth Ab33[rth] Ab00'[rth]}{L} \\ &+ \frac{6 th Ab33[rth] Ab00'[rth]}{L} + \frac{6 rth Ab02[rth] Ab30'[rth]}{L} - \frac{6 th Ab03[rth] Ab30'[rth]}{L} + \frac{3 e^{-rth+th} Ab30'[rth]}{L} \end{split}$$

This is a special, not the general solution: the AK-equations are non-linear, so the general solution cannot be built from basic solutions by linear combination. Here, all variables are functions of the coordinate  $r_{th} = \theta + log(r)$ .

The solution has to satisfy the metric boundary condition for the Minkowski spacetime, so it is desirable to bring the metric into a similar form: a function of the coordinate  $r_{th}$ .

If we use functions of the form  $exp((-a+b i)(\theta+log(r)))=exp(-a \theta)(1/r^a) exp(i b log(r)) exp(i b \theta)$ , we can see that these are *polynomials in 1/r with exponential angle-damping* combined with *almost-periodic* functions. There is no singularity in *r*, except at r=0, and no Schwarzschild-type singularity at r=1.

#### 3.2.1. The half-logarithmic Schwarzschild metric and tetrad

The special solution above has the form  $f(r_{th})$  with the new coordinate  $r_{th} = \theta + log(r)$ Under the coordinate transformation  $(r \rightarrow r_{th}, \theta \rightarrow \theta)$  the Schwarzschild metric transforms

$$\begin{split} ds^{2} &= -(1 - \frac{1}{r}) dt^{2} + \frac{1}{1 - \frac{1}{r}} dr^{2} + r^{2} \Big( d\theta^{2} + \sin^{2}(\theta) d\varphi^{2} \Big) \quad \text{into} \\ ds^{2} &= -(1 - \frac{1}{\exp(r_{th} - \theta)}) dt^{2} + \frac{\exp(3(r_{th} - \theta))}{\exp(r_{th} - \theta) - 1} \Big( dr_{th}^{2} - 2dr_{th} d\theta + d\theta^{2} \Big) + \exp(2(r_{th} - \theta)) \Big( d\theta^{2} + \sin^{2}(\theta) d\varphi^{2} \Big) \\ ds^{2} &= -(1 - \frac{1}{\exp(r_{th} - \theta)}) dt^{2} + \frac{\exp(3(r_{th} - \theta))}{\exp(r_{th} - \theta) - 1} \Big( dr_{th}^{2} - 2dr_{th} d\theta \Big) + \exp(2(r_{th} - \theta)) \sin^{2}(\theta) d\varphi^{2} + \exp(2(r_{th} - \theta)) \Big( \frac{\exp((r_{th} - \theta))}{\exp(r_{th} - \theta) - 1} + 1 \Big) d\theta^{2} \\ g_{\mu\nu} &= \begin{pmatrix} -\Big(1 - \frac{1}{\varepsilon}\Big) & 0 & 0 & 0 \\ 0 & \frac{\varepsilon^{3}}{\varepsilon - 1} & -\frac{\varepsilon^{3}}{\varepsilon - 1} & 0 \\ 0 & 0 & 0 & \varepsilon^{2} \sin^{2}(\theta) \end{pmatrix}, \text{ where} \\ \varepsilon &= \frac{\exp(r_{th})}{\exp(\theta)} , \end{split}$$

And the densitized inverse metric
$$\begin{split} &I(g_{\mu\nu}) = \frac{(g_{\mu\nu})^{-1}}{\left(-\det(g_{\mu\nu})\right)^{3/4}} = \\ &(1/(\sin[\theta]^{3/2} e^{\theta/2})) \left\{ \left\{ \frac{e}{(1-e)}, \theta, \theta, \theta \right\}, \left\{ \theta, \frac{-1+2e}{e^3}, \frac{1}{e^2}, \theta \right\}, \left\{ \theta, \frac{1}{e^2}, \frac{1}{e^2}, \theta \right\}, \left\{ \theta, \theta, \theta, \theta, \frac{1}{e^2} \sin[\theta]^2 \right\} \right\} \\ &I(g_{\mu\nu}) = \frac{(g_{\mu\nu})^{-1}}{\left(-\det(g_{\mu\nu})\right)^{3/4}} = \frac{1}{\varepsilon^{9/2} \sin^{3/2}(\theta)} \begin{pmatrix} -\frac{1}{\left(1-\frac{1}{\varepsilon}\right)} & 0 & 0 & 0 \\ 0 & \frac{2\varepsilon-1}{\varepsilon^3} \left(1+\frac{1}{\varepsilon}\right) & \frac{1}{\varepsilon^2} & 0 \\ 0 & \frac{1}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon^2} \sin^{2}(\theta) \end{pmatrix} \\ &= \left\{ \left\{ \frac{\frac{e^{th}}{2} - \frac{\theta(rth-th)}{4}}{\sqrt{e^{rth}} - e^{th}} \sin[th]^{3/4}}, \theta, \theta, \theta, \theta \right\}, \left\{ \theta, \frac{e^{-\frac{\theta}{4}(rth-th)} \sqrt{-e^{2}(rth-th)} + e^{-3rth+2th} \left(2e^{rth} - e^{th}\right)}{\sin[th]^{3/4}}, \theta \right\} \\ &= Has^{-\frac{\theta}{4}} \left\{ \left\{ \frac{e^{-\frac{\theta}{4}(rth-th)}}{\sqrt{e^{rth}} - e^{th}} \sin[th]^{3/4}}, \theta \right\}, \left\{ \theta, \frac{e^{-\frac{\theta}{4}(rth-th)} \sqrt{-e^{2}(rth-th)} + e^{-3rth+2th} \left(2e^{rth} - e^{th}\right)}{\sin[th]^{3/4}}, \theta \right\}, \left\{ \theta, \frac{e^{-\frac{\theta}{4}(rth-th)} \sqrt{-e^{2}(rth-th)} + e^{-3rth+2th} \left(2e^{rth} - e^{th}\right)}{\sin[th]^{3/4}}, \theta \right\}, \left\{ \theta, \frac{e^{-\frac{\theta}{4}(rth-th)} \sqrt{-e^{2}(rth-th)} + e^{-3rth+2th} \left(2e^{rth} - e^{th}\right)}{\sin[th]^{3/4}}, \theta \right\}$$

 $\frac{\frac{e^{\mathsf{rth}-\frac{9}{4}}(\mathsf{rth}-\mathsf{th})}{4}}{\mathsf{Sin}[\mathsf{th}]^{3/4}}, 0 \Big\}, \left\{0, 0, \frac{e^{-\mathsf{rth}-\frac{9}{4}}(\mathsf{rth}-\mathsf{th})}{\mathsf{Sin}[\mathsf{th}]^{3/4}}, 0 \Big\}, \left\{0, 0, 0, \frac{e^{\frac{13}{2}}(-\mathsf{rth}+\mathsf{th})}{\mathsf{Sin}[\mathsf{th}]^{7/4}} \right\} \Big\}$ 

$$E_{hls} = \frac{1}{\sin^{3/4}(\theta)} \begin{pmatrix} \frac{1}{\varepsilon^{7/4}\sqrt{\varepsilon-1}} & 0 & 0 & 0\\ 0 & \frac{1}{\varepsilon^{9/4}}\sqrt{-\varepsilon^2 + \frac{2}{\varepsilon^2} - \frac{1}{\varepsilon^3}} & \frac{1}{\varepsilon^{5/4}} & 0\\ 0 & 0 & \varepsilon^{5/4} & 0\\ 0 & 0 & 0 & \frac{1}{\varepsilon^{13/2}\sin(\theta)} \end{pmatrix}$$

The limit of  $g_{\mu\nu}$  for  $\varepsilon \to \infty$  is in  $O(1/\varepsilon)$ 

37

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{1}{\varepsilon}\right) & 0 & 0 & 0 \\ 0 & \varepsilon^2 \left(1 + \frac{1}{\varepsilon}\right) & -\varepsilon^2 \left(1 + \frac{1}{\varepsilon}\right) & 0 \\ 0 & -\varepsilon^2 \left(1 + \frac{1}{\varepsilon}\right) & \varepsilon^2 \left(2 + \frac{1}{\varepsilon}\right) & 0 \\ 0 & 0 & 0 & \varepsilon^2 \sin^2(\theta) \end{pmatrix}$$

and the limit of  $I(g_{\mu\nu})$  for  $\varepsilon \to \infty$  is in  $O(1/\varepsilon^2)$ 

$$I(g_{\mu\nu}) = \frac{1}{\varepsilon^{9/2} \sin^{3/2}(\theta)} \begin{pmatrix} -\left(1 + \frac{1}{\varepsilon}\right) & 0 & 0 & 0 \\ 0 & \frac{2 - \frac{1}{\varepsilon}}{\varepsilon^2} \left(1 + \frac{1}{\varepsilon}\right) & \frac{1}{\varepsilon^2} & 0 \\ 0 & \frac{1}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon^2 \sin^2(\theta)} \end{pmatrix}$$

and the limit of  $I(g_{\mu\nu})$  for  $\varepsilon \to \infty$  in  $O(1/\varepsilon)$  (Minkowski spacetime)

$$I(g_{\mu\nu}) = \frac{1}{\varepsilon^{9/2} \sin^{3/2}(\theta)} \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & \frac{2}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 0\\ 0 & \frac{1}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 0\\ 0 & 0 & 0 & \frac{1}{\varepsilon^2 \sin^2(\theta)} \end{pmatrix}$$

# 3.2.2. Solvability of the metric condition for the half-logarithmic solution

$$\{Eb_{0i}(r_{th}), Ab_{0i}(r_{th}), Ab_{3i}(r_{th})\} \to \frac{1}{\exp(\frac{9}{4}(r_{th}-\theta))\sin^{3/4}(\theta)r_{th}}$$

# 39**3.3. Behavior at Schwarzschild horizon**

At the horizon, the Schwarzschild tetrad diverges  

$$E_{dS}^{0,0} = \frac{1}{r\sqrt{r-1}\sin^{3/4}(\theta)} \rightarrow \infty$$
, so the term  $\frac{\Lambda}{3}E^{\mu\nu}$  becomes significant  
at  $r = 1 + \sqrt{\Lambda}$ ,  $dr = \sqrt{\Lambda}$ , i.e.  $E00(\theta) = \frac{1}{\Lambda^{1/4}\sin^{3/4}(\theta)}$ , the peak in the metric is  $g_{1,1} = \frac{1}{\sqrt{\Lambda}}$ 

we set the gravitational scale for the quantum realm to be  $r_{gr}$  and  $dr = \frac{l_P}{r_{gr}} = \sqrt{\Lambda} r_{gr}$ 

so 
$$r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1 \times 10^{-5} m = 31 \mu m$$

gravitation has two scales:

in the classical region the  $\Lambda$ -scale  $(\Lambda = 2.7 \ 10^{-52} m^{-2})$ :  $R_{\Lambda} = \frac{1}{\sqrt{\Lambda}} = 6.09 \times 10^{25} m$ 

in the quantum region  $r_{gr} = 3.1 * 10^{-5} m$ 

electrodynamics has one scale, the classical electron radius  $r_e = 2.8 * 10^{-15} m$ .

huge  $\Lambda$ -scale in gravitation  $R_{\Lambda}$  has consequences:

-decoupling of the A-tensor and the E-tensor, Einstein equations and the general covariance are classically valid

-this 'smears out' local structure in the classical region, allowing for invariance against arbitrary coordinate transformations, i.e. the local symmetry becomes insignificant, the symmetry is the unbroken symmetry of the metric, which is invariant under arbitrary coordinate transformations.

*objective collapse theory* links the spontaneous collapse of the wave function to quantum gravitation, this puts the limit for quantum behavior at  $r \le r_{gr}$ 

As a consequence of the gravitational quantum scale  $r_{gr}$ , we can characterize two regions of gravity: -classical region  $\Lambda \approx 0 \ r >> r_{gr}$ background equations *eqtoeivnu3b*, where the hamiltonian equations *eqham(Ab,\partial Ab)* depend only on Ab  $eqtoeivnu3b = \{eqham(Ab, \partial Ab), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\}$ wave equations eqtoeivnu3w, where in the hamiltonian equations  $\Lambda$  eqham(As,  $\partial$ As, Es, Ab)  $\Lambda$  factors out,  $eqtoeivnu3w = \{ \Lambda eqham(As, \partial As, Es, Ab), eqgaus (Es, \partial Es, Ab, Eb), eqdiff(Es, Ab, \partial Ab, Eb) \}$  $Ab \cong \frac{1}{l}$  makes EH-action  $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} d^4 x$  dimensionally renormalizable metric condition for Eb is satisfied for all r $Eb \bullet \eta \bullet Eb^{t} = I(g_{\mu\nu}) = g^{-1} / (-\det(g))^{3/4}$ , and  $Eb = E_{GS}$  resp.  $= E_{GK}$  for g = Schwarzschild resp. Kerr -quantum region  $r < < r_{gr}$ background equations *eqtoeivnu3b*, where the hamiltonian equations  $eqham(Ab, \partial Ab, Eb)$  couple weakly to Eb  $eqtoeivnu3b = \{eqham(Ab, \partial Ab, Eb), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\}$ wave equations eqtoeivnu3w, where in the hamiltonian equations  $\Lambda$  eqham(As,  $\partial$ As, Es, Ab)  $\Lambda$  factors out, and Ab is negligible, Ab < <As $eqtoeivnu3w = \{ \Lambda eqham(As, \partial As, Es, Ab), eqgaus (As, Es, \partial Es, Ab, Eb), eqdiff(As, \partial As, Es, Ab, \partial Ab, Eb) \}$  $\Lambda$  not zero,  $\Lambda \le 1$ ,  $Ab \le As$ ,  $A \approx As = (almost)$  pure wave graviton interaction via  $D_{\mu}^{\lambda} = \partial_{\mu} + \varepsilon^{\lambda}{}_{\kappa_1} \cdot A_{\mu}^{\kappa_1}$  covariant derivative, as in quantum electrodynamics metric= Minkowski metric metric condition for *Eb* for  $r \rightarrow \infty$  Minkowski  $g=\eta$ : *Eb*=*E<sub>GM</sub> Gauss-Minkowski tetrad* 

40

41 At the horizon, the Schwarzschild tetrad diverges

 $E_{dS}^{0,0} = \frac{1}{r\sqrt{r-1}\sin^{3/4}(\theta)} \rightarrow \infty$ , so the term  $\frac{\Lambda}{3}E^{\mu\nu}$  becomes significant, the coupling reappears.

 $EO[th] = EOO[th] \{1, 1, 1, 1\}, error = \frac{4 \sqrt{dr1} EOOC}{6}, EOO[th] = EOOC$ 

When the parameter dr=r-1 becomes  $dr = \sqrt{\Lambda}$ , we get in the limit  $r \rightarrow \infty$  for the E-tensor and the A-tensor a r-independent finite solution in the vicinity of r=1:  $A0_i = A00(\theta) \{1, 1, -1, 1\}, A1_i = A10c \{1, 1, -1, 1\}, A2_i = A20c \{1, 1, -1, 1\}, A3_i = A30c \{1, 1, -1, 1\}$ 

 $A0_{i} = A00(\theta) \{1, 1, -1, 1\}, A1_{i} = A10c \{1, 1, -1, 1\}, A2_{i} = A20c \{1, 1, -1, 1\}, A3_{i} = A30c \{1, 1, -1, 1\}$  $E0_{i} = E00(\theta) \{1, 1, -1, 1\}$ 

The parameters of the solution are determined by the continuity condition at  $r = 1 + \sqrt{\Lambda}$ , i.e.

 $E00(\theta) = \frac{1}{\Lambda^{1/4} \sin^{3/4}(\theta)}$ , the peak in the metric is  $g_{1,1} = \frac{1}{\sqrt{\Lambda}}$ 

This reappearance of coupling for  $dr = \sqrt{\Lambda}$  (dimensionless) results in a new scale, at which the classical character of gravity disappears and the quantum realm begins:

we set the gravitational scale for the quantum realm to be  $r_{gr}$  and  $dr = \frac{l_P}{r_{gr}} = \sqrt{\Lambda} r_{gr}$ 

so 
$$r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1 * 10^{-5} m = 31 \mu m$$

So we can say that gravitation has two scales: in the classical region the  $\Lambda$ -scale ( $\Lambda = 2.7 \ 10^{-52} m^{-2}$ ):

 $R_{\Lambda} = \frac{1}{\sqrt{\Lambda}} = 6.09 * 10^{25} m$  and in the quantum region  $r_{gr} = 3.1 * 10^{-5} m$ . The electrodynamics has, in contrast,

only one scale, the classical electron radius  $r_e = 2.8 * 10^{-15} m$ . The huge  $\Lambda$ -scale in gravitation is responsible for the decoupling of the A-tensor and the E-tensor in the classical region with the consequence that the Einstein equations and the general covariance are classically valid, again in contrast to the electrodynamics, which is only gauge-invariant, not general-covariant.

Therefore, one is tempted to explain the validity of the general covariance in GR as the consequence of the huge  $\Lambda$ -scale, which 'smears out' local structure in the classical region, allowing for invariance against arbitrary coordinate transformations.

The situation is similar to the symmetry of a n-polyhedron approximating a sphere: in the limit  $n \rightarrow \infty$  the symmetry becomes the spherical symmetry (the symmetry of the metric) and the local symmetry of edges and vertices becomes insignificant. But this is of course at best a heuristic explanation.

The *objective collapse theory* put forward by Penrose [19], links the spontaneous collapse of the wave function to quantum gravitation, the limit being one graviton. If true, this would put the limit for quantum coherence at  $r \le r_{er}$ .

As a consequence of the gravitational quantum scale  $r_{gr}$ , we can characterize two regions of gravity: -classical region  $\Lambda \approx 0 \ r >> r_{gr}$ 

background equations *eqtoeivnu3b*, where the hamiltonian equations *eqham(Ab,∂Ab)* depend only on *Ab eqtoeivnu3b={eqham(Ab,∂Ab), eqgaus (Ab,Eb,∂Eb), eqdiff(Ab, ∂Ab,Eb)}* wave equations *eqtoeivnu3w*, where in the hamiltonian equations  $\Lambda$  *eqham(As,∂As,Es,Ab)*  $\Lambda$  factors out, *eqtoeivnu3w={*  $\Lambda$  *eqham(As,∂As,Es,Ab), eqgaus (Es,∂Es,Ab,Eb), eqdiff(Es,Ab,∂Ab,Eb)}* 

$$Ab \cong \frac{1}{l_p}$$
 makes EH-action  $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} d^4 x$  dimensionally renormalizable

the scale is  $r_s = \frac{2Gm}{c^2}$  and the Schwarzschild radius depends on the mass *m* of the gravitating object metric condition for *Eb* is satisfied for all *r* 

 $Eb \bullet \eta \bullet Eb^t = I(g_{\mu\nu}) = g^{-1} / (-\det(g))^{3/4}$ , and  $Eb = E_{GS}$  resp.  $=E_{GK}$  for g = Schwarzschild resp. Kerr -quantum region  $r < <r_{gr}$ 

background equations *eqtoeivnu3b*, where the hamiltonian equations *eqham(Ab,* $\partial Ab, Eb$ ) couple weakly to *Eb eqtoeivnu3b={eqham(Ab,* $\partial Ab, Eb), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab,$  $\partial Ab, Eb)} wave equations$ *eqtoeivnu3w* $, where in the hamiltonian equations <math>\Lambda$  *eqham(As,* $\partial As, Es, Ab$ )  $\Lambda$  factors out, and Ab is negligible, Ab < <As *eqtoeivnu3w={*  $\Lambda$  *eqham(As,* $\partial As, Es, Ab$ ), *eqgaus (As,*Es, $\partial Es,$ Ab,Eb), eqdiff(As, $\partial As,$ Es,Ab, $\partial Ab,$  $Eb)}$ 

 $\Lambda$  is the same as in the classical region (?), Ab < <As,  $A \approx As = (almost)$  pure wave graviton the scale is constant =  $r_{gr}$ 

interaction via  $D_{\mu}^{\ \lambda} = \partial_{\mu} + \varepsilon^{\lambda}{}_{\kappa_{1}} \cdot A_{\mu}^{\ \kappa_{1}}$  covariant derivative, as in quantum electrodynamics metric= Minkowski metric

metric condition for *Eb* for  $r \rightarrow \infty$  Minkowski  $g=\eta$ : *Eb*=*E*<sub>GM</sub> Gauss-Minkowski tetrad

# 43 **B4. Solutions of time-dependent equations**

# 4.1. The $\Lambda$ -scaled wave ansatz for the A-tensor

classical case  $\Lambda \approx 0 \ r >> r_{gr}$   $eqtoeivnu3b = \{eqham(Ab, \partial Ab), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\}$   $eqtoeivnu3w = \{A \ eqham(As, \partial As, Es, Ab), eqgaus (Es, \partial Es, Ab, Eb), eqdiff(Es, Ab, \partial Ab, Eb)\}$   $Ab \cong \frac{1}{l_p}$  makes EH-action  $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\nu} A_{\nu}^{\mu}) R \sqrt{-g} d^4 x$  dimensionally renormalizable metric condition for Eb for all r  $Eb \bullet \eta \bullet Eb^t = I(g_{\mu\nu}) = g^{-1} / (-\det(g))^{3/4}$ ,

quantum case  $r << r_{gr}$   $\Lambda$  not zero  $\Lambda << 1$ , Ab<<1 A $\approx$ As=pure wave graviton interaction via  $D_{\mu}^{\lambda} = \partial_{\mu} + \varepsilon^{\lambda}{}_{\kappa_{1}} \cdot A_{\mu}^{\kappa_{1}}$ metric= Minkowski metric metric condition for *Eb* for  $r \rightarrow \infty$  Minkowski  $g=\eta$  : *Eb=Gauss-Minkowski tetrade eqtoeivnu3b={eqham(Ab, \partial Ab, Eb), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)}* 

 $eatoeivnu3w = \{ A eaham(As, \partial As, Es, Ab), eagaus (As, Es, \partial Es, Ab, Eb), eadiff(As, \partial As, Es, Ab, \partial Ab, Eb) \}$ 

The covariant derivative of the AK-gravitation is

$$D_{\mu}t_{\nu}^{\ \lambda} = \partial_{\mu}t_{\nu}^{\ \lambda} + \varepsilon^{\lambda}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}t^{\nu\kappa_{2}} \quad D_{\mu}^{\ \lambda} = \partial_{\mu} + \varepsilon^{\lambda}{}_{\kappa_{1}}A_{\mu}^{\ \kappa_{1}}$$

The gaussian equations have the form of the covariant derivative acting on the E-tensor

$$G^{\mu} = D_{\nu}E^{\nu\mu} = \partial_{\nu}E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda}A_{\nu}{}^{\kappa}E^{\nu\lambda}$$

One can show, that the second term in the covariant derivative cancels out only if the A-tensor vanishes, i.e. the covariant derivative is not background-independent.

Now, if we separate the static background and the wave component in the A-tensor:

 $A = A_{bg} + A_{wave}$ ,  $E = E_{bg} + E_{wave}$ 

we have to take account of the fact that in GR the gravitational wave interacts weakly with the metric, because it interacts through the energy tensor, which appears on the right side of the Einstein equations with the small factor  $\kappa$ :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Therefore, classically, we have to use some power of  $\Lambda$  as the factor in the ansatz above (setting c=1)

$$A_{\mu}^{\nu} = Ab_{\mu}^{\nu} + \Lambda^{p} \frac{As_{\mu}^{\nu}}{r} \exp(-ik(r-t))$$
, where Ab is the (static) background, As is the wave amplitude

In order to make As interact with E-tensor in the hamiltonian equations , we have to set p=1, the ansatz becomes ( $\Lambda$ -scaled ansatz for the A-tensor)

$$A_{\mu}^{\nu} = Ab_{\mu}^{\nu} + \Lambda \frac{As_{\mu}^{\nu}}{r} \exp(-ik(r-t)) \text{ and correspondingly for the E-tensor}$$
$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp(-ik(r-t))$$

This has some remarkable consequences: in the Hamiltonian equations we now have the background part of order 1 for Ab and Eb and the wave part of order  $\Lambda$  for As, and Es.

In the A-tensor and the E-tensor we now have the background part Ab and Eb and the wave part As, and Es. We insert this into the AK-equations, and separate the static part *eqtoeivnu3b* in the schematic form *eqtoeivnu3b*={*eqham*( $Ab, \partial Ab, Eb$ ), *eqgaus* ( $Ab, Eb, \partial Eb$ ), *eqdiff*( $Ab, \partial Ab, Eb$ )}

and the wave part *eqtoeivnu3w* after stripping the wave factor  $\exp(-ik(r-t))$  in the schematic form *eqtoeivnu3w*={ $\Lambda$  *eqham*(As, $\partial As$ ,Es,Ab), *eqgaus* (As,Es, $\partial Es$ ,Ab,Eb), *eqdiff*(As, $\partial As$ ,Es,Ab, $\partial Ab$ ,Eb)} where  $\partial = \{\partial_r, \partial_\theta\}$  ist the differential operator for r and  $\theta$ .

As the dimensions are [A] = [1/r] = 1/cm and [E] = 1, we get for the A-amplitude the dimension  $[As] = [r^2] = cm^2$  i.e. As becomes a cross-section, which is a sensible interpretation in the quantum limit.

