SOLUTION OF ERDÖS-MOSER EQUATION
$1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$

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ABSTRACT. I will provide the solution of Erdös-Moser equation based on the properties of Bernoulli polynomials and prove that there is only one solution satisfying the above-mentioned equation.

1. Notation

$1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$ represents Erdös-Moser equation, where $k, p \in \mathbb{N}$.
Let b_n denotes Bernoulli numbers.
Let

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$$

denotes Bernoulli polynomials for $n \geq 0$.

2. Introduction

The Erdös-Moser equation (EM equation), named after Paul Erdös and Leo Moser, has been studied by many number theorists through history since combines addition, powers and summation together. The open and very interesting conjecture of Erdös-Moser states that there is no other solution of EM equation than the trivial $1 + 2 = 3$. Investigation of the properties and identities of the EM equation and ultimately providing the proof of this conjecture is the main purpose of this article.

3. Solution

Lemma 3.1. *The EM equation is equivalent of*

$$\sum_{k=0}^{x} k^p \equiv \frac{B_{p+1}(x+1)}{p+1} = (x+1)^p \tag{3.1}$$

for $x \in \mathbb{N}$.

Proof. Sum of pth powers is defined as

$$\sum_{k=0}^{x} k^p = \frac{B_{p+