In the quantum limit  $r < r_{gr}$ , the graviton interacts via the covariant derivative, like the photon, and the metric condition for *Eb* is for the flat Minkowski metric  $(r_s=l_P)$ ,  $\Lambda \neq 0$ , the Einstein equation and the general covariance are not valid anymore.

In the classical case  $A \approx 0$ , the AK-equations separate into the background part for Ab, Eb and the wave-part with the wave factor  $\exp(-ik(r-t))$  for Es, As, Ab.

 $eqtoeivnu3b = \{eqham(Ab, \partial Ab), eqgaus (Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\}\$  $eqtoeivnu3w = \{A eqham(As, \partial As, Es, Ab), eqgaus (Es, \partial Es, Ab, Eb), eqdiff(Es, Ab, \partial Ab, Eb)\}\$ 

The background part with the metric condition is equivalent to the Einstein equations.

As we shall see below, the wave As carries the wave energy, and induces locally a tetrad (metric) wave, which is damped exponentially. The gravitational wave energy tensor depends on the wave amplitude As in a similar way as the electromagnetic wave energy depends on the photon vector  $A_{\mu}$ . Also, it satisfies the Einstein power formula for the gravitational wave.

44

eqtoiev  $\Lambda$ -scaled wave ansatz

A1i=A0i A2i=A3i, eliminate  $(E1_i, E2_i, E3_i)$ 

remaining variables Ab0<sub>i</sub>,Ab3<sub>i</sub>,Eb0<sub>i</sub>,Es0<sub>i</sub> half-logarithmic ansatz  $f(r_{th})$ ,  $r_{th} = \theta + log(r)$ 

solution  $EsOi(r, \theta) = f(AsOi(\theta + log(r)), AbOi(\theta + log(r)), exp(2 i k r), ExpIntegralEi(-2 i k r))$ free parameters  $AsO_i(r_{th})$ ,  $AbO_i(r_{th})$ ,  $Ab3_i(r_{th})$ ,  $EbO_i(r_{th})$ The 12 free parameters {AbOi, Ab3i, EbOi} have to satisfy the 10 equation of the half-logarithmic Minkowski metric condition for  $r_{th} \rightarrow \infty$ 

```
sol. eqtoiev with general L: ansatz Aij=Abij[r1,th]+L Asij[r1,th]Exp[-i k (r1-th)];
Eij=Ebij[r1,th]+Esij[r1,th]Exp[-i k (r1-th)]
```

solution:

```
A1==A0, A2==A3

MAb={Ab0[th+Log[r1]],Ab0[th+Log[r1]],Ab3[th+Log[r1]],Ab3[th+Log[r1]]},

MEb=MEb(Eb0[th+Log[r1]],Ab0[th+Log[r1]],Ab3[th+Log[r1]],Ab0',Ab3')

{E10,E20,E30}=f(E00,A0,A3,A0',A3')

{E11,E21,E31}=f(E01,A0,A3,A0',A3')

{E12,E22,E32}=f(E02,A0,A3,A0',A3')

{E13,E23,E33}=f(E03,A0,A3,A0',A3')

As3i==As2i==0,As1==As0, As0i[r1,th]== r1 As0ic1[th+Log[r1]],

Es0i== f(As0, Ab3,As0',Ab3')

e.o.
```

 $\begin{bmatrix} \text{E}_{\text{S}} \\ \text{E}_{\text{S}}$ 

We set A1i=A0i A2i=A3i, eliminate  $(E1_i, E2_i, E3_i)$  and make for the remaining variables the *half-logarithmic* ansatz  $f(r_{th})$  with  $r_{th} = \theta + log(r)$ , which satisfies the gaussian equations automatically. The results are:

 $Es0i(r, \theta) = f(As0i(\theta + log(r)), Ab0i(\theta + log(r)), exp(2 i k r), ExpIntegralEi(-2 i k r))$ with free parameters  $As0(r_{+}) = Ab0(r_{+}) = Ab3(r_{+}) = Eb0(r_{+})$ 

with free parameters  $AsO_i(r_{th})$ ,  $AbO_i(r_{th})$ ,  $Ab3_i(r_{th})$ ,  $EbO_i(r_{th})$ .

The functions  $f(\theta + log(r))$  are exponentially damped or almost-periodic for  $r \rightarrow \infty$ .

The 12 free parameters {*Ab0i*, *Ab3i*, *Eb0i*} have to satisfy the 10 equation of the *half-logarithmic Minkowski metric* condition for  $r_{th} \rightarrow \infty$  (the metric condition is required only for the static part of the solution, not for the wave part).

As in 3.2., one shows that the condition can be satisfied.

# 474.3. Wave equation in Schwarzschild spacetime

eqtoiev A-scaled wave ansatz backgrund equation eqtoeivnu3b=eqtoiv standard solution: A-tensor = constant background in the half-antisymmetric form  $A0_i = A00c \{1, 1, -1, 1\}, A1_i = A10c \{1, 1, -1, 1\}, A2_i = A20c \{1, 1, -1, 1\}, A3_i = A30c \{1, 1, -1, 1\}$ E-tensor= the Gauss-Schwarzschild-tetrad  $E_{GS}$ .

resulting wave equation  $eqtoeivnu4w = \{ eqham(As, \partial As, Es, \partial Es), \\ eqgaus (Es0i, Es1i, \partial Es1i, , \partial Es2i), eqdiff=0 \}$ eliminate (Es0, Es3, As1) multipole ansatz Es(r,  $\theta$ ) = Es(r)exp(i\*lx\* $\theta$ ), As(r,  $\theta$ ) = As(r)exp(i\*lx\* $\theta$ ) eliminate Es2 and get the gravitational wave equation for Es1

# **gravitational wave equation for E-tensor** *eqgravlxEn* =

 $\left(k \, \text{r1} \, \left(-2 - \text{i} \, k \, \text{r1}\right) + 1 x^3 \, \left(1 + \text{i} \, k \, \text{r1}\right) + 1 x^2 \left(-2 \, \text{i} + 4 \, k \, \text{r1} + 3 \, \text{i} \, k^2 \, \text{r1}^2\right) + 1 x \left(-1 - 5 \, \text{i} \, k \, \text{r1} + 4 \, k^2 \, \text{r1}^2 + 2 \, \text{i} \, k^3 \, \text{r1}^3\right)\right) \text{ Es10[r1]} + r1 \left(\left(-1 x^3 + 1 x^2 \, \left(2 \, \text{i} - 5 \, k \, \text{r1}\right) + 1 x \, \left(1 + 3 \, \text{i} \, k \, \text{r1} - 6 \, k^2 \, \text{r1}^2\right) + k \, \text{r1} \, \left(2 + \text{i} \, k \, \text{r1} - 2 \, k^2 \, \text{r1}^2\right)\right) \text{ Es10[r1]} + r1 \left(\left(-1 x^3 + 1 x^2 \, \left(2 \, \text{i} - 5 \, k \, \text{r1}\right) + 1 x \, \left(1 + 3 \, \text{i} \, k \, \text{r1} - 6 \, k^2 \, \text{r1}^2\right) + k \, \text{r1} \, \left(2 + \text{i} \, k \, \text{r1} - 2 \, k^2 \, \text{r1}^2\right)\right) \text{ Es10[r1]} + r1 \left(\left(-1 x^3 + 3 \, k \, \text{r1}\right) + 1 x \, \left(-1 + 3 \, k \, \text{r1}\right)\right) \text{ Es10[r1]} + r1 \left(1 x + k \, \text{r1}\right) \text{ Es10[r1]} + r1 \left(1 x + k \, \text{r1}\right) \text{ Es10[r1]}\right) \right)$ 

at infinity eqgravlxEninf = $2ik^2lx fs(r) - 2k^2r fs'(r) - 3ikr fs''(r) + r fs'''(r)$ 

For comparison, the **radial wave equation** for the wave factor fs(r) from the ansatz {fs[r1, th]  $\rightarrow$  fs[r1] SphericalHarmonicY[lx, 0, th, 0]}

 $fs(t, r, \theta) = \frac{fs(r)}{r} Y_{lx,m}(\theta, \varphi) \exp(-ik(r-t))$ 

 $Helmholtzwr = -1x fs[r1] - 1x^2 fs[r1] - 2 i k r1^2 fs'[r1] + r1^2 fs''[r1]$ 

 $-lx(1+lx) fs(r) - 2ikr^2 fs'(r) + r^2 fs''(r)$ 

In addition, we get the gravitational wave equations for the A-tensor variables As00, As30 depending on Es10

eqgravlxA0

3  $(1 + i lx) (lx + k r1)^{2} As00[r1] - r1 ((-1 + lx^{2} + 2 lx (-i + k r1)) Es10[r1] + 3 (lx + k r1)^{2} As00'[r1] + r1 ((1 + 2 i lx + 2 i k r1) Es10'[r1] - r1 Es10''[r1]))$ 

eqgravlxA3

 $\begin{array}{l} 6 \ k \ lx \ (1 + i \ k \ r1) \ As30 \ [ r1 ] \ + \ \left(1 + i \ k \ r1 - 2 \ k^2 \ r1^2 + lx \ (i \ - k \ r1) \ \right) \ Es10 \ [ r1 ] \ + \\ r1 \ (-6 \ k \ lx \ As30' \ [ r1 ] \ - i \ (-i \ + \ lx \ + \ 3 \ k \ r1) \ Es10' \ [ r1 ] \ + \ r1 \ Es10'' \ [ r1 ] \ ) \end{array}$ 

# Ai =const+L\*sphwave\*Asi+Abi , Ei =Ebi +sphwave\*Esi

As described in 4.1., we introduce the  $\Lambda$ -scaled ansatz for the A-tensor

$$A_{\mu}^{\nu}(t,r,\theta) = Ab_{\mu}^{\nu}(r,\theta) + \Lambda \frac{As_{\mu}^{\nu}(r,\theta)}{r} \exp(-ik(r-t)) \text{ and correspondingly for the E-tensor}$$
$$E^{\mu\nu}(t,r,\theta) = Eb^{\mu\nu}(r,\theta) + \frac{Es^{\mu\nu}(r,\theta)}{r} \exp(-ik(r-t))$$

In the A-tensor and the E-tensor we now have the background part *Ab* and *Eb* and the wave part *As*, and *Es*. We insert this into the AK-equations, let  $\Lambda \rightarrow 0$ , and separate the static part *eqtoeivnu3b* in the schematic form *eqtoeivnu3b*={*eqham*(*Ab*), *eqgaus* (*Ab*,*Eb*), *eqdiff*(*Ab*,*Eb*)}

and the wave part *eqtoeivnu3w* after stripping the wave factor  $\exp(-ik(r-t))$  in the schematic form

eqtoievnu3w: L\*eqham(As,Es,Ab),eqgaus(Ab,dEs),eqdiff(Es,dAb)  $eqtoeivnu3w = \{ \Lambda eqham(As, \partial As, Es, \partial Es, Ab), eqgaus (Ab, Es, \partial Es), eqdiff(Es, \partial Es, Ab) \}$ where  $\partial = \{\partial_r, \partial_\theta\}$  ist the differential operator for r and  $\theta$ . eqtoeivnu3b is identical with eqtoiv the static AK-equations, and the solution is as in 3.1. for the A-tensor the constant background in the half-antisymmetric form  $A0_i = A00c \{1, 1, -1, 1\}, A1_i = A10c \{1, 1, -1, 1\}, A2_i = A20c \{1, 1, -1, 1\}, A3_i = A30c \{1, 1, -1, 1\}$ and for the E-tensor the Gauss-Schwarzschild-tetrad  $E_{GS}$ . After inserting this into eqtoeivnu3w we get a new version of the wave part equations *eqtoeivnu4w* in the form  $eqtoeivnu4w = \{ eqham(As, \partial As, Es, \partial Es), eqgaus (Es0i, Es1i, \partial Es1i, \partial Es2i), eqdiff=0 \}$  $eq1 = \frac{1}{3} \left( (-3 - 3 \pm k r^{1}) As00[r^{1}, th] + r^{1} (3 \pm k As10[r^{1}, th] + Es20[r^{1}, th] + Es30[r^{1}, th] + 3 As00^{(1,0)} [r^{1}, th] \right) \right)$  $eq2 = \frac{1}{3} \left( (3 + 3 \pm k r1) As01[r1, th] + r1 \left( -3 \pm k As11[r1, th] + Es21[r1, th] + Es31[r1, th] - 3 As01^{(1,0)}[r1, th] \right) \right)$  $eq5 = (1 + i \ k \ r1) \ As20 \ [r1, \ th] - \frac{1}{3} \ r1 \ Es00 \ [r1, \ th] + \frac{1}{3} \ r1 \ Es30 \ [r1, \ th] + As10^{(0,1)} \ [r1, \ th] - r1 \ As20^{(1,0)} \ [r1, \ th]$  $eq9 = \frac{1}{3} r1 \left( -i k r1 Es00 [r1, th] + i k r1 Es10 [r1, th] - Es00^{(0,1)} [r1, th] - Es20^{(0,1)} [r1, th] + r1 Es00^{(1,0)} [r1, th] - r1 Es10^{(1,0)} [r1, th] \right)$  $eq13=i\ k\ r1\ As20\ [\ r1,\ th\ ]\ -\ \frac{1}{3}\ r1\ Es10\ [\ r1,\ th\ ]\ +\ \frac{1}{3}\ r1\ Es30\ [\ r1,\ th\ ]\ +\ As00\ ^{(0,1)}\ [\ r1,\ th\ ]$  $eq17 = \frac{1}{3} \left( (3 + 3 \pm r1) \text{ As30}[r1, th] - r1 \left( \text{Es00}[r1, th] + \text{Es20}[r1, th] + 3 \text{ As30}^{(1,0)}[r1, th] \right) \right)$  $eq21 = \frac{1}{3} \left( -i k r1 Es00[r1, th] + (1 + i k r1) Es10[r1, th] + Es20[r1, th] - r1 Es10^{(1,0)}[r1, th] - r1 Es20^{(1,0)}[r1, th] \right)$  $eq25 = -i k r1 Es00[r1, th] + (-1 - i k r1) Es10[r1, th] + Es20^{(0,1)} [r1, th] + r1 Es10^{(1,0)} [r1, th]$ eq29=0 Four consecutive equations contain consecutive variables of a row of the tensor As and Es, as shown in eq1 and eq2 in the schematic form

 $\begin{aligned} eql &= eql(As00, As10, Es20, Es30, \partial_r As00) \\ eq2 &= eq2(As01, As11, Es21, Es31, \partial_r As01) \\ \text{Now we eliminate variables algebraically} \\ Es0i \text{ from eq25..28} \\ & [ \underline{\mathsf{Esd0}(\mathsf{r1}, \mathsf{th}) + \frac{-\mathsf{kr1Es10}(\mathsf{r1}, \mathsf{th}) + i (\underline{\mathsf{Es10}}(\mathsf{r1}, \mathsf{th}) - \underline{\mathsf{r1Es20}}^{0,1}(\mathsf{r1}, \mathsf{th}) - \underline{\mathsf{r1Es12}}^{1,0}(\mathsf{r1}, \mathsf{th})) \\ & \underline{\mathsf{kr1}} \\ & \underline{\mathsf{kr$ 

r1

Asli from eq13..16

### 48

${igl\{ As10[r1,th] \rightarrow$	3 (kr1 As00[r1, th] + kr1 As20[r1, th]) + i (-3 As00[r1, th] + r1 Es10[r1, th] + r1 Es20[r1, th] - 3 As00[0,1] (r1, th] + 3 r1 As00[1,0] (r1, th]) - 3 As00[1,0] (r1, th] + 3 r1 As00[1,0] (r1, th]) = 0
	3 kr1 ,
As11[r1, th] $\rightarrow$	3 (kr1 As01(r1, th) + kr1 As21(r1, th)) + i (-3As01(r1, th) - r1 Es11(r1, th) - r1 Es21(r1, th) - 3As01(0,1)(r1, th) + 3r1 As01(1,0)(r1, th)) - 3As01(0,1)(r1, t
	,,,,
As12[r1, th] $\rightarrow$	3 (kr1As02[r1, th] + kr1As22[r1, th]) + i (-3As02[r1, th] - r1Es12[r1, th] - r1Es22[r1, th] - 3As02[0,1] (r1, th] + 3r1As02(1,0) (r1, th]) - 3As02[r1, th] + 3r1As02[r1, th] + 3r1As0[r1, th] + 3r1As
	3 kr1 ,
As13[r1, th] $\rightarrow$	3 (kr1 As03(r1, th) + kr1 As23(r1, th)) + i (-3 As03(r1, th) - r1 Es13(r1, th) - r1 Es23(r1, th) - 3 As03(0,1)(r1, th) + 3 r1 As03(1,0)(r1, th)) )
	3 k r 1

We get equations for 4x5 variables  $\{Es1i, Es2i, As2i, As3i, As0i\}$ (\* 4\*4 equs for Es1,Es2,As2,As3,As0 \*)

Now we fix the angular momentum of the wave by setting

(Es10[r1, th] → Es10[r1] Exp[I1xth], Es20[r1, th] → Es20[r1] Exp[I1xth], As20[r1, th] → As20[r1] Exp[I1xth], As30[r1, th] → As30[r1, th] → As30[r1] Exp[I1xth])

 $Es_{1i}(r, \theta) = Es_{1i}(r)exp(i*lx*\theta)$  and correspondingly for  $\{Es_{2i}, As_{2i}, As_{3i}, As_{0i}\}$ ,

where lx=0,1,2,... is the angular momentum of the wave: lx=0 for a spherical wave, lx=1 for a dipole wave, lx=2 for a quadrupole wave.

In GR one can show that the gravitational wave must be at least quadrupole waves, there are no spherical and dipole waves.

In the following, we consider the equations  $eqgravlx = \{eq5, eq9, eq17, eq21\}$ , i.e. the four first equations from the four equation groups, for the five first column variables [Es10i, Es20i, As20i, As30i, As00i]

Expand[eqtoievnu5ws0[[5]]]

Expand[eqtoievnu5ws0[[9]]]

Expand[eqtoievnu5ws0[[17]]]

Expand[eqtoievnu5ws0[[21]]]

 $\begin{aligned} & \text{As20[r1, th]} = \frac{i \, \text{Es10[r1, th]}}{3 \, k} + \frac{2}{3} \, \text{r1} \, \text{Es10[r1, th]} = \frac{i \, \text{As00}^{(\theta,1)} \, [r1, th]}{k \, \text{r1}} + \text{As20}^{(\theta,1)} \, [r1, th] + \\ & \frac{i \, \text{Es10}^{(\theta,1)} \, [r1, th]}{3 \, k} + \frac{2 \, i \, \text{Es20}^{(\theta,1)} \, [r1, th]}{3 \, k} = \frac{i \, \text{As00}^{(\theta,2)} \, [r1, th]}{k \, \text{r1}} - r1 \, \text{As20}^{(1,\theta)} \, [r1, th] + \frac{i \, \text{r1} \, \text{Es10}^{(1,\theta)} \, [r1, th]}{3 \, k} + \frac{i \, \text{As00}^{(1,1)} \, [r1, th]}{k \, \text{r1}} + \frac{i \, \text{As00}^{(\theta,2)} \, [r1, th]}{k \, \text{r1}} + \frac{i \, \text{As00}^{(1,1)} \, [r1, th]}{k \, \text{r1}} + \frac{i \,$  $-\frac{i E s 1 \theta [r1, th]}{3 k}+\frac{1}{3} r 1 E s 1 \theta [r1, th]+\frac{2}{3} i k r 1^{2} E s 1 \theta [r1, th]-\frac{i E s 1 \theta (\theta,1) [r1, th]}{3 k}+\frac{1}{3} r 1 E s 1 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}+\frac{1}{3} r 1 E s 1 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}+\frac{1}{3} r 1 E s 1 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}+\frac{1}{3} r 1 E s 2 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}+\frac{1}{3} r 1 E s 2 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}+\frac{1}{3} r 1 E s 2 \theta ^{(\theta,1)} [r1, th]+\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3 k}-\frac{i E s 2 \theta ^{(\theta,1)} [r1, th]}{3$  $\frac{2}{3}r1 Es20^{(0,1)}[r1, th] + \frac{i Es20^{(0,2)}[r1, th]}{3k} + \frac{i r1 Es10^{(1,0)}[r1, th]}{3k} - r1^{2} Es10^{(1,0)}[r1, th] + \frac{i r1 Es10^{(1,1)}[r1, th]}{3k} - \frac{i r1 Es20^{(1,1)}[r1, th]}{3k} - \frac{i r1^{2} Es10^{(2,0)}[r1, th]}{3k} - \frac{i r1^{2}$ 

 $\frac{2}{2} \text{Es10[r1, th]} + \frac{2}{2} \text{i} \text{kr1} \text{Es10[r1, th]} + \frac{1}{2} \text{Es20[r1, th]} - \frac{1}{2} \text{Es20}^{(0,1)} [r1, th] - \frac{2}{2} \text{r1} \text{Es10}^{(1,0)} [r1, th] - \frac{1}{2} \text{r1} \text{Es20}^{(1,0)} [r1, th]$ 

The four second equations are identical to these in the four second column variables *{Es11i, Es21i, As21i, As31i, As01i }* etc.

Now we combine eqgravlx[2] and eqgravlx[4] to eliminate Es20 and get from eqgravlx[4] the gravitational wave equation for the variable *Es10*,

eqgravlxEn =

```
\left(k\,r\,\mathbf{1}\,\left(-2-i\,k\,r\,\mathbf{1}\right)\,+\,\mathbf{1}x^{3}\,\left(\mathbf{1}+i\,k\,r\,\mathbf{1}\right)\,+\,\mathbf{1}x^{2}\,\left(-2\,i\,+\,4\,k\,r\,\mathbf{1}\,+\,3\,i\,k^{2}\,r\,\mathbf{1}^{2}\right)\,+\,\mathbf{1}x\,\left(-1-5\,i\,k\,r\,\mathbf{1}\,+\,4\,k^{2}\,r\,\mathbf{1}^{2}\,+\,2\,i\,k^{3}\,r\,\mathbf{1}^{3}\right)\,\right)\,\,\mathbf{Es10}\,[\,r\,\mathbf{1}\,]\,+\,\mathbf{1}x^{2}\,\left(-2\,i\,k\,r\,\mathbf{1}\,+\,3\,i\,k^{2}\,r\,\mathbf{1}^{2}\right)\,+\,\mathbf{1}x^{2}\,\left(-1-5\,i\,k\,r\,\mathbf{1}\,+\,4\,k^{2}\,r\,\mathbf{1}^{2}\,+\,2\,i\,k^{3}\,r\,\mathbf{1}^{3}\right)\,
  \texttt{r1}\left(\left(-\texttt{lx}^3+\texttt{lx}^2\;(\texttt{2}\;\texttt{i}-\texttt{5}\;\texttt{k}\;\texttt{r1})+\texttt{lx}\;\left(\texttt{1}+\texttt{3}\;\texttt{i}\;\texttt{k}\;\texttt{r1}-\texttt{6}\;\texttt{k}^2\;\texttt{r1}^2\right)+\texttt{k}\;\texttt{r1}\left(\texttt{2}+\texttt{i}\;\texttt{k}\;\texttt{r1}-\texttt{2}\;\texttt{k}^2\;\texttt{r1}^2\right)\right)\;\texttt{Es10}'\;[\texttt{r1}]+\texttt{Es10}'\;\texttt{r1}=\texttt{es10}'\;\texttt{r1}=\texttt{es10}'\;\texttt{r1}=\texttt{es10}''
                  r1\left(-i\left(2lx^{2}+5klxr1+kr1\left(-i+3kr1\right)\right)Es10''[r1]+r1\left(lx+kr1\right)Es10^{(3)}[r1]\right)\right)
```

At infinity

eqgravlxEninf=

2  $\pm$  k<sup>2</sup> lx Es10[r1] + r1 (-2 k<sup>2</sup> Es10' [r1] - 3  $\pm$  k Es10'' [r1] + Es10<sup>(3)</sup> [r1])

 $2ik^{2}lx fs(r) - 2k^{2}r fs'(r) - 3ikr fs''(r) + r fs'''(r)$ 

For comparison, the radial wave equation for the wave factor  $f_s(r)$  from the ansatz  $\{fs[r1, th] \rightarrow fs[r1] SphericalHarmonicY[lx, 0, th, 0]\}$ 

 $fs(t, r, \theta) = \frac{fs(r)}{r} Y_{lx,m}(\theta, \varphi) \exp(-ik(r-t))$ Helmholtzwr=  $-1x fs[r1] - 1x^{2} fs[r1] - 2 i k r1^{2} fs'[r1] + r1^{2} fs''[r1]$ 

 $-lx(1+lx) f_{s}(r) - 2ikr^{2} f_{s}'(r) + r^{2} f_{s}''(r)$ 

In addition, we get wave equations for the A-tensor variables As00, As20, As30

eggravlxA02

eqgrav1xA3

6 k lx (1 + i k r1) As30[r1] + (1 + i k r1 - 2 k<sup>2</sup> r1<sup>2</sup> + lx (i - k r1)) Es10[r1] + r1 (-6 k lx As30'[r1] - i (-i + lx + 3 k r1) Es10'[r1] + r1 Es10''[r1])

and setting As20=As00

eqgravlxA0

3  $(1 + i lx) (lx + k rl)^{2} As00[rl] - rl((-1 + lx^{2} + 2 lx (-i + k rl)) Es10[rl] + 3 (lx + k rl)^{2} As00'[rl] + rl((1 + 2 i lx + 2 i k rl) Es10'[rl] - rlEs10''[rl]))$ 

Equations eqgravlxA0, eqgravlxA3 and eqgravlxEn are homogeneous deq's for the A-tensor variables As0i, As3i, Es1i. Now, if there is a source, which generates an oscillation  $\delta Es$  of the metric (e.g. a binary geavitational rotator), i.e. of the tetrad Es

 $eqgravlxEn(Es1) = \delta Es$ , we can calculate  $Es1 = Es1(\delta Es)$ , and from (eqgravlxA0(Es1), eqgravlxA0(Es1))we calculate  $As0 = As0(Es1(\delta Es))$  and  $As3 = As3(Es1(\delta Es))$ .

### 50

# 51 **4.3.1. Solutions of the gravitational wave equation**

solution lx=0 spherical wave:  $\{Es10[r1] \rightarrow e^{2ir1} r1C[1] + e^{2ir1} r1C[2] ExpIntegralEi[-ir1]\}$ generates an incoming wave, which is not feasible, therefore C1=0 and Es10=0, solution lx=1 dipole wave:  $\{Es10[r1] \rightarrow \frac{2}{3} ir1C[1] Hypergeometric1F1[1-i, 2, \frac{2ir1}{3}] + C[2] MeijerG[\{\{\}, \{1+i\}\}, \{\{0, 1\}, \{\}\}, -\frac{2ir1}{3}]\}$ diverges, therefore Es10=0



The overall result is:

-the E-tensor is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

-the A-tensor components As0 and As1 are pure quadrupole waves, As2 is a linearly damped quadrupole wave,

As3 is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

This means that a classical wave source generates gravitational waves As via the metric, the energy is carried away by the As-tensor and, when the wave is absorbed, it dissipates energy and generates again a (locally damped) metric oscillation Es.

For lx=0 (spherical wave) we get the solution for *Es10* :

 $\left\{ \text{Es10[r1]} \rightarrow e^{2 \text{ i r1}} \text{ r1C[1]} + e^{2 \text{ i r1}} \text{ r1C[2]} \text{ ExpIntegralEi[-i r1]} \right\}$ 

which has the limit at infinity:

 $\left\{\texttt{Es10[r1]} \rightarrow \frac{\texttt{C1} e^{2 \texttt{ir1}} \left(-2 + \texttt{i} \texttt{r1} + \texttt{r1}^2\right)}{\pi \texttt{r1}^2}\right\}$ 

The factor  $e^{2ir}$  generates an incoming wave, which is not feasible, therefore C1=0 and Es10=0, and consequently As20=As30=Es20=0, there is only the zero solution: there are no spherical gravitational waves.

For lx=1 (dipole wave) we get the solution for Es10:  $\left\{ Es10[r1] \rightarrow \frac{2}{3} i r1C[1] \text{ Hypergeometric1F1}\left[1-i, 2, \frac{2ir1}{3}\right] + C[2] \text{ MeijerG}\left[\{\{\}, \{1+i\}\}, \{\{0, 1\}, \{\}\}, -\frac{2ir1}{3}\right] \right\}$ 

with hypergeometric and Meijer functions, the limit at infinity is

which diverges, therefore *Es10=0*, and, as before, there is only the zero solution: there are no dipole gravitational waves.

For lx=2 (quadrupole wave) we get the solution for the real part Re(Es10):  $\left\{ Es10[r1] \rightarrow C[3] - \frac{i r1^{3} C[1] \text{ HypergeometricPFQ}\left[\left\{\frac{3}{2}\right\}, \left\{2, \frac{5}{2}\right\}, \frac{r1^{4}}{2}\right] + \frac{1}{2} r1^{2} C[2] \text{ MeijerG}\left[\left\{\{0\}, \left\{-1\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \left\{-1, -1\right\}\right\}, -\frac{i r1}{\sqrt{2}}, \frac{1}{2}\right]\right\}$ and for the imaginary part Im(Es10):  $\left[ \text{Es10[r1]} \rightarrow -2 \times 6^{1/3} \text{ r1}^{2/3} \text{ BesselI} \left[ -\frac{4}{3}, \frac{4 \sqrt{\text{r1}}}{\sqrt{3}} \right] \text{C[1]} \text{ Gamma} \left[ \frac{2}{3} \right] - \frac{8 (-2)^{1/3} \text{ r1}^{2/3} \text{ BesselI} \left[ \frac{4}{3}, \frac{4 \sqrt{\text{r1}}}{\sqrt{3}} \right] \text{C[2]} \text{ Gamma} \left[ \frac{4}{3} \right] - \frac{3 \times 3^{2/3}}{3 \times 3^{2/3}} \right] \text{C[2]} \text{ Gamma} \left[ \frac{4}{3} \right] + \frac{3 \times 3^{2/3}}{3 \times 3^{2/3}} = \frac{3 \times$ calculation of the limit at infinity yields  $\left\{ \mathsf{Es10[r1]} \rightarrow \mathsf{IC1} \left( e^{-\frac{4\sqrt{r1}}{\sqrt{3}}} r1^{5/12} \right) \right\}$ , i.e. *Es10* is purely imaginary and exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ ,

the same is valid for Es20, for As20 we get

 $\left\{ As20[r1, th] \rightarrow \frac{As20c e^{2ith}}{r1} \right\}$ , i.e. a linearly damped quadrupole wave,

for As00 we get

$$\left\{\text{As00[r1, th]} \rightarrow \text{e}^{2\,\text{ith}} \left(-\frac{\text{As20c}}{2+\text{r1}} - \frac{\text{As20c}\,\text{r1}}{2\,(2+\text{r1})}\right)\right\}$$

i.e. As00 is a quadrupole wave with the amplitude  $\frac{As20c}{2}$ ,

for As10 we get

$$\frac{\text{As20c } e^{2 \text{ ith }} (-\text{i} + \text{r1})}{2 \text{ r1}}$$

 $As10[r1,th] \rightarrow$ , again a quadrupole wave, for As30 we get an exponentially damped wave again: 1 1/22

$$\left\{ \text{As30[r1]} \rightarrow \frac{\text{i C1 e}^{-\frac{4 \sqrt{11}}{\sqrt{3}}} \left(-49 - 64 \sqrt{3} \sqrt{\text{r1}} + 96 \text{ r1} + 288 \text{ r1}^2\right)}{1728 \text{ r1}^{7/12}} \right\}$$

The overall result is:

-the E-tensor is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

-the A-tensor components As0 and As1 are pure quadrupole waves, As2 is a linearly damped quadrupole wave, As3 is exponentially damped with  $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$ 

This means that a classical wave source generates gravitational waves As via the metric, the energy is carried away by the As-tensor and, when the wave is absorbed, it dissipates energy and generates again a (locally damped) metric oscillation Es.

# 544.4. Wave equation in binary rotator spacetime

binary gravitational rotator: masses  $m_1 m_2$ , distance  $r_0$ , mass ratio  $\mu = m_2/m_1 \le l$ , total mass  $m = m_1 + m_2$ ,



described by Kerr spacetime in first order approximation for  $\alpha \ll 1$ 



 $\alpha = \frac{c_0}{r_0}$ , exactly:  $\alpha = \frac{8\pi}{5r_0F} \frac{\mu}{(1+\mu)^7 (3+8i\mu-4\mu^2)}$ 

The celebrated Einstein's power formula for gravitational waves of the bgr :

 $P_{gr} = \frac{32}{5} m_1^2 m_1^2 m \frac{G^4}{r_0^5 c^5}$ 

The binary gravitational rotator, abbreviated bgr, (two masses rotating around their center-of-mass in their own gravitational field) is the simplest source of gravitational waves, a single rotating mass (i.e. with axial symmetry) does not emit gravitational waves.

Bgr has an axial symmetry and can be described by a Kerr-spacetime with an appropriate Kerr-parameter  $\alpha$ , which determines the power of the generated gravitational wave as shown in [11].

The exact formula derived there is

 $\alpha = \frac{8\pi}{5r_0F} \frac{\mu}{(1+\mu)^7 (3+8i\mu-4\mu^2)} \text{, where } F \approx I \text{ is the relativistic velocity factor, } \mu = \frac{m_2}{m_1} \le 1 \text{ is the mass ratio}$ 

and  $r_0$  is the mean distance of the masses., masses  $m_1 m_2$ , total mass  $m = m_1 + m_2$ , Schwarzschild radius

$$r_s = \frac{2Gm}{c^2}$$

The celebrated Einstein's power formula for gravitational waves of the bgr is [2]:

$$P_{gr} = \frac{32}{5} m_1^2 m_1^2 m \frac{G^4}{r_0^5 c^5} \text{ or formulated with } \alpha$$
$$P_{\alpha} = \frac{\Delta E_{\alpha}}{T} = \frac{(1+i\mu)\alpha}{r_0^5} \frac{F}{4\pi} (3+8i\mu-4\mu^2)$$

The gravitational waves of the bgr have the (dimensionless) wave number  $k = \frac{1}{\sqrt{2r_0^3}}$ , with dimension

$$k = \frac{\sqrt{r_s}}{\sqrt{2r_0^3}}$$

In the following, we need only  $\alpha = \frac{c_0}{r_0}$  with a constant  $c_0$  and the bgr to be described by a Kerr-spacetime to be

exact of order  $O\left(\frac{\alpha^2}{r^2}\right)$  .

# 56 4.4.1. Wave equations for the binary gravitational rotator



series in  $r_0$ :  $eqtoievnu3wdAs = eqtoievnu3wdAs0 + eqtoievnu3wdAs1/r_0 + eqtoievnu3wdAs1/r_0^{3/2} + ...$ goal: calculate  $As00(r, \theta, r_0)$  analytically in  $\{\theta, r_0\}$  as a series in r, then all others  $As00(r, \theta, r_0) = (As00n00(\theta) + \frac{As00n01(\theta)}{r_0} + \frac{As00n02(\theta)}{r_0^{3/2}} + ...) + \frac{(As00n10(\theta) + \frac{As00n11(\theta)}{r_0} + \frac{As00n12(\theta)}{r_0^{3/2}} + ...)}{r} + ...$ result:  $As00n00(\theta) = 0$ , i.e. for  $r_0 \rightarrow \infty$   $As00(r, \theta, r_0) = \frac{As00n01(\theta)}{r_0}$  57 First, we make a  $\Lambda$ -scaled ansatz for the A-tensor

$$A_{\mu}^{\nu} = Ab_{\mu}^{\nu} + \Lambda \frac{As_{\mu}^{\nu}}{r} \exp(-ik(r-t)) \text{ and correspondingly for the E-tensor}$$
$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp(-ik(r-t))$$

Then, calculate the tetrad of the Kerr spacetime, which satisfies the gaussian equation (Gauss-Kerr-tetrad)  $E_{GK}$ .

We start with the (dimensionless) Kerr spacetime with  $\frac{\alpha}{r_s} \ll 1$ , i.e. dimensionless (setting  $r_s=1$ )  $\alpha \ll 1$ :

original line element with  $r_s$ :

$$-ds^{2} = \left(1 - \frac{rr_{s}}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)(dt)^{2} + \left(\frac{2rr_{s}\alpha\sin^{2}\theta}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)dt \,d\varphi$$
$$-\left(\frac{r^{2} + \alpha^{2}\cos^{2}\theta}{r^{2} - rr_{s} + \alpha^{2}}\right)dr^{2} - \left(r^{2} + \alpha^{2} + \frac{rr_{s}\alpha^{2}\sin^{2}\theta}{r^{2} + \alpha^{2}\cos^{2}\theta}\right)\sin^{2}\theta \,d\varphi^{2} - \left(r^{2} + \alpha^{2}\cos^{2}\theta\right)(d\theta^{2})$$

In matrix form dimensionless:

$$g_{\mu\nu} = -\begin{pmatrix} \left(1 - \frac{r}{r^{2} + \alpha^{2} \cos^{2} \theta}\right) & 0 & 0 & \frac{r \alpha \sin^{2} \theta}{r^{2} + \alpha^{2} \cos^{2} \theta} \\ & - \frac{r^{2} + \alpha^{2} \cos^{2} \theta}{r^{2} - r + \alpha^{2}} & 0 & 0 \\ & & -\left(r^{2} + \alpha^{2} \cos^{2} \theta\right) & 0 \\ & & -\sin^{2} \theta \left(r^{2} + \alpha^{2} + \frac{r \alpha^{2} \sin^{2} \theta}{r^{2} + \alpha^{2} \cos^{2} \theta}\right) \end{pmatrix}$$

and in first order approximation for  $\alpha \ll 1$ 

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{1}{r}\right) & 0 & 0 & -\frac{\alpha \sin^2 \theta}{r} \\ & \frac{1}{\left(1 - \frac{1}{r}\right)} & 0 & 0 \\ & & r^2 & 0 \\ & & & r^2 \sin^2 \theta \end{pmatrix}$$

Calculation of the Gauss-Kerr-tetrad with the metric condition  $E \eta E^{t} = g^{-1} / (-\det(g))^{3/4}$  and satisfying the gaussian equations  $\frac{\partial_{\theta} E^{2\nu}}{r} + \partial_{r} E^{1\nu} = 0$ ,

yields  $E_{GK} = E_{GS}$  except  $(E_{GK})_{03} = \frac{\alpha}{r^{9/2} \sin^{3/4} \theta}$ 

This non-diagonal element  $(E_{GK})_{03}$  causes (in first approximation order) a perturbation  $dAb_{\mu\nu}$  of the *constant* half-antisymmetric background solution  $A_{hab}$  for the A-tensor. This perturbation is calculated from the static part of the AK-equations eqtoievnu3b (see 4.3.), inserting  $Eb=E_{GK}$  and  $Ab=A_{hab}+dAb$ . The result for the perturbed solution Ab is

 $\left\{ Ab00[r1, th] \rightarrow A00c + \frac{alphax}{r1^3}, Ab01[r1, th] \rightarrow A00c + \frac{alphax}{r1^3}, Ab02[r1, th] \rightarrow -A00c + \frac{alphax}{r1^3}, Ab03[r1, th] \rightarrow A00c + \frac{alphax}{r1^3}, Ab03[r1, th$ 

with constant parameters  $Aijc = \{A00c, A10c, A20c, A30c\}$  and  $alphax = \alpha$ 

58  $dAb0 = \frac{\alpha}{r^3}$ ,  $dAb2 = \frac{i}{\sin\theta}$ , dAb1 = 0, dAb3 = 1

After inserting this result into *eqtoievnu3w* the wave part of the AK-equations, we get new equations *eqtoievnu3wdA*, which do not depend on the constants *Aijc*, only on *As*, *Es*,  $\alpha$ , *k*:

```
eo1 = \frac{3 r1^2 (-1 - i k r1) As00[r1, th] + 3 i k r1^3 As10[r1, th] - 6 alphax As11[r1, th] + 6 alphax As13[r1, th] + r1^3 Es20[r1, th] + r1^3 Es30[r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 6 alphax As13[r1, th] + r1^3 Es20[r1, th] + r1^3 Es30[r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 6 alphax As13[r1, th] + r1^3 Es20[r1, th] + r1^3 Es30[r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 6 alphax As13[r1, th] + r1^3 Es20[r1, th] + r1^3 Es30[r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 6 alphax As13[r1, th] + r1^3 Es20[r1, th] + r1^3 Es30[r1, th] + 3 r1^3 As00^{(1,0)} [r1, th] + 3 r1^3 As00^{(1,0)
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               3 r1<sup>4</sup>
                                                   (3+3 i k r1) As20[r1, th] +6 i r1 As11[r1, th] Csc[th] -6 i r1 As13[r1, th] Csc[th] -r1 Es00[r1, th] +r1 Es30[r1, th] +3 As10<sup>(0,1)</sup> [r1, th] -3 r1 As20<sup>(1,0)</sup> [r1, th]
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     3 r1²
            eq9-\frac{1}{3 r l} (-6 i k r 1 A s 21 [r 1, th] + 6 i k r 1 A s 23 [r 1, th] - 6 k r 1 A s 31 [r 1, th] Csc [th] + 6 k r 1 A s 33 [r 1, th] Csc [th] + i k r 1 E s 00 [r 1, th] - i k r 1 E s 10 [r 1, th] - 6 A s 11^{(0,1)} [r 1, th] + 6 A s 13^{(0,1)} [r 1, th] + 6 A s 14^{(0,1)} [r 1, th] + 6 A s 14^{(0,1)
                                              Es00<sup>(0,1)</sup> [r1, th] + Es20<sup>(0,1)</sup> [r1, th] + 6 r1 As21<sup>(1,0)</sup> [r1, th] - 6 r1 As23<sup>(1,0)</sup> [r1, th] - 6 r1 Csc[th] As31<sup>(1,0)</sup> [r1, th] + 6 r1 Csc[th] As33<sup>(1,0)</sup> [r1, th] + r1 Es10<sup>(1,0)</sup> [r1, th] + 
       eq13: \frac{3 \pm k r^{13} \text{ As20}[r1, th] - 6 \text{ alphax As21}[r1, th] + 6 \text{ alphax As23}[r1, th] + 6 \pm r^{13} \text{ As01}[r1, th] \text{ Csc}[th] - 6 \pm r^{13} \text{ As03}[r1, th] \text{ Csc}[th] - r^{13} \text{ Es10}[r1, th] + r^{13} \text{ Es30}[r1, th] + 3 r^{12} \text{ As00}[^{0,1}][r1, th]
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     3 r1<sup>4</sup>
          eq17=- -6 r1 As11[r1, th] +6 r1 As13[r1, th] -3 As30[r1, th] -3 ik r1 As30[r1, th] + r1 Es00[r1, th] + r1 Es20[r1, th] +3 r1 As30<sup>(1,0)</sup>[r1, th]
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            3 r1²
          eq21 = \frac{1}{3 r 1^5} \left( 6 r 1^3 (-1 - i k r 1) As01 [r1, th] + 6 r 1^3 (1 + i k r 1) As03 [r1, th] + 6 i k r 1^4 As11 [r1, th] - 6 i k r 1^4 As13 [r1, th] + 6 r 1^4 As13 [r
                                      24 alphax As31[r1, th] + 6 i alphax k r1 As31[r1, th] - 24 alphax As33[r1, th] - 6 i alphax k r1 As33[r1, th] - i k r1<sup>4</sup> Es00[r1, th] + r1<sup>3</sup> Es10[r1, th] + i k r1<sup>4</sup> Es10[r1, th] +
                                         r1<sup>3</sup> Es20[r1, th] +6 r1<sup>4</sup> As01<sup>(1,0)</sup>[r1, th] -6 r1<sup>4</sup> As03<sup>(1,0)</sup>[r1, th] -6 alphax r1 As33<sup>(1,0)</sup>[r1, th] +6 alphax r1 As33<sup>(1,0)</sup>[r1, th] -r1<sup>4</sup> Es10<sup>(1,0)</sup>[r1, th] -r1<sup>4</sup> Es20[r<sup>1</sup>, th] - r1<sup>4</sup> Es10<sup>(1,0)</sup>[r<sup>1</sup>, th] - r<sup>1</sup> Es20[r<sup>1</sup>, th
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               <sup>)</sup>[r1, th])
          eq25 = \frac{1}{r1^4} \left( -\frac{1}{2} k r1^3 Es00 [r1, th] + 2 alphax Es01 [r1, th] - 2 alphax Es03 [r1, th] - r1^2 Es10 [r1, th] - r1^2 [r1, th
                                         i k r1<sup>3</sup> Es10[r1, th] - 2 i r1<sup>3</sup> Csc[th] Es21[r1, th] + 2 i r1<sup>3</sup> Csc[th] Es23[r1, th] - 2 r1<sup>3</sup> Es31[r1, th] + 2 r1<sup>3</sup> Es33[r1, th] + r1<sup>2</sup> Es20<sup>(0,1</sup>[r1, th] + r1<sup>3</sup> Es10<sup>(1,0)</sup>[r1, th])
            eq29= <u>3 alphax (Es10[r1, th] + Es11[r1, th] + Es12[r1, th]</u> + Es13[r1, th])
                                                                                                                                                                                                                                                                                                                                                                                                             r1<sup>5</sup>
            \mathsf{eq30} = \frac{3\,\mathsf{alphax}\,\,(\mathsf{Es00}\,[\,\mathsf{r1,\,th}\,]\,+\mathsf{Es01}\,[\,\mathsf{r1,\,th}\,]\,+\mathsf{Es02}\,[\,\mathsf{r1,\,th}\,]\,+\mathsf{Es03}\,[\,\mathsf{r1,\,th}\,]\,)}{(\,\mathsf{r1,\,th}\,]\,(\,\mathsf{r1,\,th}\,]\,+\mathsf{Es03}\,[\,\mathsf{r1,\,th}\,]\,+\mathsf{Es03}\,[\,\mathsf{r1,\,th}\,]\,)}
                                                                                                                                                                                                                                                                                                                                                                                                             r1<sup>5</sup>
eq31=0
  eq32=0
For simplicity, we show only the first equation of a 4-group, but here the symmetry
column-index\leftrightarrowgroup-index compared to the eqtoievnu3w in 4.3. is lost,
e.g. eq1=eq1(As00,As10,As11,As13,Es20,Es30) depends also on \{As11,As13\}, not only on As10,
as in eqtoievnu3w.
  We eliminate Es3, Es0, Es1
       repEs3 = Solve[EquList[eqtoievnu3wdA[[Range[1, 4]]], {0, 0, 0, 0}], MEs[[4]]][[1]]
       repEsOnO = Solve[EquList[eqtoievnu3wdAsO[[Range[5, 8]]], {0, 0, 0, 0}], MEs[[1]]][[1]]
               repEs1n0 = Solve[EquList[eqtoievnu3wdAs1[[Range[13, 16]]], {0, 0, 0, 0}], MEs[[2]]][[1]]
and are left with 18 equs eqtoievnu3wdAs2s3 for 20 variables Es2, As :
eq1...8, eq13...16, eq31...32 vanish identically,
we show here resp. the first equaton from the respective 4-group
                    eq9=-\frac{As20[r1, th]}{r1} = \frac{i k As20[r1, th]}{r1} + 2 i k As21[r1, th] - \frac{6 alphax As21[r1, th]}{r14} = \frac{2 i alphax k As21[r1, th]}{r14} = \frac{2 i alphax k As21[r1, th]}{r14} = \frac{2 i alphax k As23[r1, th]}{r
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              r1<sup>4</sup>
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                r1<sup>3</sup>
                                 \frac{r_{1}^{r_{1}}}{2k \text{ As03}[r_{1}, t_{1}] (sc(t_{1}) + 2k \text{ As11}[r_{1}, t_{1}] (sc(t_{1}) - 2k \text{ As13}[r_{1}, t_{1}] (sc(t_{1}) - 2k \text{ As33}[r_{1}, t_{1}] (sc(t_{1}) + 2k \text{ As33}[r_{1}, t_{1}] (sc(t_{1}) +
                                 \frac{A_{21}}{r_{12}} - \frac{1}{r_{12}} -
                                 2 \text{ AS21}^{(1,0)} [r1, \text{ th}] + \frac{2 \text{ alphax AS21}^{(1,0)} [r1, \text{ th}]}{r1^3} + 2 \text{ AS23}^{(1,0)} [r1, \text{ th}] - \frac{2 \text{ alphax AS21}^{(1,0)} [r1, \text{ th}]}{r1^3} + 2 \text{ alphax AS21}^{(1,0)} [r1, \text{ th}] - 2 \text{ alphax AS21}^{(1,0)} [r1, \text{ th}] + \frac{4 \text{ AS10}^{(1,1)} [r1, \text{ th}]}{r1} + \frac{4 \text{ AS20}^{(1,1)} [r1, \text{ th}]}{r1} - \text{ AS20}^{(2,0)} [r1, \text{ th}]
            eq17 = -i k As00 [r1, th] - \frac{As00 [r1, th]}{r1} + i k As10 [r1, th] + 2 As11 [r1, th] - \frac{2 alphax As11 [r1, th]}{r1^3} - 2 As13 [r1, th] + \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As13 [r1, th]}{r1^3} - i k As20 [r1, th] - \frac{2 alphax As
                               \frac{As20[r1, th]}{r1} + i k As30[r1, th] + \frac{As30[r1, th]}{r1} - 2 i As11[r1, th] Csc[th] + 2 i As13[r1, th] Csc[th] - \frac{As10^{(0,1)}[r1, th]}{r1} + As00^{(1,0)}[r1, th] + As20^{(1,0)}[r1, th] - As30^{(1,0)}[r1, th] + As30[r1, th] 
                                                   \frac{2 \text{As00}[r1, \text{th}]}{r^2} + \frac{i \text{ k As00}[r1, \text{th}]}{r^1} - 2 \text{ i k As01}[r1, \text{th}] - \frac{2 \text{As01}[r1, \text{th}]}{r^1} + 2 \text{ i k As03}[r1, \text{th}] + \frac{2 \text{As03}[r1, \text{th}]}{r^1} - \frac{i \text{ k As10}[r1, \text{th}]}{r^1} + 2 \text{ i k As11}[r1, \text{th}] + \frac{8 \text{ alphax As11}[r1, \text{th}]}{r^4} - 2 \text{ i k As13}[r1, \text{th}] - \frac{8 \text{ alphax As13}[r1, \text{th}]}{r^4} + \frac{2 \text{ alphax As13}[r1, \text{th}]}{r^4} + \frac{8 \text{ alphax As
                               \frac{8 \text{ alphax AS21}[r], \text{ th}}{r1^4} = \frac{2 \text{ i alphax kAS21}[r], \text{ th}}{r1^3} + \frac{8 \text{ alphax kAS21}[r], \text{ th}}{r1^4} + \frac{2 \text{ i alphax kAS21}[r], \text{ th}}{r1^3} + \frac{8 \text{ alphax kAS31}[r], \text{ th}}{r1^4} + \frac{2 \text{ i alphax kAS31}[r], \text{ 
                               2kAs01[n1, th] Csc[th] + \frac{2iAs01[n1, th] Csc[th]}{n1} + 2kAs03[n1, th] Csc[th] - \frac{2iAs03[n1, th] Csc[th]}{n1} + 2kAs11[n1, th] Csc[th] + 2kAs11[n1, th] Csc[th] + \frac{2As00[n1, [n1, th]}{n^4} + \frac{kAs00[n1, [n1, th]}{n^
                               \frac{\pm k A S 10^{(\theta_1,1)} (r1, th)}{r} = \pm k A S 00^{(1,\theta)} (r1, th) = \frac{2 A S 00^{(1,\theta)} (r1, th)}{r} + 2 A S 01^{(1,\theta)} (r1, th) = 2 \pm C S C (th) A S 01^{(1,\theta)} (r1, th) = 2 A S 03^{(1,\theta)} (r1, th) + 2 \pm C S C (th) A S 03^{(1,\theta)} (r1, th) = 1 + 2 A S 00^{(1,\theta)} (r1, th) = 2 A S 03^{(1,\theta)} (r1, th) = 2 A S 03
                               \frac{r1}{2alphax}\frac{r1}{As11^{(1,0)}\left[r1,t\right]} + \frac{2alphax}{r1^3} + \frac{2al
            eq25 \cdot 6 k^{2} As00[r1, th] - \frac{6 As00[r1, th]}{r1} - \frac{9 \pm k As00[r1, th]}{r1} - \frac{9 \pm k As00[r1, th]}{r1} + 6 \pm k As01[r1, th] - \frac{6 alphax As01[r1, th]}{r4} - \frac{6 \pm alphax k As01[r1, th]}{r4} + \frac{6 \pm alphax k As01[r1, th]}{r4} + \frac{6 \pm alphax k As03[r1, th]
                         \frac{24 \text{ abphar Ass2}[r1, th]}{r1} - \frac{6 \pm 830[r1, th]}{r1} - \frac{6 \pm 830[r1, th]}{r1} - \frac{64 \text{ base}[r1, th]}{r1} - \frac{24 \text{ alphar Ass1}[r1, th]}{r1}
                         \frac{1}{r^1} - 6 \stackrel{i}{k} As10 [r^1, th] + \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} + \frac{1}{r^2} - 6 \stackrel{i}{k} As11 [r^1, th] - \frac{1}{r^2} - 6 \stackrel{i}{k} As11 [r^1, th] - \frac{1}{r^2} - \frac{1}{r^
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         r1<sup>4</sup>
                         \frac{nl^3}{18 \text{ alphax As23}(n1, \text{th})} + 6 \text{ kAs01}(n1, \text{th}) \text{ Csc}(\text{th}) - \frac{6 \text{ i As01}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ alphax}} = 6 \text{ kAs03}(n1, \text{th}) \text{ Csc}(\text{th}) + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ alphax}} + \frac{24 \text{ i alphax As10}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(\text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, \text{th})}{16 \text{ csc}(10, \text{th})} + \frac{6 \text{ i As03}(n1, \text{th}) \text{ Csc}(10, 
                         \frac{-1}{r_1^4} = -6kAsi3[r_1, t_1] \cdot Sc[t_1] = \frac{-1}{r_1} = -6kAsi3[r_1, t_1] \cdot Sc[t_1] = \frac{-1}{r_1} = \frac{-1}{r_1^3} = -6kAsi3[r_1, t_1] \cdot Sc[t_1] = -6kAsi3[r_1, t_1] \cdot Sc[
                    \frac{1}{r_{1}^{13}} - 6 k A S I [r], th] (S C [th]) = \frac{1}{r_{1}^{12}} + 2 k E S 20 [r], th] + \frac{2 l k E S 20 [r], th] + \frac{2 2 l k A S 10^{(0,1)} [r], th}{r_{1}^{13}} + 2 k S C [th] E S 21 [r], th] - \frac{6 A S 00^{(0,1)} [r], th]}{r_{1}^{2}} - \frac{3 k A S 00^{(0,1)} [r], th]}{r_{1}} - \frac{3 k A S 00^{(0,1)} [r], th]}{r_{1}} - \frac{4 k A S 10^{(0,1)} [r], th}{r_{1}^{4}} - \frac{4 l k A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 11^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{(0,1)} [r], th}{r_{1}^{4}} + \frac{6 l p h a A S 10^{
                       \frac{1}{r_{1}} = 
                       \frac{1}{r_1} \rightarrow 2 + k ASSO^{-1} [r_1, t_1] + \frac{1}{r_1} - 0 ASSI^{-1} [r_1, t_1] + \frac{1}{r_1} - 0 ASSI^{-1} [r_1, t_1] + \frac{1}{r_1} - 0 ASSI^{-1} [r_1, t_1] + 0 ASSI^{-1} [r_1, t_1] + 0 ASSI^{-1} [r_1, t_1] + 0 ASSI^{-1} [r_1, t_1] - \frac{1}{r_1} - \frac{1}{
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             r1<sup>3</sup>
```

 $eq29=3 \pm k \ As00 \ [r1, th] + \frac{3 \ As00 \ [r1, th]}{r1} - 3 \pm k \ As01 \ [r1, th] - \frac{3 \ As01 \ [r1, th]}{r1} - 3 \pm k \ As02 \ [r1, th] - \frac{3 \ As02 \ [r1, th]}{r1} - 3 \pm k \ As03 \ [r1, th] - \frac{3 \ As03 \ [r1, th]}{r1} - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [r1, th] - 3 \pm k \ As03 \ [$ 

 $\frac{34803[r1, th]}{r1} = -3 \pm kAs20[r1, th] = -\frac{1}{r1} = -3 \pm kAs20[r1, th] = -\frac{1}{r1} = -3 \pm kAs03[r1, th] = -\frac{3}{r1} = -3 \pm kAs03[r1, th] = -\frac{3}{r1} \pm kAs20[r1, th]$ 

 $\frac{r_{1}^{3}}{r_{1}} = \frac{-3.1 \times A522[r_{1}, t_{1}] - 3.1 \times A525[r_{1}, t_{1}] + \frac{r_{1}^{3}}{r_{1}} + \frac{r_{1}^{3}}{r_{1}^{3}} + 12.1 \times S01[r_{1}, t_{1}] \times Sc[t_{1}] - 12.1 \times S05[r_{1}, t_{1}] \times Sc[t_{1}] - 12.1 \times S$ 

#### The solution at infinity {*Esi2i*, *Asi0i*, *Asi1i*, *Asi2i*, *Asi3i*}, i.e. order O(1) in r-powers is

 $\left[\text{Es20[r1, th]} \rightarrow 0, \text{Es21[r1, th]} \rightarrow 0, \text{Es22[r1, th]} \rightarrow 0, \text{Es23[r1, th]} \rightarrow 0, \text{As00[r1, th]} \rightarrow \text{As00[th]}, \text{As01[r1, th]} \rightarrow \text{As00[th]}, \text{As02[r1, th]} \rightarrow -\text{As00[th]}, \text{As03[r1, th]} \rightarrow \text{As00[th]}, \text{As03[r1, th]} \rightarrow \text{As00[th]}, \text{As03[r1, th]} \rightarrow \text{As00[th]}, \text{As00[r1, th]} \rightarrow \text{As00[th]}, \text{As00[r1, th]} \rightarrow \text{As00[th]}, \text{As00[r1, th]} \rightarrow \text{As00[r1, th]} \rightarrow$  $[cs20[r1, tn] \rightarrow 0, cs22[r1, tn] \rightarrow 0, cs22[r1, tn] \rightarrow 0, cs22[r1, tn] \rightarrow 0, cs22[r1, tn] \rightarrow 0, ass0[r1, tn] \rightarrow 0, ass0[r1, tn] \rightarrow ass0[r1, tn] \rightarrow$ 

with the free parameter variable  $As00(\theta)$ , which describes the wave  $As_{00}(r,\theta) = \frac{As_{00}(\theta)}{r} \exp(-ik(r-t))$ 

generated by the bgr.

Now we insert for the parameters  $k = \frac{1}{\sqrt{2r_0^3}}$  and  $\alpha = \frac{c_0}{r_0}$ , so the equations depend now only on

the bgr-parameter  $r_0$ 

 $eqtoievnu3wdAs = eqtoievnu3wdAs(r_0, Es2, As0, As1, As2, As3)$ 

and in powers of  $r_0$  the dependence is

 $eqtoievnu3wdAs = eqtoievnu3wdAs0 + eqtoievnu3wdAs1/r_0 + eqtoievnu3wdAs1/r_0^{3/2} + \dots$ 

the parameter variable  $AsOO(r, \theta, r_0)$  has to satisfy the equations also in  $r_0$ .

Our goal in the following subsection is to calculate  $AsOO(r, \theta, r_0)$  analytically in  $\{\theta, r_0\}$  as a series in r, and all other variables, too.

$$As00(r, \theta, r_0) = (As00n00(\theta) + \frac{As00n01(\theta)}{r_0} + \frac{As00n02(\theta)}{r_0^{3/2}} + \dots) + \frac{(As00n10(\theta) + \frac{As00n11(\theta)}{r_0} + \frac{As00n12(\theta)}{r_0^{3/2}} + \dots)}{r} + \dots$$

The first term in As00 describes the wave at infinity and the dependence of the amplitude on the parameter  $r_0$  of the bgr. An important result of the following subsection is :  $As00(r, \theta, r_0)$ 

 $As00n00(\theta) = 0$ , i.e. in first approximation for  $r_0 \rightarrow \infty$   $As00(r, \theta, r_0) = \frac{As00n01(\theta)}{r_0}$ 

we transform  $r \rightarrow 1/z$  and develop in a series in z around z=0: Es20[z, th]  $\rightarrow$  K1n0[th] + z K1n1[th] +  $z^2$  K1n2[th] +  $z^3$  K1n3[th] +  $z^4$  K1n4[th] ... As33[z, th]  $\rightarrow$  K20n0[th] + z K20n1[th] +  $z^2$  K20n2[th] +  $z^3$  K20n3[th] +  $z^4$  K20n4[th] {K1n0( $\theta$ ),..., K20n0( $\theta$ )} is the solution at infinity {Esi2i, Asi0i, Asi1i, Asi2i, Asi3i} from above equations in a series in z and get 5 groups of equations, each for a coefficient of  $z^k$ , k=0,1,2,3,4  $eqtoievnu3wdAzKcn0=coef(z^0)$   $eqtoievnu3wdAzKcn1=coef(z^1)$   $eqtoievnu3wdAzKcn3=coef(z^3)$  $eqtoievnu3wdAzKcn4=coef(z^4)$ 

In order to develop the 20 variables {*Es2i*, *As0i*, *As1i*, *As2i*, *As3i*} and the equations in a series in *r*-powers around  $r=\infty$ , we transform  $r \rightarrow 1/z$  and develop in a series in z around z=0:

 $\mathsf{Es20[z, th]} \rightarrow \mathsf{K1n0[th]} + \mathsf{z}\,\mathsf{K1n1[th]} + \mathsf{z}^2\,\mathsf{K1n2[th]} + \mathsf{z}^3\,\mathsf{K1n3[th]} + \mathsf{z}^4\,\mathsf{K1n4[th]}$ 

As33[z, th]  $\rightarrow$  K20n0[th] + z K20n1[th] + z<sup>2</sup> K20n2[th] + z<sup>3</sup> K20n3[th] + z<sup>4</sup> K20n4[th]

 $\{K1n0(\theta), \dots, K20n0(\theta)\}$  is the solution at infinity  $\{Esi2i, Asi0i, Asi1i, Asi2i, Asi3i\}$  from 4.4.1. Then we develop the equations in a series in z and get 5 groups of equations, each for a coefficient of  $z^k$ , k=0,1,2,3,4  $eqtoievnu3wdAzKcn0=coef(z^{0})$ 

 $eqtoievnu3wdAzKcn1=coef(z^{1})$  $eqtoievnu3wdAzKcn2=coef(z^2)$  $eqtoievnu3wdAzKcn3=coef(z^3)$  $eqtoievnu3wdAzKcn4=coef(z^4)$ 

eqtoievnu3wdAzKcn0 vanishes identically, because the ansatz already solves the equations at infinity. The remaining 4 equation groups have to be solved consecutively and the solution inserted in the next equation group, until eqtoievnu3wdAzKcn4 is solved.

The solution of *eqtoievnu3wdAzKcn1* :

#### rKvardAn1=

[K1n1[th] →	$\frac{3 \text{ iAs00} + 3 \text{ kK10n1[th]} + 3 \text{ kK6n1[th]}}{i + Csc(th)}, \text{ K2n1[th]} \rightarrow \frac{3 \text{ (-iAs00} - \text{ kK10n1[th]} + \text{ kK6n1[th]})}{i + Csc(th)}, \text{ K4n1[th]} \rightarrow \frac{3 \text{ (-iAs00} - \text{ kK10n1[th]} + \text{ kK6n1[th]})}{i + Csc(th)}, \text{ K5n1[th]} \rightarrow -\text{K10n1[th]} + \text{ K6n1[th]} + \text{ K6n1[th]}), \text{ K6n1[th]} \rightarrow -\text{K10n1[th]} + \text{ K6n1[th]}), \text{ K6n1[th]} \rightarrow -\text{K10n1[th]} \rightarrow -\text{K10n1[th]} + \text{ K6n1[th]}), \text{ K6n1[th]} \rightarrow -\text{K10n1[th]} \rightarrow -\text{K10n1[th]} + \text{ K6n1[th]}), \text{ K6n1[th]} \rightarrow -\text{K10n1[th]} \rightarrow -K$
K12n1[th]	$ + \texttt{K10n1[th]} - \texttt{K6n1[th]} + \texttt{K8n1[th]}, \texttt{K13n1[th]} \rightarrow \frac{\texttt{Csc[th]} (\texttt{iAs00} + \texttt{K10n1[th]} - (\texttt{2i} + \texttt{k} + \texttt{2Csc[th]}) \texttt{K6n1[th]} + \texttt{2iK8n1[th]} + \texttt{2Csc[th]} \texttt{K8n1[th]} + \texttt{2Csc[th]} \texttt{Kn11[th]} + \texttt{Kn11[th]} +$
K14n1[th]	$\xrightarrow{\text{Csc[th]}(i \text{ As00} + (-2i + k - 2 \text{ Csc[th]}) \text{ K10n1[th]} + (2i - k + 2 \text{ Csc[th]}) \text{ K6n1[th]} - 2 \text{ K8n1[th]} - 2 \text{ Csc[th]} \text{ K8n1[th]} + 2 \text{ i K9n1[th]} + 2 \text{ Csc[th]} \text{ K9n1[th]})}{k (i + \text{Csc[th]})},$
K15n1[th]	$ + \texttt{K11n1[th]} = \frac{3\texttt{i}\texttt{As00} - \texttt{i}\texttt{K3n1[th]} + 3\texttt{k}\texttt{K7n1[th]}}{3\texttt{k}}, \texttt{K16n1[th]} \rightarrow \frac{\texttt{Csc[th]}(\texttt{i}\texttt{As00} + (\texttt{2}\texttt{i} + \texttt{k} + \texttt{2}\texttt{Csc[th]})\texttt{K10n1[th]} - \texttt{k}\texttt{K6n1[th]} - \texttt{2}(\texttt{i} + \texttt{Csc[th]})\texttt{K9n1[th]})}{\texttt{k}(\texttt{i} + \texttt{Csc[th]})}, \texttt{K10n1[th]} \rightarrow \frac{\texttt{Csc[th]}(\texttt{i}\texttt{As00} + (\texttt{2}\texttt{i} + \texttt{k} + \texttt{2}\texttt{Csc[th]})\texttt{K10n1[th]} - \texttt{k}\texttt{Kn11[th]} - \texttt{k}\texttt{Kn11[th]} - \texttt{k}\texttt{Kn11[th]})}{\texttt{k}(\texttt{i} + \texttt{Csc[th]})}, \texttt{Kn111[th]} \rightarrow \frac{\texttt{Kn111[th]} - \texttt{k}\texttt{Kn11[th]}}{\texttt{k}(\texttt{i} + \texttt{Csc[th]})}, \texttt{Kn111[th]} - \texttt{k}\texttt{Kn11[th]} - \texttt{k}\texttt{Kn11[th]} - \texttt{k}\texttt{Kn11[th]})}{\texttt{k}(\texttt{i} + \texttt{csc[th]})}, \texttt{Kn111[th]} \rightarrow \frac{\texttt{k}\texttt{Kn11[th]}}{\texttt{k}(\texttt{i} + \texttt{k} +$
K17n1[th]	$\xrightarrow{\text{ASOO}-ik\text{KION1[th]}+i(2i+k+2\text{Csc[th]})\text{KON1[th]}+2\text{KSN1[th]}-2i\text{Csc[th]}\text{KSN1[th]}}_{k(i+\text{Csc[th]})},$
K40 4 (11)	As00 - i (-2 i + k - 2 Csc[th]) K10n1[th] + (2 + i k - 2 i Csc[th]) K6n1[th] - 2 K8n1[th] + 2 i Csc[th] K8n1[th] + 2 K9n1[th] - 2 i Csc[th] K9n1[th]

$$\begin{split} \kappa & (1 + Csc[th]) \\ \text{K19n1[th]} \rightarrow \frac{i \text{K3n1[th]}}{3 \, k} \text{, K20n1[th]} \rightarrow \frac{\text{As00} - i \left(2 \, i + k + 2 \, \text{Csc[th]}\right) \text{K10n1[th]} + i \text{ K6n1[th]} - 2 \, \text{K9n1[th]} + 2 \, i \, \text{Csc[th]} \text{ K9n1[th]} + 2 \, i \, \text{Csc[th]} + 2 \,$$
k (i + Csc[th]

### remaining free variables *KvardAn1f*

{K3n1[th], K6n1[th], K7n1[th], K8n1[th], K9n1[th], K10n1[th], K11n1[th]}

### The solution of *eqtoievnu3wdAzKcn2* :

#### rKvardAn2=

 $\left[ \text{K6n1[th]} \rightarrow \frac{i \; (\text{As00-}i \; k \; \text{K10n1[th]})}{i \; \text{K7n1[th]}} \text{, } \text{K7n1[th]} \rightarrow -\frac{i \; (\text{As00+}i \; k \; \text{K11n1[th]})}{i \; \text{K1n1[th]}} \text{, } \text{K3n1[th]} \rightarrow 0 \text{, } \text{K1n1[th]} \right] = 0 \text{, } \text{K1n1[th]} = 0 \text$ 

 $\frac{\kappa}{\text{Kin2[th]}} + \frac{3 \text{ KKin2[th]}}{i + \text{Csc[th]}} - \frac{3 \text{ KKon2[th]}}{i + \text{Csc[th]}} + \frac{1}{4 \text{ } (i + \text{Csc[th]})^2} \frac{\kappa}{3 \text{ } (a \text{ i A S00 - 8 A S00 Csc[th] - A S00 Cot[th] Csc[th] + i A S00 \text{ } Cot[th] Csc[th] + i \text{ } (a \text{ } a \text{ } a \text{ } b \text{ } a \text{ } a \text{ } b \text$ 

 $\pm k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{K8n1}(\texttt{th}) = k^2 \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{K8n1}(\texttt{th}) = k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th})^2 \operatorname{K8n1}(\texttt{th}) + 2 \pm k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{K9n1}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th})^2 \operatorname{K9n1}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{Csc}(\texttt{th})^2 \operatorname{K9n1}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th})^2 \operatorname{K9n1}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th})^2 \operatorname{K9n1}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{Csc}(\texttt{th}) + 2 k \operatorname{Cot}(\texttt{th}) \operatorname{Csc}(\texttt{th}) \operatorname{Csc$ 

 $K2n2[th] \rightarrow -\frac{3kK10n2[th]}{i+Csc[th]} + \frac{3kK0n2[th]}{i+Csc[th]} - \frac{1}{4k(i+Csc[th])^2} 3\left(-8iAs00-8As00Csc[th] + As00Cot[th]Csc[th] + iAs00kCot[th]Csc[th] - iAs00Cot[th]Csc[th]^2\right)$ 

 $8 \, \texttt{K10n1[th]} + 8 \, \texttt{i} \, \texttt{K} \, \texttt{Csc[th]} \, \texttt{K10n1[th]} + \texttt{i} \, \texttt{K} \, \texttt{Cot[th]} \, \texttt{Csc[th]} \, \texttt{K10n1[th]} + \texttt{k}^2 \, \texttt{Cot[th]} \, \texttt{Csc[th]} \, \texttt{K10n1[th]} + \texttt{k} \, \texttt{K10n1[th]} + \texttt{k} \, \texttt{Cot[th]} \, \texttt{K10n1[th]} + \texttt{k} \, \texttt{k} \, \texttt{K10n1[th]} + \texttt{k} \, \texttt{k} \, \texttt{K10n1[th]} + \texttt{k} \, \texttt$ 

 $\pm k \cot(th) \csc(th) K8n1(th) - k^2 \cot(th) Csc(th) K8n1(th) + k \cot(th) Csc(th)^2 K8n1(th) - 2 \pm k \cot(th) Csc(th) K9n1(th) - 2 k \cot(th) Csc(th)^2 K9n1(th))$ 

k (i + Csc[th])

K13n2[th]

 $\frac{\mathsf{Csc[th]}\left(2\,i-k+2\,\mathsf{Csc[th]}\right)\,\mathsf{K10n2[th]}}{k\,\left(i+\mathsf{Csc[th]}\right)}+\frac{2\,\mathsf{Csc[th]}\,\mathsf{K12n2[th]}}{k}=\frac{\mathsf{Csc[th]}\,\mathsf{K6n2[th]}}{i+\mathsf{Csc[th]}}$ 

 $\frac{1}{4\,k^2\,(i+Csc[th])^2}\,\left(16\,As\theta\theta\,Csc[th]+8\,i\,As\theta\theta\,k\,Csc[th]-i\,As\theta\theta\,k\,Cot[th]\,Csc[th]-As\theta\theta\,k^2\,Cot[th]\,Csc[th]-32\,i\,As\theta\theta\,Csc[th]^2+8\,As\theta\theta\,k\,Csc[th]^2-As\theta\theta\,k\,Cot[th]\,Csc[th]^2+20\,Csc[th]^2+8\,Csc[$ 

 $\left(1 + \operatorname{Sc}(\operatorname{In})^{2} + \operatorname{I6} i k \operatorname{Sc}(\operatorname{Ih}) \operatorname{Kl0n1}(\operatorname{Ih}) + 8 k^{2} \operatorname{Cs}(\operatorname{Ih}) \operatorname{Kl0n1}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Kl0n1}(\operatorname{Ih}) + i k^{3} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Kl0n1}(\operatorname{Ih}) - 32 k \operatorname{Sc}(\operatorname{Ih})^{2} \operatorname{Kl0n1}(\operatorname{Ih}) + 8 i k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih})^{2} \operatorname{Kl0n1}(\operatorname{Ih}) + 16 i k \operatorname{Sc}(\operatorname{Ih})^{2} \operatorname{Kl0n1}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{K0n1}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih})^{2} \operatorname{K0n1}(\operatorname{Ih}) + 16 i k \operatorname{Sc}(\operatorname{Ih})^{3} \operatorname{K0n1}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih})^{2} \operatorname{K0n1}(\operatorname{Ih}) - 1 k^{2} \operatorname{Cot}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}(\operatorname{Ih}) + k^{2} \operatorname{Sc}(\operatorname{Ih}) \operatorname{Sc}$ 

i k<sup>3</sup> Cot[th] Csc[th] K8n1[th] +

SL kost(un) = Sint(un) = 1 + k cost(un) est(un) = 1 + k cost(un) est(un) = 1 + k cost(un) est(un) est(un) = 1 + k cost(un) est(un) est(un) est(un) = 1 + k cost(un) est(un) est(un $16 \pm k \operatorname{Csc}(th)^3 \operatorname{K8n1}(th) + 16 \pm k \operatorname{Csc}(th) \operatorname{K9n1}(th) + 2k^2 \operatorname{Cot}(th) \operatorname{Csc}(th) \operatorname{K9n1}(th) + 32k \operatorname{Csc}(th)^2 \operatorname{K9n1}(th) - 2\pm k^2 \operatorname{Cot}(th) \operatorname{Csc}(th)^2 \operatorname{K9n1}(th) - 16\pm k \operatorname{Csc}(th)^3 \operatorname{K9n1}(th) + \frac{2\operatorname{Csc}(th) \operatorname{K9n2}(th)}{\iota},$ 

 $K16n2[th] \rightarrow \frac{Csc[th] (2i+k+2Csc[th]) K10n2[th]}{k(i+Csc[th])} = \frac{Csc[th] K6n2[th]}{i+Csc[th]} = \frac{1}{4k^2(i+Csc[th])^2} \left[ 8iAs00Csc[th] + 3iAs00Cot[th] Csc[th] - As00kCot[th] Csc[th] + 8As00Csc[th]^2 + 2As00Csc[th]^2 + 2As00kCot[th] Csc[th] + 2As00kCot[th] + 2As00kCot[th]$ 

 $\frac{1 + \cos(\omega_1)}{3 \operatorname{SeO}(\operatorname{Cot})} = \frac{4 \times (1 + \cos(\omega_1))}{4 + \cos(\omega_1)} + 4 \times (1 + \cos(\omega_1)) + 1 + \frac{1}{2} \operatorname{Cot}(\operatorname{Th}) \operatorname{Csc}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{SeO}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) \operatorname{Csc}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{Csc}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{Csc}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{Csc}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{SeO}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{SeO}(\operatorname{Th}) \operatorname{SeO}(\operatorname{Th}) + 3 \operatorname{S$ 3 i k Cot [th] Csc [th]<sup>2</sup> K8n1 [th]

 $16 \text{ i} Csc[th] \text{ K9n1}[th] + 2 \text{ k} Cot[th] Csc[th] \text{ K9n1}[th] - 32 Csc[th]^2 \text{ K9n1}[th] - 2 \text{ i} \text{ k} Cot[th] Csc[th]^2 \text{ K9n1}[th] + 16 \text{ i} Csc[th]^3 \text{ k} Csc[th]$ 

 $K16n2[th] \rightarrow \frac{Csc[th] (2i+k+2Csc[th]) K10n2[th]}{k(i+Csc[th])} = \frac{Csc[th] K6n2[th]}{i+Csc[th]} = \frac{1}{4k^2(i+Csc[th])^2} \left[ 8iAs00 Csc[th] + 3iAs00 Csc[th] + 3iAs00 Csc[th] - As00 kCsc[th] - Csc[th] + 8As00 Csc[th]^2 + 2iAs00 Csc[th]^2 +$ 

 $\begin{array}{l} \begin{array}{c} x_{k} + cs_{k}(m) \\ x_{k} + cs_{k$  $16 \pm Csc[th] K9n1[th] + 2kCot[th] Csc[th] K9n1[th] - 32Csc[th]^2 K9n1[th] - 2\pm kCot[th] Csc[th]^2 K9n1[th] + 16\pm Csc[th]^3 K9n1[th] \right) = \frac{2Csc[th] K9n2[th]}{k},$ 

 $K17n2[th] \rightarrow -\frac{(2+ik-2iCsc[th])}{k(i+Csc[th])} - \frac{2iK12n2[th]}{k} - \frac{2iK12n2[th]}{k} + \frac{iK6n2[th]}{i+Csc[th]} - \frac{1}{4k^3(i+Csc[th])^2} \left(-16iAs\theta\theta + 8As\theta\theta k - 32As\theta\theta Csc[th] - 8iAs\theta\theta k Csc[th] - iAs\theta\theta k Cot[th] Csc[th] - 2iK12n2[th] + \frac{1}{4k^3(i+Csc[th])^2} \left(-16iAs\theta\theta + 8As\theta\theta k - 32As\theta\theta Csc[th] - 8iAs\theta\theta k Csc[th] - iAs\theta\theta k Cot[th] Csc[th] + \frac{1}{4k^3(i+Csc[th])^2} \left(-16iAs\theta\theta + 8As\theta\theta k - 32As\theta\theta Csc[th] - 8iAs\theta\theta k Csc[th] - 8iAs\theta\theta k Csc[th] - 8iAs\theta\theta k Csc[th] - \frac{1}{4k^3(i+Csc[th])^2} \left(-16iAs\theta\theta + 8As\theta\theta k - 32As\theta\theta Csc[th] - 8iAs\theta\theta k Cs$ 

 $\begin{array}{l} \kappa & (1+CSC(In)) & \kappa & (1+CSC(Inn)) & 4K \\ AS08\,k^2\,Cot(Ih)\,Csc(Ih)\,+16\,i\,As00\,Csc(Ih)^2\,-As00\,k\,Cot(Ih)\,Csc(Ih)^2\,-16\,k\,KI0n1(Ih)\,-8\,i\,k^2\,KI0n1(Ih)\,+32\,i\,k\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,+8\,k^2\,Csc(Ih)\,KI0n1(Ih)\,-8\,k^2\,KI0n1($ 

 $K18n2[th] \rightarrow \frac{i \ K10n2[th]}{i + Csc[th]} + \frac{i \ K12n2[th]}{k} + \frac{i \ K0n2[th]}{k} + \frac{i \ K0n2[th]}{k} + \frac{i \ K0n2[th]}{i + Csc[th]} - \frac{1}{4k^3 \ (i + Csc[th])^2} \left(16 \ i \ As00 + 8 \ As00 \ k + 32 \ As00 \ csc[th] - 8 \ i \ As00 \ k \ Csc[th] - i \ As00 \ k \ Cot[th] \ Csc[th] - As00 \ k^2 \ Cot[th] \ Csc[th] - 4k^3 \ Cot[th] \ Csc[th] \ Csc[th$  $16 i A S 0 \theta C S c [th]^2 - A S 0 \theta K C d [th] C S c [th]^2 - \theta i k^2 K I 0 n 1 (th] - 8 k^2 C S c [th] K 1 0 n 1 (th] - 3 k^2 C O t [th] C S c [th] K I 0 n 1 [th] + i k^3 C O t [th] C S c [th] K I 0$ 

i k<sup>2</sup> Cot(th) Csc(th)<sup>2</sup> K8n1(th) + 16 k K9n1(th) - 32 i k Csc(th) K9n1(th) + 2 k<sup>2</sup> Cot(th) Csc(th) K9n1(th) - 16 k Csc(th)<sup>2</sup> K9n1(th) - 2 i k<sup>2</sup> Cot(th) Csc(th) Csc(th) - 2 i k<sup>2</sup> Cot(th) Csc(th) - 2 i k<sup>2</sup> Cot(th) Csc(th) - 2 i k<sup>2</sup> Cot(th) - 2 i k<sup>2</sup>

$$\begin{split} \text{K19n2[th]} \rightarrow & -\frac{2\left(\text{As00} + i\;k\;\text{K11n1[th]}\right)}{k^2} - \text{K11n2[th]} + \text{K15n2[th]} + \text{K7n2[th]}\text{, K20n2[th]} \rightarrow & -\frac{i\left(2\;i+k+2\;\text{Csc[th]}\right)\;\text{K10n2[th]}}{k\;(i+\text{Csc[th]})} + \frac{i\;\text{K6n2[th]}}{i+\text{Csc[th]}} + \frac{i\;\text{K6n2[th$$

1 4 k<sup>2</sup> (i+Csc[th])<sup>2</sup> [8As00 - 8 i As00 Csc[th] + 3 i As00 Cot[th] Csc[th] - As00 k Cot[th] Csc[th] + 3 As00 Cot[th] Csc[th]<sup>2</sup> + 16 K10n1[th] - 8 i k K10n1[th] - 32 i Csc[th] K10n1[th] - 8 k Csc[th] K10n1[th] + 16 k Csc[th] K10n1[th] - 8 k Csc  $\frac{1}{1000} \left[ \frac{1}{1000} \left[$  $3 \\ i \\ k \\ Cot[th] \\ Csc[th]^2 \\ Kont[th] - 16 \\ Kont[th] - 16 \\ Kont[th] + 32 \\ i \\ Csc[th] \\ Kont[th] + 2 \\ k \\ Cot[th] \\ Kont[th] + 16 \\ Csc[th]^2 \\ Kont[th] - 2 \\ i \\ k \\ Cot[th] \\ Csc[th]^2 \\ Kont[th] - 2 \\ i \\ k \\ Csc[th]^2 \\ Kont[th] \\ Csc[th] \\$ 

#### remaining free variables *KvardAn12f*

(K8n1[th], K9n1[th], K10n1[th], K11n1[th], K6n2[th], K7n2[th], K9n2[th], K10n2[th], K11n2[th], K12n2[th], K15n2[th])

#### The solution of *eqtoievnu3wdAzKcn3* :

#### {K9n2[th], K10n2[th], K11n2[th]}

#### *rKvardAn31*= 3 variables

 $[89n2[th] \rightarrow [24\,800\,r0x^3 - 192\,As00\,r0x^6 + 384 \pm As00\,r0x^6 Csc[th] + 192\,As00\,r0x^6 Csc[th]^2 - 12\pm i\sqrt{2} r0x^{3/2} K12n2[th] - 48\,r0x^3 K12n2[th] - 12\sqrt{2} r0x^{3/2} Csc[th] K12n2[th] + 96\pm r0x^3 Csc[th] K12n2[th] + 192\,As00\,r0x^6 - 12\pm i\sqrt{2} r0x^{3/2} Csc[th] - 12\pm i\sqrt{2}$ 

 $48 \text{ rex}^3 \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] - 4 \text{ K12n3}[\text{th}] + 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ K12n3}[\text{th}] + 4 \pm \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] + 16 \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] - \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K13n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K13n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K13n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K13n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K13n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K12n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - 8 \pm \sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}]^2 \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K12n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ K12n3}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ Csc}[\text{th}] \text{ K12n3}[\text{th}] - \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ rex}^{3/2} \text{ csc}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ csc}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ csc}[\text{th}]}{\sqrt{2} \text{ rex}^{3/2} \text{ csc}[\text{th}] + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ csc}[\text{th}]}{\sqrt{2} \text{ csc}[\text{th}]} + \frac{1}{\sqrt{2}} \frac{\sqrt{2} \text{ csc}[\text{th}]}{\sqrt{2} \text{ csc}$ 

4 K14n3 [th] - 4 i Csc [th] K14n3 [th] +  $\frac{i \sqrt{2} K17n3 [th]}{\sqrt{2} k17n3 [th]}$  - 4 K18n3 [th] + 4 i Csc [th] K18n3 [th] -  $\frac{i \sqrt{2} K5n3 [th]}{\sqrt{2} k5n3 [th]}$  + 4 K6n3 [th] - 4 i Csc [th] K6n3 [th] + 48 R9x<sup>3</sup> K8n1 [th] - 96 i  $\sqrt{2} r9x^{9/2}$  K8n1 [th] - 96 i  $\sqrt{2} r9x^{9/2}$  K8n1 [th] - 4 i Csc [th] K6n3 [th] - 4 i 4 Klans[th] - 4 1 CSc[th] Klans[th] + - 4 Klans[th] - 4 1 CSc[th] Klans[th] - 4 1 CSc[th] Kans[th] - 4 1 CSc[th]

 $96 \pm \sqrt{2} \ r\theta x^{9/2} \ Csc[th]^2 \ K9n1[th] + \frac{\pm \sqrt{2} \ K9n3[th]}{r\theta x^{3/2}} - 8 \pm \sqrt{2} \ r\theta x^{3/2} \ K9n3[th] - 16 \ \sqrt{2} \ r\theta x^{3/2} \ Csc[th] \ K9n3[th] + 8 \pm \sqrt{2} \ r\theta x^{3/2} \ Csc[th]^2 \ K9n3[th] \right) / \left(6 - 48 \ r\theta x^3 + 96 \pm r\theta x^3 \ Csc[th] + 48 \ r\theta x^3 \ Csc[th]^2 \right),$ 

K1002 [th] → (24 As00 r0x<sup>3</sup> - 192 As00 r0x<sup>6</sup> + 384 ± As00 r0x<sup>6</sup> Csc [th] + 192 As00 r0x<sup>6</sup> Csc [th]<sup>2</sup> - 12 ± √2 r0x<sup>3/2</sup> K10n1 [th] + 96 ± √2 r0x<sup>9/2</sup> K10n1 [th] + 192 √2 r0x<sup>9/2</sup> Csc [th] K10n1 [th] +

 $96 \pm \sqrt{2} \ r\Theta x^{9/2} \ Csc[th]^2 \ K10n1[th] + \frac{i \sqrt{2} \ K10n3[th]}{2} - 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K10n3[th] - 16 \ \sqrt{2} \ r\Theta x^{3/2} \ Csc[th] \ K10n3[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] - 48 \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ Csc[th]^2 \ K10n3[th] + 12 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ r\Theta x^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ K12n2[th] + 8 \pm \sqrt{2} \ R^{3/2} \ K12n2[th] + 8 \pm \sqrt{2} \ R^$  $12\sqrt{2} - 69x^{3/2} \operatorname{Csc}[th] \operatorname{Kl2n2}[th] + 96 \pm 69x^{3} \operatorname{Csc}[th] \operatorname{Kl2n2}[th] + 48 - 69x^{3} \operatorname{Csc}[th]^{2} \operatorname{Kl2n2}[th] + 4 \operatorname{Kl2n3}[th] + 8 \pm \sqrt{2} - 69x^{3/2} \operatorname{Kl2n3}[th] - 4 \pm \operatorname{Csc}[th] \operatorname{Kl2n3}[th] + 16\sqrt{2} - 69x^{3/2} \operatorname{Csc}[th] \operatorname{Kl2n3}[th] - 4 \pm \operatorname{Csc}[th] - 4 \pm \operatorname{$  $8 \pm \sqrt{2} \ r \Theta x^{3/2} \ Csc[th]^2 \ K12n3[th] - 4 \ K13n3[th] + 4 \pm Csc[th] \ K13n3[th] - \frac{1}{\sqrt{2} \ K14n3[th]} + 4 \ K17n3[th] - 4 \pm Csc[th] \ K17n3[th] + \frac{1}{\sqrt{2} \ K13n3[th]} - 4 \ K5n3[th] + 4 \ LSc[th] \ K5n3[th] + 4 \ LSc[th] \ K5n3[th] - \frac{1}{\sqrt{2} \ K5n3[th]} - \frac{1}{\sqrt{2} \ K5n3[th]} + \frac{1}{\sqrt{2} \ K5n3[th]$ 

 $48 \text{ m} 2^3 \text{ KBn1}(\text{th}) - 96 \text{ i} \sqrt{2} \text{ r} 2^{8} 2^{9/2} \text{ KBn1}(\text{th}) + 48 \text{ i} \text{ r} 2^3 \text{ Csc}(\text{th}) \text{ KBn1}(\text{th}) - 192 \sqrt{2} \text{ r} 2^{9/2} \text{ Csc}(\text{th}) \text{ KBn1}(\text{th}) + 96 \text{ i} \sqrt{2} \text{ r} 2^{8} 2^{9/2} \text{ Csc}(\text{th})^2 \text{ KBn1}(\text{th}) \right) / \left(6 - 48 \text{ r} 2^{3/2} \text{ Csc}(\text{th}) + 48 \text{ r} 2^{3/2} \text{ Csc}(\text{th}) + 48 \text{ r} 2^{3/2} \text{ Csc}(\text{th})^2 \text{ s} \right) = 10^{10} \text{ cm}^{3/2} \text{$ 

 $\texttt{K11n2[th]} \rightarrow -\frac{24\,\texttt{As00\,r0x^{9/2}+12\,i\,\sqrt{2}\,r0x^3\,\texttt{K11n1[th]}-i\,\sqrt{2}\,\texttt{K11n3[th]}+i\,\sqrt{2}\,\texttt{K15n3[th]}-i\,\sqrt{2}\,\texttt{K19n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K7n3[th]}+i\,\sqrt{2}\,\texttt{K1n3[th]$ 

#### {K15n2[th], K12n2[th], K6n2[th], K7n2[th]}

# rKvardAn32 = 4 variables

 $\left\{ \text{K15n2[th]} \to \frac{1}{3} \text{ i } \sqrt{2} \text{ r0x}^{3/2} \text{ (K11n1'[th] + K11n1''[th])} \right. \label{eq:K15n2}$ 

- K1222[th] → (i  $\sqrt{2}$  K1233[th] 24 i  $\sqrt{2}$  r $\theta x^3$  K1233[th] 48  $\sqrt{2}$  r $\theta x^3$  Csc[th] K1233[th] + 24 i  $\sqrt{2}$  r $\theta x^3$  Csc[th] <sup>2</sup> K1233[th] + 4 r $\theta x^{3/2}$  K1333[th] + 8 i  $\sqrt{2}$  r $\theta x^3$  K1333[th] 4 i r $\theta x^{3/2}$  Csc[th] K1333[th] 4 i r $\theta x^{3/2}$  Csc[th] K1333[th] + 4 r \theta x^{3/2} Csc[th] <sup>2</sup> K1333[th] + 4 r \theta x^{3/2} K1333[th] + 4 r $\theta x^{3/2}$  Csc[th] K1333[th] + 4 r \theta x^{3/2} Csc[th] <sup>2</sup> K1333[th] + 4 r \theta x^{3/2} K1333[th] + 4 r \theta x^{3/2} Csc[th] <sup>2</sup> K1333[th] + 4 r \theta x^{3/2} Csc[th] K1333[th] + 4 r \theta x^{3/2}

$$\begin{split} & \text{K12n2}(\text{th}) \rightarrow (\text{i}\sqrt{2} \text{ K12n3}(\text{th}) - 24 \sqrt{2} \text{ rev}^2 \text{K12n3}(\text{th}) - 48 \sqrt{2} \text{ rev}^2 \text{ rev}^2 \text{ K12n3}(\text{th}) + 24 \text{ i}\sqrt{2} \text{ rev}^2 \text{ csc}(\text{th})^2 \text{ K12n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^2 \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^2 \text{ K13n3}(\text{th}) - 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th})^2 \text{ K13n3}(\text{th}) - 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th})^2 \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 81 \sqrt{2} \text{ rev}^3 \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 81 \sqrt{2} \text{ rev}^3 \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) - 81 \sqrt{2} \text{ rev}^3 \text{ K13n3}(\text{th}) + 41 \text{ rev}^{3/2} \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81 \sqrt{2} \text{ rev}^3 \text{ Csc}(\text{th}) \text{ K13n3}(\text{th}) + 81$$
 $\frac{1}{44 \sqrt{2}} + \frac{1}{2} + \frac{1}{2}$ 144 1/2 A300 r0x<sup>6</sup> Cot [th] Csc [th]<sup>2</sup> - 1296 A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>2</sup> - 72 /2 A300 r0x<sup>6</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 432 A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th]<sup>3</sup> + 1296 i A300 r0x<sup>15/2</sup> Cot [th] Csc [th] C  $\frac{144\sqrt{2} \ rek^2 (sc(th)^2 \ KBn_3(th) - 284i \ rek^2 ^2 (sc(th)^2 \ KBn_3(th) - 48i \ \sqrt{2} \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 ^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ (sc(th)^2 \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ sc(th) \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ sc(th) \ KBn_3(th) - 384i \ rek^2 \ rek^2 \ sc(th)^2 \ KBn_3(th) \ rek^2 \ rek^2 \ sc(th)^2 \ KBn_3(th) \ rek^2 \ rek^2 \ sc(th) \ KBn_3(th) \ rek^2 \ rek^2 \ sc(th)^2 \ rek^2 \ sc(th)^2 \ rek^2 \ rek^2 \ sc(th)^2 \ rek^2 \$ 

 $\left(12\,\text{r}\theta x^{3/2}\,\left(\text{i}+\text{Csc[th]}\right)\,\left(1-24\,\text{r}\theta x^3+48\,\text{i}\,\text{r}\theta x^3\,\text{Csc[th]}+24\,\text{r}\theta x^3\,\text{Csc[th]}^2\right)\right)\right)\text{, }\text{K7n2[th]}\rightarrow \frac{1}{36\,\text{r}\theta x^{3/2}\,\left(\text{i}+24\,\text{r}\theta x^3+48\,\text{i}\,\text{r}\theta x^3+48\,\text{i}\,\text{r}\theta$ + Csc[th])

 $\left( 12 \ \text{r0x}^{3/2} \ (\text{i} + \text{Csc}[\text{th}]) \ \left( 1 - 24 \ \text{r0x}^3 + 48 \ \text{i} \ \text{r0x}^3 \ \text{Csc}[\text{th}] + 24 \ \text{r0x}^3 \ \text{Csc}[\text{th}]^2 \right) \right) \right) \text{, } \text{K7n2}(\text{th}) \rightarrow \frac{1}{36 \ \text{r0x}^{3/2} \ (\text{i} + \text{Csc}[\text{th}])^2 \ \text{csc}^{3/2} \ (\text{i} + \text{Csc}[\text{th}])^2 \ \text{csc}^{3/2} \ \text{csc$ 

 $Csc[th] \left(-9\sqrt{2} \text{ As00 r0x}^3 \text{ Cos}[th] + 54 \text{ i As00 r0x}^{9/2} \text{ Cos}[th] + 54 \text{ As00 r0x}^{9/2} \text{ Cot}[th] + 9 \text{ i r0x}^{3/2} \text{ Cos}[th] \text{ Kl0n1}[th] - 27\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] \text{ Kl0n1}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{ i }\sqrt{2} \text{ r0x}^3 \text{ Cos}[th] + 27 \text{$ 

 $\int \left\{ \frac{1}{\sqrt{2}} \left\{ \frac{1}{113} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2$ 

 $24\sqrt{2} \text{ rox}^3 \text{Sin[th] K11n1'[th] } + 12 \pm \sqrt{2} \text{ rox}^3 \text{Sin[th]}^2 \text{ K11n1'[th] } - 12 \pm \sqrt{2} \text{ rox}^3 \text{ K11n1''[th] } + 24\sqrt{2} \text{ rox}^3 \text{ Sin[th] K11n1''[th] } + 12 \pm \sqrt{2} \text{ rox}^3 \text{ Sin[th]}^2 \text{ K11n1''[th] } \}$ 

{K3n3[th], K4n3[th]} rKvardAn33 = 2 variables

 $\left( \texttt{K3n3[th]} \rightarrow -\frac{3 \text{ i } \left(\sqrt{2} \text{ K19n3[th]} + 2 \sqrt{2} \text{ rex}^3 \text{ K11n1'[th]} + \sqrt{2} \text{ rex}^3 \text{ K11n1''[th]} \right)}{2 \text{ rex}^{3/2}} \right)$ 

 $\mathsf{K4n3[th]} \rightarrow \frac{1}{4 \sqrt{2} \; \mathsf{r0x}^{3/2} \; (\texttt{i} + \mathsf{Csc[th]})} \; \mathsf{Sin[th]} \; \left( -18 \; \mathsf{As00} \; \mathsf{r0x}^3 \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} \; \mathsf{Csc[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \sqrt{2} \; \mathsf{As00} \; \mathsf{r0x}^{9/2} \; \mathsf{Cot[th]} + 54 \; \texttt{i} \; \mathsf{i} \; \mathsf{i}$ 

 $4\sqrt{2} \cdot r\theta x^{1/2} (i + Csc(th))$   $9i\sqrt{2} \cdot r\theta x^{3/2} Cot(th) Csc(th) K10n(th) - 54 \cdot r\theta x^3 Cot(th) Csc(th) K10n(th) + 54 i \cdot r\theta x^3 Cot(th) Csc(th)^2 K10n(th) - 12 Csc(th) K17n3(th) + 12 i Csc(th)^2 K17n3(th) + 12 Csc(th) K10n3(th) - 12 Csc(th)^2 K17n3(th) + 12 Csc(th) K10n3(th) - 12 i Csc(th)^2 K10n3(th) - 4i\sqrt{2} \cdot r\theta x^{3/2} Csc(th) K10n3(th) - 12 i Csc(th)^2 K10n3(th) - 12 i Csc(th)^2 K20n3(th) - 4i\sqrt{2} \cdot r\theta x^{3/2} Csc(th) K10n3(th) + 48 \sqrt{2} \cdot r\theta x^{3/2} Csc(th)^2 K10n3(th) + 12 Csc(th)^2 K0n3(th) - 12 i Csc(th)^2 K20n3(th) - 4i\sqrt{2} \cdot r\theta x^{3/2} Csc(th) K20n3(th) - 44 i \sqrt{2} \cdot r\theta x^{3/2} Csc(th) K10n3(th) + 84 \sqrt{2} \cdot r\theta x^{3/2} Csc(th)^2 K0n3(th) - 9i\sqrt{2} \cdot r\theta x^{3/2} Csc(th) Csc(th) K8n1(th) - 54 \cdot r\theta x^3 Cot(th) Csc(th) K8n1(th) + 54 i r\theta x^3 Cot(th) Csc(th)^2 K8n3(th) - 48 \sqrt{2} \cdot r\theta x^{3/2} Csc(th)^2 K8n3(th) + 108 \cdot r\theta x^3 Cot(th) Csc(th) K9n1(th) - 108 i r\theta x^3 Cot(th) Csc(th)^2 K9n1(th) + 24 i r\theta x^3 K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 48 \cdot r\theta x^3 Csc(th) K11n1'(th) - 24 i r\theta x^3 Csc(th)^2 K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) - 12 i r\theta x^3 Csc(th)^2 K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) - 12 i r\theta x^3 Csc(th)^2 K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) + 24 i r\theta x^3 Csc(th) K11n1'(th) - 12 i r\theta x^3 Csc(th)^2 K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) + 24 i r\theta x^3 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) + 24 i r\theta x^3 Csc(th) K11n1'(th) - 24 i r\theta x^3 Csc(th)^2 K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n1'(th) + 24 i r\theta x^3 Csc(th) K11n1'(th) + 36 \cdot r\theta x^3 Csc(th) K11n$ 

#### eqtoievnu3wdAzKcn33

63

eqtoievnu3wdAzKcn33: 8 equs, {9,10,12,21,24,25,26,28} 18 vars= 4 dvars Kin1 14 Kin3 :{1,2,5,6,8,9,10,12,13,14,16,17,18,20}

solution rKvardAn33s2 = 8 variables

{K11n1"[th], K1n3[th], K2n3[th], K12n3[th], K14n3[th], K17n3[th], K18n3[th], K20n3[th]}

complexity(*rKvardAn33s2*) = 10309497

The first replacement in rKvardAn33s2 is a differential equation (deq): this deq has to appended to the next equation group, the remaining replacements will be carried out. partial solution eqtoievnu3wdAzKcn3:

rKvardAn3 = Join[rKvardAn31, rKvardAn32, rKvardAn33];

```
eqtoievnu3wdAzKcn4 :
eqtoievnu3wdAzKcn4s =
(eqtoievnu3wdAzKcn4 /. rKvardAn12s1 /. rKvardAn3s) /. rKvardAn3s /. repkalphax
```

## 64 4.4.3. Solution of $coef(1/r^4)$ as a series in $r_0$

eqtoievnu3wdAzKcn0 vanishes identically, because the ansatz already solves the equations at infinity we solve eqtoievnu3wdAzKcn1, eqtoievnu3wdAzKcn2, and parts of eqtoievnu3wdAzKcn3 and are left with eqtoievnu3wdAzKcn33 eqtoievnu3wdAzKcn33
eqtoievnu3wdAzKcn33 e equs, {9,10,12,21,24,25,26,28}
18 vars= 4 dvars Kin1 14 Kin3 : {1,2,5,6,8,9,10,12,13,14,16,17,18,20}
solution rKvardAn33s2= 8 variables
{K11n1"[th], K1n3[th], K2n3[th], K12n3[th], K14n3[th], K17n3[th], K18n3[th], K20n3[th]}
eqtoievnu3wdAzKcn4s is separated in different  $r_0$ -powers :
sr0Kcn4s = (sr03Kcn4s, sr02Kcn4s, sr01n5Kcn4s, sr00n5Kcn4s, sr0n1n5Kcn4s, sr0n1n5Kcn4s, sr0n2n5Kcn4s, sr0n3Kcn4s})
solution highest coefficient( $r_0^{3}$ ) sr03Kcn4s: only solvable if  $As00n00(\theta) = 0$ , i.e. for  $r_0 \rightarrow \infty$   $As00(r, \theta, r_0) = \frac{As00n01(\theta)}{r_0}$ solution coefficient( $r_0^{3/2}$ ) sr1n5Kcn4s:
K11n1=0,
sr1n5Kcn4s is solvable and a Ritz-Galerkin solution resr01n5Kcn4 in  $\theta$  is calculated in variables
{K8n1[th], K9n1[th], K10n1[th], K10n3[th], K13n3[th], K16n3[th], K5n3[th], K6n3[th], K8n3[th], K9n3[th]}

*eqtoievnu3wdAzKcn4s* is separated in different  $r_0$ -powers :

sr0Kcn4s = {sr03Kcn4s, sr02Kcn4s, sr01n5Kcn4s, sr00n5Kcn4s, sr0n0Kcn4s, sr0n1Kcn4s, sr0n1n5Kcn4s, sr0n2n5Kcn4s, sr0n3Kcn4s}

(\* vars eqtoievnu3wdAzKcn4s: 4 Kin1, 18 Kin3, fixed(rKvardAn33s2) K11n1'' 7 Kin3 (K1n3,K2n3,K12n3,K14n3,K17n3,K18n3,K20n3) iweakvarsKcn4s=(7,11,15,19); iindepvarsD2Kin3Kcn4s={1,2}: K1n3' K2n3' D1th only \*)

# -solution highest coefficient( $r_0^3$ ) sr03Kcn4s

*esr03Kcn4tv*= {*simplify*[*sr03Kcn4s*], *deq*(*rKvardAn33s2*[1])} : *sr03Kcn4s* is simplified and *K11n1*''-deq from *rKvardAn33s2* appended.

 $\begin{aligned} MatrixRank[mDdvsr03Kcn4tloce] = 10 & \text{derivative-matrix} \quad \frac{\partial esr03Kcn4tv}{\partial v(esr03Kcn4tv)} & \text{with } As00\text{-term as last column} \\ \text{where } v(esr03Kcn4tv) & \text{are all variables in } esr03Kcn4tv & \text{including derivatives} \\ MatrixRank[mDdvsr03Kcn4tloc] = 9 & \text{derivative-matrix} & \frac{\partial esr03Kcn4tv}{\partial v(esr03Kcn4tv)} & \text{without } As00\text{-term} \end{aligned}$ 

This proves that

 $As00(r, \theta, r_0) = (As00n00(\theta) + \frac{As00n01(\theta)}{r_0} + \frac{As00n02(\theta)}{r_0^{3/2}} + \dots) + \frac{(As00n10(\theta) + \frac{As00n11(\theta)}{r_0} + \frac{As00n12(\theta)}{r_0^{3/2}} + \dots)}{r} + \dots$ with  $As00n00(\theta) = 0$  and  $coeff(r(r_0, 0.2K_0, r_0) + \dots) + \frac{(r_0, r_0)}{r_0^{3/2}} + \dots$ 

with  $As00n00(\theta)=0$ , and  $coeff(v(esr03Kcn4tv),r_0^0)=0$  i.e. the constant terms in  $r_0$ -power-series in the variables of esr03Kcn4tv are all zero.

-solution coefficient( $r_0^{3/2}$ ) sr1n5Kcn4s

(\* solution extended esr01n5Kcn4t, esr01n5Kcn4tAz(As00==0) with rKvardAn33t from sr0n0 sr0n1n5; rank=10 (full) Kn11n1==0 → solvable with 3 Kin1 7 true Kin3 for As00>0 \*)

ansatz As00=|As00c1/r0x, Kin=K0sin+K1sin\*r0x^1/2  $\rightarrow$  0(r^2) == sr01n5Kcn4t(K1sin)+As00(sr03n5Kcn4t) solvable We set K11n1=0.

sr1n5Kcn4s is solvable and a Ritz-Galerkin solution resr01n5Kcn4 in  $\theta$  is calculated in variables (K8n1[th], K10n1[th], K10n3[th], K13n3[th], K16n3[th], K5n3[th], K6n3[th], K8n3[th], K9n3[th])

# 66 **4.4.4. Complete solution of the** *r***-powers-series ansatz for** r<sub>0</sub>=1

0.0

3.0

complete solution (in order  $1/r^4$  and for *Kcn4* in highest order in  $r_0^{3/2}$ ) for  $r_0=1$ ,  $As00n01(\theta)=1$  $Sin^2(\theta) As00(r,\theta)$ 



0.0

3.0

# 67 The solution resr01n5Kcn4 is inserted in all previous replacements and we get the complete solution (in order $1/r^4$ and for *Kcn4* in highest order in $r_0^{3/2}$ ) for $r_0=1$ , *As00n01(\theta)=1*

MEs3wdA for Es

# MAs3wdA for As

## e.g. Mas3wdA[1,1]=As00 is

 $1 + \frac{1}{nt^2} \left[ \left( -0.214277 + 0.401952 \, i \right) - \left( 0.135318 - 0.627472 \, i \right) \, \text{Cos}[th] - \left( 0.479204 + 0.199967 \, i \right) \, \text{Sin}[th] + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th] - \left( 0.583865 - 0.620115 \, i \right) \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th] - \left( 0.583865 - 0.620115 \, i \right) \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th] - \left( 0.583865 - 0.620115 \, i \right) \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th] \, \text{Sin}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th]^2 \, \text{Cos}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th]^2 \, \text{Cos}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th]^2 \, \text{Cos}[th]^2 + \left( 0.526395 - 0.0617833 \, i \right) \, \text{Cos}[th]^2 \, \text{Cos}[th]^2$ 

- $\frac{12^{-1}}{(0.14498 + 0.851698 \pm)} \cos[th] \sin[th]^2 + (0.636613 0.153096 \pm) \sin[th]^3 + (0.690141 1.1528 \pm) \cos[th] \sin[th]^3 (0.984446 0.653405 \pm) \sin[th]^4 (0.6582292 + 0.467155 \pm) \cos[th] \sin[th]^4 + (0.0194688 + 0.556248 \pm) \sin[th]^5 (0.641894 0.017984 \pm) \sin[th]^7 + (0.0194688 0.734558 \pm) \cos[th] \sin[th]^5 (0.641894 0.017984 \pm) \sin[th]^7 + (0.415868 0.734558 \pm) \cos[th] \sin[th]^7 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.117675 + 0.137756 \pm) \cos[th] \sin[th]^9 + (0.511572 0.615912 \pm) \sin[th]^8 (0.599515 0.033346 \pm) \cos[th] \sin[th]^8 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.251391 + 0.884293 \pm) \sin[th]^9 (0.251391 + 0.884293 \pm) \sin[th]^8 (0.251391 + 0.884293$
- $\frac{1}{rt^2} \left(-\text{Csc[th]}^2 \left((-0.0747558-0.919005\,i)+(0.0465821+0.879729\,i)\,\text{Cos[th]}+(1.71841-0.000776895\,i)\,\text{Sin[th]}-(1.03522-1.48218\,i)\,\text{Cos[th]}\,\text{Sin[th]}+(0.852942-0.240585\,i)\,\text{Sin[th]}^2+(0.879729\,i)\,\text{Cos[th]}^2\right)\right)$
- $(1.00934 + 0.682846 i) \cos[th] \sin[th]^2 + (0.787998 + 0.311227 i) \sin[th]^3 (1.92443 + 0.722468 i) \cos[th] \sin[th]^3 + (2.29427 + 0.434501 i) \sin[th]^4 + (1.20618 + 0.72136 i) \cos[th] \sin[th]^4 + (2.87399 0.0337082 i) \sin[th]^5 (1.9403 + 0.963655 i) \cos[th] \sin[th]^5 (1.11581 0.546401 i) \sin[th]^6 (0.670097 1.6609 i) \cos[th] \sin[th]^6 (0.827886 0.212639 i) \sin[th]^7 + (0.613785 + 0.849093 i) \cos[th] \sin[th]^7 (0.432301 + 0.820567 i) \sin[th]^6 (1.31468 1.6337 i) \cos[th] \sin[th]^8 (0.964969 + 0.105833 i) \sin[th]^9 (1.96996 0.80871 i) \cos[th] \sin[th]^9 + (0.964969 + 0.105833 i) \sin[th]^9 (1.96996 0.80871 i) \cos[th] \sin[th]^9 + (0.964969 + 0.105833 i) \sin[th]^9 (1.96996 0.80871 i) \cos[th] \sin[th]^9 + (0.964969 + 0.105833 i) \sin[th]^9 (0.96496 + 0.105836 i)$
- $\frac{(0.51763 + 0.649953 + 0.649953 + 0.649953 + 0.62957 + 0.62957 + 0.62957 + 0.62957 + 0.1657 + 0.657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.1657598 + 0.16223 i / Cos [th] Sin(th] (0.519812 0.112087 i ) Sin[th]^2 (0.34144 0.438765 i) Cos [th] Sin(th]^2 + (0.74679 + 0.675998 + 0.16253 0.6471345 i) Cos [th] Sin(th]^3 + (0.26959 + 0.679598 + 0.69599 + 0.679598 + 0.695799 + 0.679598 + 0.691679 i) Cos [th] Sin(th]^2 (0.12087 i ) Sin[th]^2 (0.12087 i ) Sin[th]^2 + (0.74679 + 0.679598 + 0.26924 i) Sin[th]^4 (0.1908453 0.6195791 i) Cos [th] Sin[th]^4 (0.1908453 0.6195791 i) Cos [th] Sin[th]^4 (0.269453 0.6471345 i) Cos [th] Sin[th]^6 (0.920518 0.847814 i) Cos [th] Sin[th]^6 (0.87542 + 0.572627 i) Sin[th]^7 (1.29347 + 0.22974 i) Cos [th] Sin[th]^7 (1.29347 + 0.22974 i) Cos [th] Sin[th]^6 (1.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (1.7952 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (1.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.6633593 0.463605 i) Sin[th]^9 (0.7652 + 0.287437 i) Cos [th] Sin[th]^9 + (0.7652 + 0.287437 i)$  $Csc(th)^{2} \left( (0.0280851 + 0.682847 i) - (0.241026 + 0.581927 i) Cos(th) + (2.53494 - 0.332641 i) Sin(th) + (0.0567167 + 0.504413 i) Cos(th) Sin(th) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.0280851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682847 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682857 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682857 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682857 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028851 + 0.682857 i) - (0.719888 - 2.0174 i) Sin(th)^{2} (0.028857 i) - (0.719888 - 2.0174 i) - (0.719888 i) - (0.71988$
- $\left[ (h)^{-2} ((h) \cdot 220051 + 0 \cdot 62624^{-1}) (h)^{-2} ((h)^{-2} + (h)^{-2} + (h)^{-2$
- $\left[ \left( -0.33725 0.36236 + 1 \right) \left( -0.51247 + 0.613646 + 1 \right) \sin(\ln 1) \left( -1.2744 + 1.1352 + 1 \right) \cos(\ln 1) \sin(\ln 1) (-4.65324 1.5615) + 1 \sin(\ln 1) (-1.14666 + 0.675656) + 1 \cos(\ln 1) \sin(\ln 1) (-1.14666 + 0.67566) + 1 \sin(\ln 1) (-1.14666 + 0.6756$

 $\frac{1}{10} \left[ (0.437719 + 0.869184 \, i) - (0.532478 + 0.619845 \, i) \, \text{Cos}[\text{th}] - (1.8992 + 0.629465 \, i) \, \text{Sin}[\text{th}] + (0.456469 + 0.079466 \, i) \, \text{Cos}[\text{th}] \, \text{Sin}[\text{th}] + (0.741301 - 0.374262 \, i) \, \text{Sin}[\text{th}]^2 + (0.741301 - 0.3742$ 

 $\begin{array}{c} (0.276706-0.251115 i) \ Cos[th] \ Sin[th]^2-(0.79357+1.73568 i) \ Sin[th]^3+(0.815736+1.19872 i) \ Cos[th] \ Sin[th]^3-(1.38453+1.19319 i) \ Sin[th]^4-(0.476357-0.0179296 i) \ Cos[th] \ Sin[th]^5-(0.496618+0.750255 i) \ Sin[th]^6-(0.496618+0.750255 i) \ Sin[th]^5-(0.496618+0.750255 i) \ Sin[th]^6-(0.496618+0.750255 i) \ Sin[th]^7-(0.597067-0.812797 i) \ Sin[th]^7-(1.27859-0.0992431 i) \ Cos[th] \ Sin[th]^7-(0.597067-0.812797 i) \ Sin[th]^6-(0.443353+0.92112 i) \ Cos[th] \ Sin[th]^8+(0.8032039-0.755353 i) \ Sin[th]^9+(0.121395-1.63319 i) \ Cos[th] \ Sin[th]^9+i\sqrt{2} \end{array}$ 

 $\left(1 - \frac{1}{\sqrt{2}} i \left((0.180954 - 0.71601 i) - (0.382319 - 0.70612 i) \cos[th] + (1.19742 - 0.184172 i) \sin[th] - (0.330503 - 0.0322737 i) \cos[th] \sin[th] - (0.964347 - 0.237862 i) \sin[th]^2 - (0.409494 - 0.131849 i) \sin[t$ 

/4 Cos(th) Sin(th)<sup>2</sup> - (0.44212 - 1.10058 i) Sin(th)<sup>3</sup> - (0.902813 + 0.540556 i) Cos(th) Sin(th)<sup>3</sup> + (0.756807 + 0.3089 i) Sin(th)<sup>4</sup> + (0.873209 - 0.168857 i) Cos(th) Sin(th)<sup>4</sup> - (0.955147 + 0.0843793 i) Sin(th)<sup>5</sup> + (0.490497 + 0.0368275 i) Cos(th) Sin(th)<sup>5</sup> - (0.466096 - 0.0232202 i) Sin(th)<sup>6</sup> + (0.561349 - 0.35045 i) Cos(th) Sin(th)<sup>6</sup> - (0.0839046 + 0.776249 i) Sin(th)<sup>7</sup> + (0.284662 - 0.486605 i)  $Cos(th)Sin(th)^{7} + (0.593765 - 0.385675i)Sin(th)^{8} - (0.609705 + 0.100691i)Cos(th)Sin(th)^{8} + (0.6081245 + 0.487544i)Sin(th)^{9} + (0.6417809 + 0.91225i)Cos(th)Sin(th)^{9})$ 



 $\begin{array}{l} 69\\ Sin^2(\theta) \ As00(r,\theta) \end{array}$ 



 $Sin^{2}(\theta) Cos^{2}(\theta) As03(r,\theta)$ 



## 70 B5. Numeric solutions of time-independent equations with coupling $\Lambda$ =1

We consider the time-independent equations *eqtoiv* with full coupling (A=I). In this case the Einstein equations are no longer valid, the metric condition at infinity is the flat Minkowski metric.

The calculation is carried out by Ritz-Galerkin method with trigonometric polynomials in  $\theta$ 

 $\{\cos(\theta), \sin(\theta), \frac{1}{\sin(\theta)^{3/4}}\}$  and in r with polynomials of  $\{\frac{1}{\sqrt{r-1}}, \sqrt{r-1}\}$ , which can approximate the

Schwarzschild-singularity at r=1, in total 49 base functions.

The lattice is here a  $30x12 \{r, \theta\}$ -lattice and the Ritz-Galerkin minimization runs in parallel with 8 processes on random sublattices with 100 points.

The processing time on standard 4GHz-processors was 58000s, minimal *RG-deviation*=0.0117, median equation error *mederr*=0.0034.

The resulting solution  $\{A00v(r, \theta), ..., A33v(r, \theta), E00v(r, \theta), ..., E33v(r, \theta)\}$  is shown below for some variables:





The overall behavior of the A-tensor and the E-tensor is as follows.

Some components (e.g. A02, E11, E33) diverge like  $1/\sin(\theta)^{\kappa}$  for  $\theta \rightarrow 0$ , as in the Gauss-Schwarzschild tetrad  $E_{GS}$ . But there is no apparent singularity for  $r \rightarrow 1$ , there are only some numerical artefacts near r=1, because some of the Ritz-Galerkin base functions are divergent at r=1.

# 5.1. The metric in AK-gravity with coupling: no horizon and no singularity

From the resulting solution  $\{A00v(r, \theta), ..., A33v(r, \theta), E00v(r, \theta), ..., E33v(r, \theta)\}$  the generated metric  $fgijv(r, \theta)$  is calculated.

 $(v/c)^{2} = (1/goo-1)/g11 \le 1$ 

$$\left(\frac{1}{g_{00}}-1\right)$$

Using this metric we can approximately calculate the velocity  $v \approx \frac{g_{00}}{g_{11}} \le 1$  during th free fall to the

horizon r=1. The result is

max=0.43 at r1=2. max(v)=0.43 at r=2., i.e. there is no horizon, the velocity reaches a maximum, then there is a rebound. This is to be expected, if we consider the absence of Schwarzschild-like singularity at r=1 for the couplingsolution of *eqtoiv* in 3.2.

Now we calculate the Christoffel symbols 
$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right)$$
 from the metric and solve the

equations-of-motion for the free fall from r=10.

In GR, we have the following picture:

The proper time  $\tau(r)$  of fall in dependence of radius r: the fall time is  $\tau_f = \tau(r=1) = 48.98$  and of course v(r=1)=1 and  $\tau(r=10)=0$ .

The proper fall-time from 
$$r=r02x$$
 to  $r=1$  is

$$r\sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r\sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \sqrt{r\theta 2x} + 2 \ r \ \sqrt{\frac{1}{r}-\frac{1}{r\theta 2x}} \ r\theta 2x+\text{i} \ r\theta 2x^{3/2}\Big] -\frac{1}{2} \ \text{i} \ r\theta 2x^{3/2} \ \text{Log} \Big[-2 \ \text{i} \ r \ \theta 2x^{3/2} \$$

The inverse function radius in dependence on the fall time  $\tau$  is  $r1tOs(\tau)$ :



In AK-metric we have the following picture: The radius in dependence on the fall time  $\tau$  is  $rltOs(\tau)$ :



The fall-time is here  $\tau_f = 51$  reached at  $r_f = 1.75$ , the maximal velocity is  $v_{max} = 0.60$ , then there is a rebound. So we see that in AK-gravitation with coupling  $(\Lambda = 1)$  there is no horizon and no singularity.
## 73

# **B6.** Numeric solutions of time-independent equations with weak coupling and binary gravitational rotator

We consider the time-dependent equations *eqtoiev* with weak coupling ( $\Lambda = 0.001$ ) and binary gravitational rotator (bgr).

We start, as in 4.1., with the A-scaled ansatz for the A-tensor  

$$A_{\mu}^{\nu} = Ab_{\mu}^{\nu} + \Lambda \frac{As_{\mu}^{\nu}}{r} \exp(-ik(r-t)) \text{ and correspondingly for the E-tensor}$$

$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp(-ik(r-t))$$

We introduce the disturbance dAb and  $Ab = A_{hab} + dAb$  from the bgr, as in 4.4. With this ansatz we derive from *eqtoiev* the static part *eqtoievnu3b*(dAb, Eb) and the wave part *eqtoievnu3w*(As, Es, dAb), but without the limit  $\Lambda \rightarrow 0$ , we set  $\Lambda = \Lambda 0 = 0.001$  and the wave number

 $k = k_0 = \frac{1}{\sqrt{2r_0^3}}$  with  $r_0 = 1$  mean distance from the bgr.

At  $r \rightarrow \infty$  {*dAb,Eb,As,Es*} take the values derived for the bgr in 4.4.

 $\{As, Es\} \rightarrow \{Asinfv, Esinfv\} = \{Asoo \rightarrow \frac{cox}{rox}, Aso2 \rightarrow \frac{cox}{rox}\}$ 

 $resAsKc = \left\{ Es20[r1, th] \rightarrow 0, Es21[r1, th] \rightarrow 0, Es22[r1, th] \rightarrow 0, Es23[r1, th] \rightarrow 0, As00[r1, th] \rightarrow As00, As01[r1, th] \rightarrow As00, As02[r1, th] \rightarrow -As00, As03[r1, th] \rightarrow As00, As10[r1, th] \rightarrow 0, As21[r1, th] \rightarrow 0, As22[r1, th] \rightarrow 0, As22[r1, th] \rightarrow 0, As30[r1, th] \rightarrow 0, As30[r1, th] \rightarrow 0, As30[r1, th] \rightarrow 0, As30[r1, th] \rightarrow 0, Es00[r1, th] \rightarrow \frac{3As00}{r1}, Es01[r1, th] \rightarrow -\frac{3As00}{r1}, Es02[r1, th] \rightarrow \frac{3As00}{r1}, Es01[r1, th] \rightarrow -\frac{3As00}{r1}, Es01[r1, th] \rightarrow -\frac{3As00}{r1}, Es01[r1, th] \rightarrow -\frac{3As00}{r1}, Es10[r1, th] \rightarrow -\frac{As00}{r1}, Es10[r1, th] \rightarrow -\frac{As00}{r1}, Es10[r1, th] \rightarrow -\frac{$ 

```
dAb \rightarrow dAbinfv =
```

 $\begin{cases} dAb00 \rightarrow \frac{alphax}{r1^3}, dAb01 \rightarrow \frac{alphax}{r1^3}, dAb02 \rightarrow \frac{alphax}{r1^3}, dAb03 \rightarrow \frac{alphax}{r1^3}, \\ dAb10 \rightarrow 0, dAb11 \rightarrow 0, dAb12 \rightarrow 0, dAb13 \rightarrow 0, dAb20 \rightarrow i \ Csc[th], dAb21 \rightarrow i \ Csc[th], \\ dAb22 \rightarrow i \ Csc[th], dAb23 \rightarrow i \ Csc[th], dAb30 \rightarrow 1, dAb31 \rightarrow 1, dAb32 \rightarrow 1, dAb33 \rightarrow 1 \end{cases}$ 

 $Eb \rightarrow Ebinfv = E_{GK}$  the Gauss-Kerr-tetrad from 4.4.1.

The calculation is carried out by Ritz-Galerkin method with trigonometric polynomials in  $\theta$ 

 $\{\cos(\theta), \sin(\theta), \frac{1}{\sin(\theta)^{3/4}}\}$  and in r with polynomials of  $\{\frac{1}{r}\}$ , in total 40 base functions.

The lattice is here a 201x31  $\{r, \theta\}$ -lattice and the Ritz-Galerkin minimization runs in parallel with 8 processes on random sublattices with 20 points.

The processing time on standard 4GHz-processors was 150000s, minimal *RG-deviation*=0.032, median equation error mederr(eqtoievnu3b)=0.016 mederr(eqtoievnu3w)=0.012.

The metric  $g^{\mu\nu}(Eb)$  generated by the background Eb has a horizon at  $r\approx 1.9$  for the free fall, that means that for weak coupling  $(\Lambda = 0.001)$  the singularity of GR still exists. So there is a  $\Lambda$ ,  $(0.001 \le \Lambda \le 1)$ , where the singularity disappears.

The resulting solution  $\{dAb(r,\theta), Eb(r,\theta), As(r,\theta), Es(r,\theta)\}$  is shown below for some variables:





 $Sin(\theta)^{3/4}$  Es11 :



# 76 **B7.** The energy tensor for the gravitational wave

#### electromagnetic energy tensor

$$\epsilon_0=rac{1}{4\pi}, \quad \mu_0=4\pi$$

in cgs units

$$T^{\mu\nu} = \frac{1}{4\pi} \begin{bmatrix} F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \end{bmatrix}.$$

$$[T]== \text{energy/r}^3=\text{endensity}$$

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{8\pi} (E^2 + B^2) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

 $\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}.$ Poynting vector [S]=energy/(r^2 \*t)=energy-flux, [S/c]= energy/r<sup>3</sup>=endensity

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \delta_{ij}$$
 Maxwell stress tensor

conservation of momentum and energy

 $\partial_
u T^{\mu
u} + \eta^{\mu
ho} \, f_
ho = 0$ 

where is the (4D) Lorentz force per unit volume on matter.

electromagnetic energy density

$$u_{
m em} = rac{\epsilon_0}{2} E^2 + rac{1}{2\mu_0} B^2$$

electromagnetic momentum density

$$\mathbf{p}_{ ext{em}} = rac{\mathbf{S}}{c^2}$$

It is a symmetric tensor:

$$T^{\mu
u} = T^{
u\mu}$$

- The tensor  $T^{
  u}{}_{lpha}$  is traceless:  $T^{\alpha}{}_{\alpha} = 0.$
- The energy density is positive-definite:

$$T^{00} \ge 0$$

# gravitational Ashtekar-Kodama energy

GR grav. wave energy density (plane wave)  $t_{\mu\nu} = k_{\mu}k_{\nu}\left(\left(e^{\lambda\kappa}\right)^*e_{\lambda\kappa} - \frac{1}{2}\left(e^{\lambda}e_{\lambda}\right)^2\right)\frac{\hbar c}{16\pi l_n^2}$ ,

$$t_{\mu\nu}^{\text{grav}} = \frac{c^4}{16\pi G} k_{\mu} k_{\nu} \left( e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2} |e^{\lambda}_{\lambda}|^2 \right)$$
  
$$t_{\mu\nu} = \frac{\hbar c}{16\pi l_p^2} k_{\mu} k_{\nu} \left( e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2} |e^{\lambda}_{\lambda}|^2 \right)$$
 ([2] 34.23)

dimension  $[t_{\mu\nu}] = energy/r^3 = endensity$  ([2]),  $e^{\lambda\kappa}$  is the polarization. when the metric wave is spherical  $h_{\mu\nu} = \frac{e_{\mu\nu}}{r} \exp(-ik_{\mu}x^{\mu})$ 

(transition from spherical wave  $A_r$  to plane wave  $A_p$  via energy condition:  $4\pi r^2 |A_r|^2 = r_s^2 |A_p|^2$ )

AK grav. wave energy density  $t_{\mu\nu} = D_{\kappa}A_{\mu}^{\kappa}D_{\lambda}A_{\nu}^{\lambda}\hbar c \left(\frac{1}{l_{p}^{2}\Lambda^{2}r_{*}^{2}}\right)$ , dimension  $[t_{\mu\nu}] = energy/r^{3} = endensity$ 

(the dimensionless factor  $\frac{r_{P\Lambda}^2}{r^2} = \left(\frac{1}{l_P^2 \Lambda^2 r^2}\right)$  is inserted for compatibility with GR and to account for the  $\Lambda$ -

Planck-lambda scale

scaled wave ansatz) , where  $r_{P\Lambda} = \frac{1}{l_P\Lambda} = 5.64 \ 10^{86} m$ 

second term: gravitational stress energy:  $t^{e}_{\mu\nu} = D_{\kappa}E_{\mu}^{\kappa}D_{\lambda}E_{\nu}^{\lambda}\Lambda\hbar c$  ( $\Lambda$  must be inserted for dimensional reasons), which is normally negligible

for the standard spherical wave  $k_{\mu} = (-k_0, k_0, 0, 0)$  x-y-polarization unit amplitude

AK energy density

for a standard scaled spherical wave with a single r-t-amplitude

a dimensionless factor  $\frac{1}{A}$ , which can be incorporated in  $As_{00}$ 

The AK energy density has the form:  $t_{\mu\nu} = s_{\mu}s_{\nu}\hbar c$ , where  $s_{\mu} = D_{\kappa}A_{\mu}^{\kappa}$  and dimension  $[s_{\mu}] = 1/r^2$ 

The current is  $j_{\nu} = c \frac{x^{\kappa}}{|x^{\kappa}|} t_{\kappa\nu}$ , where  $n^{\kappa} = \frac{x^{\kappa}}{|x^{\kappa}|}$  is a unit direction 4-vector,

the energy flux in the direction  $n_i$  is then ([2] 41.11)

$$S = \sum c t_{0i} n_i$$
, dimension [S]=energy/r<sup>2</sup> t

the total power of gravitational radiation for a quadrupole Q is in GR ([2] 42.21)

$$P = \frac{32G\omega^6 Q}{5c^5}$$

in the special case of a binary gravitational rotator with masses  $m_1$  and  $m_2$  (total mass  $m = m_1 + m_2$ ) and the mean orbit radius  $r_0$  we get

$$r_{s} = \frac{2mG}{c^{2}} \quad k = \frac{\omega}{c} = \frac{\sqrt{r_{s}}}{\sqrt{2r_{0}^{3}}} \quad P_{GR} = \frac{r_{s}^{2}c^{5}}{2G}k^{6}r_{0}^{4}\left(\frac{m_{1}m_{2}}{m^{2}}\right)^{2} = P_{0}\frac{r_{s}^{5}}{r_{0}^{5}}\left(\frac{m_{1}m_{2}}{m^{2}}\right)^{2}, \text{ where } P_{0} = \frac{\hbar c^{2}}{2l_{p}^{2}} \text{ is a constant with }$$

dimension of power.

In 4.4.2. we have shown that for bgr  $As00(r, \theta, r_0) = \frac{As00n01(\theta)}{r_0}$ ,

we get 
$$t_{00} = k_0^2 \frac{As_{00}^2}{r^2 r_s^2} \hbar c \left(\frac{1}{l_p^2}\right)$$
,  $P_{KA} = t_{00} c 4\pi r^2 = k_0^2 A s_{00}^2 4\pi \hbar c^2 \left(\frac{1}{l_p r_s}\right)^2$   
Setting  $As_{00} = \frac{c_0}{r_0}$ , with  $k_0^2 = \frac{r_s}{2r_0^3}$  it follows from  $P_{KA} = P_{GR}$ ,  $c_0^2 = r_s^6 \frac{\left(\frac{m_1 m_2}{m^2}\right)^2}{32\pi}$ ,  $c_0 = r_s^3 \frac{\left(\frac{m_1 m_2}{m^2}\right)^2}{4\sqrt{2\pi}}$ 

So the amplitude of the gravitational wave of the binary gravitational rotator becomes

$$As_{00} = \frac{\left(\frac{m_1m_2}{m^2}\right)}{4\sqrt{2\pi}} \frac{r_s^3}{r_0}$$
, where  $r_s = \frac{2Gm}{c^2}$  is the Schwarzschild radius of the total mass  $m$ , and

 $f_m = \frac{m_1 m_2}{m^2} = \frac{m_r}{m} = \frac{\mu}{(1+\mu)^2}$  is the ratio of the reduced mass to the total mass  $\mu = \frac{m_1}{m_2} \le 1$ .

This formula can be easily generalized to multiple masses rotating around their common center-of mass:

$$As_{00} = \frac{f_m}{4\sqrt{2\pi}} \frac{r_s^3}{r_0} \text{ with } f_m = \frac{m_1 m_2 \dots m_n}{m^n} = \frac{m_r}{m} \text{ and } r_0 \text{ the mean diameter of the rotator.}$$

# **B8.** Quantum AK-gravitation

79

# We recall the Ashtekar-Kodama equations

spatial spacetime curvature  $F_{\mu\nu}^{\ \kappa} = \partial_{\mu}A_{\nu}^{\ \kappa} - \partial_{\nu}A_{\mu}^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}A_{\nu}^{\ \kappa_{2}}$ 4 gaussian constraints  $G^{\mu} = \partial_{\nu}E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda}A_{\nu}^{\ \kappa}E^{\nu\lambda}$  (covariant derivative of  $E^{\mu\nu}$  vanishes) 4 diffeomorphism constraints  $I_{\mu} = E^{\kappa}{}_{\nu}F_{\mu\kappa}^{\ \nu}$ 

24 hamiltonian constraints  $H_{(\mu,\nu)}^{\ \kappa} = F_{\mu\nu}^{\ \kappa} + \frac{\Lambda}{3} \varepsilon_{\mu\nu\rho} E^{\rho\kappa}$ 

In this section, we will find the lagrangian, from which the AK equations can be derived.

# 8.1. Lagrangian of the hamiltonian equations

In electrodynamics, the lagrangian of the fundamental Maxwell equations is

$$L_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Therefore we make at first the analogous ansatz for the AK-lagrangian of the Hamiltonian equations  $L_F = F_{\mu\nu}{}^{\kappa}F^{\mu\nu}{}_{\kappa}$ 

The formal expression for the variation of action for the variables  $A_{\mu}^{\nu}$  is:

$$\frac{\delta L}{\delta A_{\rho}^{\sigma}} = \frac{\partial L}{\partial A_{\rho}^{\sigma}} - \partial_{\tau} \frac{\partial L}{\partial A_{\rho}^{\sigma},\tau} \text{, where } A_{\rho}^{\sigma},\tau = \frac{\partial A_{\rho}^{\sigma}}{\partial x^{\tau}}$$

We have 4 intermediate results

$$\begin{aligned} \frac{\partial F_{\mu\nu}^{\ \kappa}}{\partial A_{\rho}^{\ \sigma}} &= \delta_{\mu\rho} \varepsilon_{\sigma\kappa_{2}}^{\ \kappa} A_{\nu}^{\ \kappa_{2}} + \delta_{\nu\rho} \varepsilon_{\kappa_{1}\sigma}^{\ \kappa} A_{\mu}^{\ \kappa_{1}} \\ \partial_{\tau} \frac{\partial F_{\mu\nu}^{\ \kappa}}{\partial A_{\rho}^{\ \sigma},\tau} &= (\delta_{\mu\tau} \delta_{\nu}^{\ \rho} - \delta_{\nu\tau} \delta_{\mu}^{\ \rho}) \delta^{\kappa}{}_{\sigma} \\ \frac{\partial L_{F}}{\partial A_{\rho}^{\ \sigma}} &= 2\varepsilon_{\sigma\kappa_{1}}^{\ \kappa} (\delta_{\mu}^{\ \rho} A_{\nu}^{\ \kappa_{2}} - \delta_{\nu}^{\ \rho} A_{\mu}^{\ \kappa_{1}}) F^{\mu\nu}{}_{\kappa} = 4\varepsilon_{\sigma\lambda_{1}\kappa} F^{\rho\nu\kappa} A_{\nu}^{\ \lambda_{1}} \\ \partial_{\tau} \frac{\partial L_{F}}{\partial A_{\rho}^{\ \sigma}} &= 2\partial_{\tau} (\delta_{\mu\tau} \delta_{\nu}^{\ \rho} - \delta_{\nu\tau} \delta_{\mu}^{\ \rho}) \delta_{\kappa\sigma} F^{\mu\nu\kappa} = 4\partial^{\tau} F_{\tau}^{\ \rho}{}_{\sigma} \end{aligned}$$

and the result of the variation follows

$$\frac{\delta L_F}{\delta A_{\rho}^{\sigma}} = -4\partial^{\tau} F_{\tau}^{\rho}{}_{\sigma} + 4\varepsilon_{\sigma\lambda_1\kappa} F^{\rho\nu\kappa} A_{\nu}^{\lambda_1}$$

This is a *derived* equation  $\tilde{H}^{\rho}{}_{\sigma} = 4(-\partial^{\tau}H_{\tau}{}^{\rho}{}_{\sigma} + \varepsilon_{\sigma\lambda_{1}\kappa}H^{\rho\nu\kappa}A_{\nu}{}^{\lambda_{1}})$  from a 3-tensor H=F, which is the first term in the AK hamiltonian equations.

Now consider the following lagrangian

$$L_{\Lambda} = \varepsilon^{\kappa}{}_{\mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_2 \lambda_2 \kappa_2} \varepsilon^{\mu_1 \lambda_1}{}_{\kappa_1} E^{\kappa_1 \kappa_2} A_{\lambda_1}{}^{\lambda_2} A_{\mu_1}{}^{\mu_2}$$

One can show easily that for  $H_{\Lambda}(E)_{\rho\sigma}^{\ \tau} = \varepsilon_{\rho\sigma\lambda} E^{\lambda\tau}$ 

$$\frac{\partial L_F}{\partial A_{\rho}^{\sigma}} = -\partial^{\tau} H_{\Lambda}(E)_{\tau}^{\rho}{}_{\sigma} + \varepsilon_{\sigma\lambda_l\kappa} H_{\Lambda}(E)^{\rho\nu\kappa} A_{\nu}^{\lambda_l})$$

So the complete Lagrangian for the *derived* hamiltonian equations is

$$L_{H} = -(\frac{1}{4}L_{F} + \frac{\Lambda}{3}L_{\Lambda}) = -\left(\frac{1}{4}F_{\mu\nu}^{\ \kappa}F^{\mu\nu}_{\ \kappa} + \frac{\Lambda}{3}(\varepsilon^{\kappa}_{\ \mu\lambda}E^{\lambda\nu}\partial_{\kappa}A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}}\varepsilon^{\mu_{1}\lambda_{1}}_{\ \kappa_{1}}E^{\kappa_{1}\kappa_{2}}A_{\lambda_{1}}^{\ \lambda_{2}}A_{\mu_{1}}^{\ \mu_{2}})\right)$$

The corresponding derived Hamiltonian equations are  

$$\frac{\delta L_{H}}{\delta A_{\rho}^{\sigma}} = -\partial^{\tau} H(A, \frac{\Lambda}{3}E)_{\tau}^{\rho}{}_{\sigma} + \varepsilon_{\sigma\lambda_{l}\kappa} H(A, \frac{\Lambda}{3}E)^{\rho\nu\kappa} A_{\nu}^{\lambda_{l}}) , \text{ where}$$

 $H(A, \frac{\Lambda}{3}E)_{\mu\nu}^{\ \kappa} = F_{\mu\nu}^{\ \kappa} + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa}$  are the AK hamiltonian equations.

Furthermore, we follow the ansatz of Smolin in [5] and let  $\Lambda$  be generated by a scalar field  $\varphi_{\Lambda}$  with the constraint  $\overline{\varphi}_{\Lambda}\varphi_{\Lambda} = \Lambda$ 

$$L_{\mu} = -\hbar c \left( \frac{1}{4} F_{\mu\nu}^{\ \kappa} F^{\mu\nu}_{\ \kappa} + \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}_{\ \mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}_{\ \kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}^{\ \lambda_{2}} A_{\mu_{1}}^{\ \mu_{2}}) \right)$$
, which brings the action to the

correct dimension  $[L_H] = \frac{energy * r}{r^4}$ , because  $[\varphi_\Lambda] = \frac{1}{r}$  and  $[\Lambda] = \frac{1}{r^2}$ , therefore this action is formally

renormalizable.

If we carry out the variation for  $\varphi_{\mu\nu}$ , we get the following expression

$$\frac{\delta L_{H}}{\delta \varphi_{\rho\sigma}} = -\hbar c \left( \frac{\varphi_{\Lambda}}{3} \left( \varepsilon^{\kappa}{}_{\mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}{}_{\kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}{}^{\lambda_{2}} A_{\mu_{1}}{}^{\mu_{2}} \right) \right), \text{ which becomes the } \Lambda\text{-gauge condition for the}$$

AK equations in the form

 $G_{\Lambda} = \varepsilon^{\mu\nu}{}_{\lambda}E^{\lambda\kappa}\partial_{\mu}A_{\nu\kappa} + \varepsilon_{\mu_{2}\mu\kappa_{2}}\varepsilon^{\nu}{}_{\mu\kappa_{1}}E^{\kappa_{1}\kappa_{2}}A_{\mu}{}^{\lambda_{2}}A_{\nu}{}^{\mu_{2}} , G_{\Lambda} != 0$ 

We use the Hamiltonian equations, and after some algebra we get the expression

$$G_{\Lambda} = -\frac{\Lambda}{3} \sum_{\lambda,\rho} \left( E \bullet \eta \bullet E^{t} \right)^{\lambda\rho} + \sum_{\kappa,\lambda} E^{\lambda\kappa} \sum_{(\mu,\nu)=C(\kappa,\lambda)} \left( A_{\mu\mu}A_{\nu\nu} - A_{\mu\nu}A_{\nu\mu} \right) \text{, where } (\mu,\nu) = C(\kappa,\lambda) \text{ is the complementary index noise.}$$

pair.

For the classical case with  $\Lambda \approx 0$  with the *constant half-antisymmetric background*  $A_{hab}$  and the Gauss-Schwarzschild tetrad  $E_{GS}$  the first term in  $G_A$  is negligible and the second vanishes for  $A = A_{hab}$ . In the general case,  $G_A$  is a single gauge condition, which fixes one free parameter of the AK-solution.

#### 8.2. Lagrangian of the remaining equations

For the diffeomorphism equations  $I_{\mu} = E^{\kappa_{\nu}} F_{\mu\kappa}^{\nu}$ , we set simply the variable  $C_{\mu} := E^{\kappa_{\nu}} F_{\mu\kappa}^{\nu}$ and take the lagrangian  $L_{I} = \hbar c C_{\mu} C^{\mu} = \hbar c E^{\kappa_{1}} F_{\mu\kappa_{1}} F_{\mu\kappa_{1}} E^{\kappa_{2}} F^{\mu} F_{\kappa_{2}} F^{\mu}$  as the corresponding lagrangian

As for the gaussian equations  $G^{\mu} = \partial_{\nu} E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda} A_{\nu}{}^{\kappa} E^{\nu\lambda}$ , they can be derived from the fact, that this is the covariant derivative for the tetrad E , so it must vanish.

With 
$$L_{H} = -\hbar c \left( \frac{1}{4} F_{\mu\nu}^{\ \kappa} F^{\mu\nu}_{\ \kappa} + \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}_{\ \mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}^{\ \lambda_{2}} A_{\mu_{1}}^{\ \mu_{2}}) \right)$$

the complete AK lagrangian is then

$$L_{gr} = L_{H} + L_{I} = \hbar c \left( -\frac{1}{4} F_{\mu\nu}^{\ \kappa} F^{\mu\nu}_{\ \kappa} - \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}_{\ \mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}_{\ \kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}^{\ \lambda_{2}} A_{\mu_{1}}^{\ \mu_{2}} ) + E^{\kappa_{1}}_{\ \nu_{1}} F_{\mu\kappa_{1}}^{\ \nu_{1}} E^{\kappa_{2}}_{\ \nu_{2}} F^{\mu}_{\ \kappa_{2}}^{\ \nu_{2}} \right)$$

#### 8.3. Dirac lagrangian for the graviton

The Dirac lagrangian for the photon reads

$$L_{Dem} = \overline{\psi} \left( -\hbar c i \gamma^{\mu} D_{\mu} - m c^{2} \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \text{, where } D_{\mu} = \partial_{\mu} + \frac{i e}{\sqrt{\hbar c}} A_{\mu} = \partial_{\mu} + i \sqrt{\alpha} A_{\mu}$$

is the covariant derivative of the photon (note the negative sign in the first term: we use here the metric  $\eta = diag(-1, 1, 1, 1)$ )

This describes the interaction of the photon with a fermion and yields the corresponding Feynman diagrams and cross sections.

The Dirac lagrangian for the graviton reads  $L_{Dgr} = \overline{\psi} \left( -\hbar c i \gamma^{\mu} D_{\mu} - m c^{2} \right) \psi + L_{gr} \text{, where } (D_{\mu})^{\lambda}{}_{\kappa} = \partial_{\mu} + (\varepsilon_{a})^{\lambda}{}_{\kappa} A_{\mu}{}^{a} \text{ is the covariant derivative of the graviton , where the generator matrix } (\varepsilon_{a})^{\lambda}{}_{\kappa} = \varepsilon^{\lambda}{}_{\kappa_{1}\kappa}$ 

The electron-graviton interaction term is

$$\delta_{I}L_{Dgr} = -\hbar c i \overline{\psi} \left( \gamma^{\mu} \left( \varepsilon_{a} \right)^{\lambda} \kappa A_{\mu}^{a} \right) \psi \quad \text{, where} \quad A_{\mu}^{\nu} = \Lambda \frac{A s_{\mu}^{\nu}}{r} \exp(2i\theta) \exp(-ik(r-t)) \quad \text{is the graviton quadrupole}$$

wave function, so (background  $Ab \approx 0$ ), so the term is linear in As, like in the electromagnetic case. The presence of  $\Lambda$  makes the term very small.

Let us compare this to the GR-Dirac lagrangian

$$L_{GRD} = -\frac{\sqrt{\det(-g)}}{2\kappa} (R - 2\Lambda) + \sqrt{-g} \overline{\psi} (i\hbar c \gamma^{\mu}(x) \nabla_{\mu} - mc^{2}) \psi$$

where

$$\nabla_{\mu}\psi = (\partial_{\mu} - \frac{i}{4}\omega_{\mu}^{ab}\sigma_{ab})\psi$$
 and  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$  are the Dirac  $\sigma$ -matrices

and  $\omega$  the GR connection field in tetrad-expression

$$\omega_{\mu}^{ab} = \frac{1}{2}e^{a\nu}(\partial_{\mu}e_{\nu}^{b} - \partial_{\nu}e_{\mu}^{b}) + \frac{1}{4}e^{a\rho}e^{b\sigma}(\partial_{\sigma}e_{\rho}^{c} - \partial_{\rho}e_{\sigma}^{c})e_{\mu}^{c} - (a \leftrightarrow b)$$

with the tetrad  $e^{\mu}_{\mu}e^{\nu}_{\nu} = g_{\mu\nu}$  i.e.  $e \bullet \eta \bullet e = g$ 

compared to the metric condition for the inverse densitized background tetrad Eb

$$Eb \bullet \eta \bullet Eb' = g^{-1} / (-\det(g))^{3/4}$$
, so  $e = (Eb^{-1})' / (-\det(g))^{3/8}$   
Here the interaction term is

$$\delta_I L_{GRD} = -\frac{\hbar c}{4} \sqrt{\det(-g)} \overline{\psi} (\gamma^{\mu} \omega_{\mu}^{\ ab} \sigma_{ab}) \psi = -\frac{\hbar c}{4} \sqrt{\det(-g)} \overline{\psi} (\sum_{\mu} \gamma^{\mu} f^{\mu} (Eb^{-1})) \psi$$

where the middle term  $\sum_{\mu} \gamma^{\mu} f^{\mu} (Eb^{-1})$  is a sum of  $\gamma$ -matrices with coefficients, which are quadratic functions of  $Eb^{-1}$  so  $\delta_I L_{GRD}$  is quite different from the AK-interaction term  $\delta_I L_{Dgr}$ .

#### 8.4. The graviton wave function and cross-sections

For the Compton effect, i.e. electron-photon scattering

the Thompson cross-section for small energies is  $\sigma_{Th}$ =

$$\frac{8\pi\alpha^2}{3m^2} = 0.665 \times 10^{-24} \,\mathrm{cm}^2 = \alpha^2 \left(\frac{\hbar c}{mc^2}\right)^2 \frac{8\pi}{3} \,\,\mathrm{, \, where} \,\, m = m_e$$

is the electron mass,

$$\alpha = \frac{e^2}{\hbar c}$$
 is the fine-structure-constant and

the reduced de-Broglie wavelength of the electron 
$$\tilde{\lambda}_e = \frac{\hbar c}{m_e c^2} = 0.38 * 10^{-12} m$$
.

So the electron-photon Thompson cross-section is with these denominations  $\sigma_{th} = \alpha^2 \frac{1}{\tilde{\lambda}_e^2} \frac{8\pi}{3}$ 

The photon wave function is here [20, 7.53]

$$(A_e)^{\mu} = \frac{\varepsilon^{\mu}}{\sqrt{2kV}} \left( \exp(-ik \bullet x) + \exp(ik \bullet x) \right)$$

where  $\varepsilon^{\mu}$  is unit-polarization vector,  $k^{\mu}k_{\mu} = 0$  and  $\varepsilon^{\mu}k_{\mu} = 0$ .  $A^{\mu}$  is normalized to give the energy  $E(A^{\mu}) = \hbar c \int (\nabla \times A)^2 d^3x = \hbar \omega = \hbar c k$ 

The covariant derivative of the photon is  $D_{\mu} = \partial_{\mu} + i\sqrt{\alpha}A_{\mu}$ We use the results from 4.3.1

$$As30 = As30c \ i\exp(-\frac{4\sqrt{r}}{\sqrt{3}})\frac{r^{17/12}}{6} \to 0$$

$$As10 = -\frac{As20c}{2} \exp(2i\theta)$$
$$As00 = -\frac{As20c}{2} \exp(2i\theta)$$
$$As20 = \frac{As20c}{r} \exp(2i\theta) \rightarrow 0$$

and from 7 and write the graviton wave function as a plane wave analogous to the photon (the quadrupole characteristics disappear in the plane wave, therefore  $exp(2i\theta)$  is skipped)

 $(A_g)_{\mu}^{\nu} = \Omega_{\mu}^{\nu} \frac{1}{2} \Lambda f_m \frac{r_s^2 \sqrt{\pi}}{2\sqrt{2}r_0} \frac{r_s^{3/2}}{\sqrt{2V}} \left( \exp(-ik \bullet x) + \exp(ik \bullet x) \right), \text{ with the polarization matrix according to the polarization matrix according to the second sec$ 

results from 4.3.1

 $\Omega_{\mu}^{\nu} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 

According to 7 we get now for the energy density  $t_{00} = t_{11} = (2k As 00)^2 \frac{\hbar c}{\Lambda^2 l_p^2 r_s^2}$ 

$$t_{\mu}^{\ \mu} = 8k^2 A s 00^2 \frac{\hbar c}{\Lambda^2 l_p^2 r_s^2} , \text{ and as } \int_V \frac{1}{2V} \left( \exp(-ik \bullet x) + \exp(ik \bullet x) \right)^2 d^3 x = 1 \text{ and } k = \sqrt{\frac{r_s}{2r_0^3}} ,$$

we get for the energy

 $E(A_g) = \int_V t_{\mu}^{\mu} d^3x = \frac{r_s \hbar c}{l_p^2} \frac{\pi f_m^2}{4^{2/3}} (kr_s)^{10/3} \text{, now we demand that } E(A_g) = \hbar ck \text{, so the normalization factor is}$   $c_n = \frac{1}{\frac{r_s}{l_p} \frac{\sqrt{\pi} f_m}{4^{1/3}} (kr_s)^{7/6}} \text{ and the normalized wave function becomes}$   $(A_{gn})_{\mu}^{\nu} = (A_g)_{\mu}^{\nu} c_n = \Omega_{\mu}^{\nu} \sqrt{\alpha_{gr}} \frac{1}{\sqrt{kr_s}} \frac{r_s^{1/2}}{\sqrt{2V}} (\exp(-ik \bullet x) + \exp(ik \bullet x)) \text{, where}$   $\sqrt{\alpha_{gr}} = \frac{r_s \Lambda l_p}{\sqrt{2}} = 0.55 * 10^{-91} \text{ and } \alpha_{gr} \text{ is the gravitational fine structure constant and the photon-like wave function can be written}$ 

$$(A_{gn})_{\mu}^{\nu} = \sqrt{\alpha_{gr}} (A_p)_{\mu}^{\nu}$$

The covariant derivative is then  $(D_{\mu})^{\lambda}{}_{\kappa} = \partial_{\mu} + (\varepsilon_{a})^{\lambda}{}_{\kappa}\sqrt{\alpha_{gr}}(A_{p})_{\mu}{}^{a}$ 

where  $A_p$  is completely analogous to the photon wave function  $A_e$ , and matrices  $(\varepsilon_a)^{\lambda}{}_{\kappa} = \varepsilon^{\lambda}{}_{a\kappa}$  a=0,1,2,3 in analogy to the Dirac gamma-matrices.

By analogy we can then assess the electron-graviton scattering cross-section  $\sigma_{eg} \approx \alpha_{gr}^2 \frac{1}{\tilde{\lambda}_e^2}$ ,

ignoring the tensor form and the  $\theta$ -dependence .

#### 8.5. The graviton propagator

As is well known, the photon propagator in QFT is [20]

$$D_F(q^2) = \frac{-1}{q^2 + i\epsilon}$$
, which follows from the Maxwell equations  $\Box A^{\mu}(x) = J^{\mu}(x)$ 

We consider the wave equations eqgravlxA0, eqgravlxA3 and eqgravlxEn in the momentum representation, i.e. in the *k*-space. Then the *r*-derivatives transform into *k*-powers

$$\begin{split} &\hat{\partial}_{r}^{\ n} f(r,k) = (i\,k)^{n} f(r,k) \\ &\text{We write the equations as polynomials in } k: \\ &eqgravlxEninf \quad P(Es1) = 2ik^{3}lx - 2ik^{4}r + 3ik^{4}r - ik^{4}r = 2ik^{3}lx \\ &eqgravlxA0 \quad P(As0) = 3(1 + ilx - irk)(lx + kr)^{2} + \\ &P(As0Es1) = r((-1 + lx^{2} + 2lx(-i + kr)) + ikr(1 + 2ilx + 2ikr) + k^{2}r^{2}) \\ &\text{at } r\text{-infinity: } P_{inf}(As0) = -3ik^{3}r^{3} + P_{inf}(As0Es1) = -k^{2}r^{3}, \text{ so} \\ &As0 = -P_{inf}(As0)^{-1}P_{inf}(As0Es1)Es1 = \frac{i}{3k}Es1 \\ &Es1 = P(Es1)^{-1}\delta Es = \frac{\delta Es}{2ilxk^{3}} \quad As0 = \frac{\delta Es}{6lx\,k^{4}}, \text{ so the } As0\text{-propagator is } D_{F}(As0,q^{2}) = \frac{1}{6lx(q^{4} + i\varepsilon)} \\ &eqgravlxA3 \quad P(As3) = 6klx(1 + ikr - ikr) = 6klx + \\ &P(As3Es1) = (-2k^{2}r^{2} + 3k^{2}r^{2} - k^{2}r^{2}) + ((i - lx)kr - (i - lx)kr) + (1 + ilx) = 1 + ilx , \text{ so} \\ &As3 = \frac{i(1 + ilx)\delta Es}{12lx^{2}\,k^{4}}, \text{ so the } As3\text{-propagator is } D_{F}(As3,q^{2}) = \frac{(i - lx)}{12lx^{2}(q^{4} + i\varepsilon)} \end{split}$$

The As-propagators are identical apart from a constant factor and are finite-integrable in  $q^2 dq$ .

#### 8.6. The gravitational Compton cross section

For the Compton effect, i.e. electron-photon scattering



the total Klein-Nishina cross-section [22]  $\sigma$ =

$$= \left(\frac{8\pi\alpha^2}{3m^2}\right)(3/4) \left[\frac{1+a}{a^3}\left(\frac{2a(1+a)}{1+2a} - \log(1+2a)\right) + \frac{\log(1+2a)}{2a} - \frac{1+3a}{(1+2a)^2}\right]$$

where a = k/m, for small energies it becomes the

$$\frac{8\pi\alpha^2}{3m^2} = 0.665 \times 10^{-24} \text{ cm}^2$$
$$= \alpha^2 \left(\frac{\hbar c}{mc^2}\right)^2 \frac{8\pi}{3} \text{ , where } m = m_e \text{ is the electron mass,}$$

Thompson cross-section  $\sigma_{Th}=$ 

$$\alpha = \frac{e^2}{\hbar c}$$
 is the fine-structure-constant and

the reduced de-Broglie wavelength of the electron 
$$\tilde{\lambda}_e = \frac{\hbar c}{m_e c^2} = 0.38 \times 10^{-12} m$$
.

so  $\sigma_{th} = \alpha^2 \tilde{\lambda}_e^2 \frac{8\pi}{3}$  with these de nominations.

The start formula for the calculation of the differential cross-section according to the Feynman rules is [20 7.7.2], [21 4.218]

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2} \sum_{\pm s_i, s_f} \frac{d\sigma}{d\Omega}$$
$$= \frac{\alpha^2}{2} \left(\frac{k'}{k}\right)^2 \operatorname{Tr} \frac{p_f + m}{2m} \left(\frac{\epsilon'\epsilon k}{2k \cdot p_i} + \frac{\epsilon \epsilon' k'}{2k' \cdot p_i}\right) \frac{p_i + m}{2m}$$
$$\times \left(\frac{k\epsilon \epsilon'}{2k \cdot p_i} + \frac{k'\epsilon' \epsilon}{2k' \cdot p_i}\right)$$

with the initial and final momenta  $p_i p_f$  of the electron, k k' momenta of the photon and polarizations  $\varepsilon \varepsilon'$  of the photon. The following conditions have to be satisfied:

 $p_i \bullet p_i = m^2 \quad p_f \bullet p_f = m^2 \quad k \bullet k = k' \bullet k' = 0$  energy relations  $p_f + k' = p_i + k$  4-momentum conservation

 $k_0' = \frac{k_0}{1 + \frac{k_0}{m}(1 - \cos\theta)}$  Compton condition for the photon energy

There is 3 degrees of freedom in the choice of the polarization, the choice is made to simplify the expression above

 $\varepsilon \bullet \varepsilon = \varepsilon' \bullet \varepsilon' = -1 \quad \varepsilon \bullet k = \varepsilon' \bullet k' = 0 \quad \varepsilon \bullet p_i = \varepsilon' \bullet p_i = 0$ 

After some manipulations using the conditions and commutation rules for Dirac matrices, the famous Klein-Nishina formula results [20 7.74]

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{4m^2} \left(\frac{k'}{\bar{k}}\right)^2 \left[\frac{k'}{\bar{k}} + \frac{k}{\bar{k}'} + 4(\epsilon'\cdot\epsilon)^2 - 2\right]$$

, where the scalar denomination k k' is used for the energy  $k_0 k_0'$ . We get the total cross-section using the Compton condition and integrating over  $z = \cos \theta$ 

$$\bar{\sigma} = \frac{\pi \alpha^2}{m^2} \int_{-1}^{1} dz \left\{ \frac{1}{[1 + (k/m)(1-z)]^3} + \frac{1}{[1 + (k/m)(1-z)]} - \frac{1-z^2}{[1 + (k/m)(1-z)]^2} \right\}$$
  
and averaging over polarizations [21 4.221] 
$$\overline{(\varepsilon'.\varepsilon)^2} = -\frac{1}{2} \left( \delta_{ij} - \frac{k^i k^j}{k^2} \right) \left( \delta_{ij} - \frac{k'^i k'^j}{k'^2} \right)$$

for small energies  $\frac{k}{m} \rightarrow 0$ , the Thomson cross section arises

$$\sigma_{th} = \alpha^2 \left(\frac{\hbar c}{mc^2}\right)^2 \frac{8\pi}{3} = \alpha^2 \tilde{\lambda}_e^2 \frac{8\pi}{3}$$

For the graviton, we insert the photon-like (dimensionless, dropping the scale  $r_s = r_{gr}$ ) wave function  $(A_p)_{\mu}^{\nu} = \Omega_{\mu}^{\nu} \frac{1}{\sqrt{2Vk}} \left( \exp(-ik \bullet x) + \exp(ik \bullet x) \right)$ 

with the covariant derivative  $(D_{\mu})^{\lambda}{}_{\kappa} = \partial_{\mu} + (\varepsilon_{a})^{\lambda}{}_{\kappa}\sqrt{\alpha_{gr}}(A_{p})_{\mu}^{a}$ 

and again the starting formula above, where the only change is in the polarization terms  $\xi = \varepsilon_{\mu} \gamma^{\mu}$  and

 $\boldsymbol{\xi}' = \boldsymbol{\varepsilon}'_{\mu} \gamma^{\mu}$ , which, with the setting  $\Omega_{\mu}^{\nu} = \begin{pmatrix} e_0 & e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 & e_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  and the new initial polarization  $e_{\mu} = (e_0, e_1, e_2, e_3)$ 

and final polarization  $e'_{\mu} = (e'_0, e'_1, e'_2, e'_3)$ 

 $\|e\| = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2} = \|e'\| = 1$ , and the totally antisymmetric matrices  $\varepsilon_a$ , become  $\varepsilon_a^{\lambda_1} \omega_\mu^{\alpha} (\gamma^{\mu})^{\kappa_{\lambda_2}} = e^{\alpha} (\varepsilon_{\alpha}) (\gamma^0 + \gamma^1) = e^{\alpha} g_{\alpha}$ , where  $g_{\alpha} = \varepsilon_{\alpha} (\gamma^0 + \gamma^1)$  are the matrices analogous to the  $\gamma$ -matrices in the "Dirac-dagger"  $\notin = \varepsilon_{\mu} \gamma^{\mu}$  in the quantum-electrodynamics. 85

After going into the rest frame of the electron p = (m,0,0,0) and some manipulations we get

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{\alpha_{gr}^2}{32m^2} \left(\frac{k_0'}{k_0}\right)^2 \left(d_{s0}(e, e', \theta) + \frac{k_0}{m}d_{s1}(e, e', \theta) + O\left(\frac{k_0^2}{m^2}\right)\right), \text{ where the functions } d_{s0}(e, e', \theta) \text{ and } d_{s0}(e, e', \theta) = O\left(\frac{k_0^2}{m^2}\right)$$

 $d_{s1}(e, e', \theta)$  are series-coefficients in the  $\frac{k_0}{m}$ -series.

Now perform the integration over  $\theta$  and averaging over  $e_{\mu}$  and  $e'_{\mu} = e_{\mu}$ 

to get the total cross-section

 $\overline{\sigma} = 8\pi \,\alpha_{gr}^{2} \widetilde{\lambda}_{e}^{2} \left( 1.170 + \frac{k_{0}}{m} 0.400 + \dots \right) \approx 9.36 \,\pi \,\alpha_{gr}^{2} \widetilde{\lambda}_{e}^{2} \quad \text{, where the last expression is the gravitational low-}$ 

energy Thomson cross-section .

The different form of the bracket expression in the differential cross-section as compared to the electromagnetic cross-section is due to the different nature of polarization: for the photon the polarization is *transversal to the momentum*, so the averaging depends on k and k', for the graviton the polarization is a *free parameter* independent of momentum.

# 86 **B9. Unified QFT**

electromagnetic: gauge-group=Lie group SU(1) (trivial) commutative, long-range, classical and quantum, scale  $r_e = 2.8 \times 10^{-15} m$ , energy scale  $E_{em} = \frac{\hbar c}{r} = 0.07 MeV$ , the interaction-boson=photon is a vector with spin=1

covariant derivative  $D_{\mu} = \partial_{\mu} + \frac{ie}{\sqrt{\hbar c}} A_{\mu}$ , fundamental equations Maxwell equations  $\partial_{\mu} F^{\mu\nu} = 0$ wave: photon  $A_{\mu}$  spin=1 one polarization, modes: spherical (l = 0) and and all l > 0, angular function  $Y_{lm}$ , wave equation, radial Helmholtz equation, lagrangian  $L_{em} = -\frac{1}{\Lambda} F_{\mu\nu} F^{\mu\nu}$  with the Maxwell field tensor  $F_{\mu\nu} = \partial_{\mu}A_{\nu}^{\kappa} - \partial_{\nu}A_{\mu}^{\kappa}$ 

gravitation: gauge-group=extended SU(2) with 4 generators  $\tau_{\mu}$ , additional invariance: background independence, long-range, classical and quantum, its interaction-boson=graviton is a 2-tensor with spin=2 scale  $r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1 \times 10^{-5} m = 31 \mu m$ , energy scale  $E_{gr} = \frac{\hbar c}{r_{gr}} = 6.34 \times 10^{-3} eV$ 

fundamental equations: Ashtekar-Kodama equations  $AK(A_{\mu}^{\nu}, E^{\mu\nu}, \Lambda)=0$  for the graviton tensor  $A_{\mu}^{\nu}$ , the tetrad  $E^{\mu\nu}$  and the cosmological constant  $\Lambda$ , in the limit  $\Lambda ->0$  the Einstein-equations are valid (except at the horizon) and  $A_{\mu}^{\nu}$  =const (i.e. trivial background)

 $\Lambda$  generated by the tensor field  $\varphi_{\Lambda}$  is scale dependent, classical scale  $r >> r_{gr}$   $\Lambda = \Lambda_0 = 2.7 \ 10^{-52} m^{-2}$ wave: graviton  $A_{\mu}^{\nu}$  spin=2 two polarizations, modes: quadrupole and higher  $l \ge 2$ , angular function  $\exp(i l \theta)$ , gravitational radial wave equation  $\operatorname{weq}_{\nu}(E^{1\nu}, \partial_r^3 E^{1\nu})$  for the tetrad component  $E^{l\nu}$ 

determines  $A_{\mu}^{\nu}, E^{\mu\nu}$  with 4 free parameters for  $E^{\mu\nu}$  and 4 free parameters for  $A_{\mu}^{\nu}$ ,  $E^{\mu\nu}$  damped  $\exp(-4\sqrt{\frac{r}{3}})$ for  $r \le r_{gr} = 31 \mu m$  QFT gravity

covariant derivative  $D_{\mu}t^{\nu\lambda} = \partial_{\nu}t^{\nu}{}_{\mu} + \varepsilon_{\mu\kappa\lambda}A_{\nu}{}^{\kappa}t^{\nu\lambda}$ ,  $D_{\mu} = \partial_{\mu} - iA_{\mu}{}^{a}\tau^{a}$ , where  $\tau^{a} = i\varepsilon_{\nu}{}^{a}{}_{\lambda}$  are the generators of the extended SU(2)-algebra with 4 generators  $\tau^{a}$  and the Lie-algebra  $[\tau^{a}, \tau^{b}] = i\varepsilon_{c}^{ab}\tau^{c}$ (renormalizable) lagrangian

$$L_{gr} = L_{H} + L_{I} = \hbar c \left( -\frac{1}{4} F_{\mu\nu}^{\ \kappa} F^{\mu\nu}_{\ \kappa} - \frac{\overline{\varphi}_{\Lambda} \varphi_{\Lambda}}{3} (\varepsilon^{\kappa}_{\ \mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_{2}\lambda_{2}\kappa_{2}} \varepsilon^{\mu_{1}\lambda_{1}}_{\ \kappa_{1}} E^{\kappa_{1}\kappa_{2}} A_{\lambda_{1}}^{\ \lambda_{2}} A_{\mu_{1}}^{\ \mu_{2}} ) \right) + E^{\kappa_{1}}_{\ \nu_{1}} F_{\mu\kappa_{1}}^{\ \nu_{1}} E^{\kappa_{2}}_{\ \nu_{2}} F^{\mu}_{\ \kappa_{2}}^{\ \nu_{2}} \right)$$

with the constraint  $\overline{\varphi}_{\Lambda} \varphi_{\Lambda} = \Lambda$ 

and with the field tensor  $F_{\mu\nu}^{\ \kappa} = \partial_{\mu}A_{\nu}^{\ \kappa} - \partial_{\nu}A_{\mu}^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_{1}\kappa_{2}}A_{\mu}^{\ \kappa_{1}}A_{\nu}^{\ \kappa_{2}}$ 

weak interaction: gauge-group=Lie-group SU(2) 3 generators Pauli matrices  $\sigma_i$ , corresponding 3 massive gauge bosons

 $Z, W^+, W^-$ , short range, pure quantum cut-off energy  $M_Z = 91.17 GeV$ , length scale  $r_{weak} = 10^{-17} m$ energy scale  $E_{weak} \approx 10^{-4} E_{em} = 7 eV$ 

 $\mathcal{L}_{1} = -\frac{1}{4}W_{\mu\nu}^{a}W^{a\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad W^{\mu\nu} = \partial_{\mu}W_{\nu}^{a} - \partial_{\nu}W_{\mu}^{a} + gf^{abc}W_{\mu}^{b}W_{\nu}^{c}$   $F_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}$ (where hysical gauge fields Z. W<sup>+</sup> W<sup>+</sup> or constants), where  $F_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}$ 

lagrangian

where the physical gauge fields Z,  $W^+$ ,  $W^-$  are

$$Z_{\mu} = \frac{gW_{\mu}^{3} + g'B_{\mu}}{(g^{2} + g'^{2})^{1/2}} \equiv \cos\theta_{W}W_{\mu}^{3} + \sin\theta_{W}B_{\mu}$$

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}} (W^{1}_{\mu} \pm i W^{2}_{\mu})$$

covariant derivatives left and right

$$D_{\mu}R = (\partial_{\mu} + ig'B_{\mu})R$$

$$D_{\mu}L = \left[\partial_{\mu} + (i/2)g'B_{\mu} - (i/2)g\sigma_{i}W_{\mu}^{i}\right]L$$

**strong (color) interaction**: gauge-group=Lie-group SU(3) 8 generators Gell-Mann-matrices , corresponding 8 massless spin-1 gauge bosons, 3 charges (colors) r g b, short range, pure quantum, asymptotically free (confinement),

confinement length scale  $r_{col} = r_{pr} = 0.84 * 10^{-15} m$ , energy scale  $E_{col} = 210 MeV$ 

cut-off energy 
$$M_{col} = \frac{\hbar c}{r_{col}} = 2.33 * 10^{10} eV \approx 23 GeV$$

lagrangian  $L_{col} = -\frac{1}{4}G^{a}{}_{\mu\nu}G^{a}{}^{\mu\nu}$ 

with the field tensor  $G^a_{\mu
u} = \partial_\mu \mathcal{A}^a_
u - \partial_
u \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_
u$ 

$$D_\mu:=\partial_\mu - ig\,A^lpha_\mu\,\lambda_lpha$$

covariant derivative  $p_{\mu} = p_{\mu} + p_{\mu} + p_{\mu} + q_{\mu}$ , where is the <u>coupling constant</u>, is the gluon <u>gauge field</u>, for eight different gluons and where is one of the eight <u>Gell-Mann matrices</u>,

(hypothetical preon [25]) **hypercolor interaction**: gauge-group= Lie-group SU(4), 15 generators massless bosons spin=1, short range, pure quantum

4 charges  $r_L^+ r_R^+ r_L^- r_R^-$  for r-preon and  $q_L^+ q_R^+ q_L^- q_R^-$  for q-preon resp. with charge + or - and helicity L or R, antiparticle  $C(r_L^+) = r_R^-$ ,  $C(r_L^-) = r_R^+$ 

the weak-interaction is the Yukawa-limit of the hypercolor interaction

the angular momentum l = 1 determines the flavor  $l_z = 0$  for flavor 0,  $l_z = +1$  for flavor 1 and  $l_z = -1$  for flavor 2

L-R-symmetry breaking  $SU(4) = SU(2)_L \otimes SU(1)_R \otimes \overline{SU}(1)_R$ 

GUT (approximate) SU(5):

R=(3 rgb *d*-antiquark, 2 leptons e  $\vee$ ) =  $d(\overline{3},1) \oplus ev(1,\overline{2})$ 

L=(3x2 rgb *u*-quark *d*-quark, 3 rgb *u*-antiquark, 1 antilepton *e*)= $ud(3,2) \oplus \overline{u}(\overline{3},1) \oplus \overline{e}(\overline{1},1)$ 

#### 88 **References**

[1] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press 1992

[2] T. Fliessbach, Allgemeine Relativitätstheorie, Bibliographisches Institut 1990

[3] M. Visser, The Kerr spacetime: a brief introduction [arXiv: gr-qc/07060622],2008

[4] L. Freidel, The linearization of the Kodama state [arXiv hep-th/0310224], 2003

[5] L. Smolin, Quantum gravity with a positive cosmological contant [arXiv hep-th/0209079], 2002

[6] T. Thiemann, Lectures on loop quantum gravity, MPI Gravitationsphysik AEI-2002-087, 2002

[7] C. Wüthrich, Approaching the Planck scale from a general relativistic point of view, Thesis University of Pittsburgh, 2006

[8] H. Nicolai, K. Peeters, M. Zamaklar, Loop quantum gravity: an outside view [arXiv hep-th/0501114], 2005 [9] J. Helm, New solutions of the Tolman-Oppenheimer-Volkov (TOV) equation and of Kerr space-time with matter and the corresponding star models, viXra 1404.0037, 2017

[10] J. Helm, New solutions of gravitational collapse in General Relativity and in the Newtonian limit, viXra 1408.0117, 2014

[11] J. Helm, An exact GR-solution for the relativistic rotator, viXra 1706.0200, 2017

[12] M. Hersch, The Wheeler-DeWitt equation, wordpress.com, 2017

[13] C. Kiefer, Quantum Gravity, Oxford University Press, 2012

[14] D.Vey, Gravity, self duality, BF theory, Lecture Notes, Paris, 2013

[15] L. Smolin, S. Speziale, A note on the Plebanski action with cosmological constant and an Immirzi parameter [arXiv gr-qc/0908.2288], 2010

[16] L. Smolin, The Plebanski action extended to a unification of gravity and Yang-Mills theory, [arXiv hep-th/0712.0977], 2008

[17] R. Durka, Constrained BF theory as gravity, Max Born Symposium , 2010

[18] H. Kalka, G. Soff, Supersymmetrie, B.G.Teubner, 1997

[19] R. Penrose, On Gravity's Role in Quantum State Reduction . General Relativity and Gravitation. **28** (5): 581–600. 1996

[20] J.D.Bjorken, S.D.Drell, Relativistic Quantum Mechanics, McGraw-Hill 1964

[21] H.-K. Quang, Elementary Particles and their Interactions, Springer 1998

[22] M.Kaku, Quantum Field Theory, Oxford University Press 1993

[23] Mathematica-notebook WdWcov4Thnceta.nb, <u>www.janhelm-works.de</u>: calculation and derivation of AK-gravity

[24] Mathematica-notebook QFTFeynDiag.nb, <u>www.janhelm-works.de</u>: calculation and derivation of Compton cross-section for AK-gravity

[25] I.A. D'Souza, C.S. Kalman, Preons: models of leptons, quarks and gauge bosons as composite objects, World Scientific 1992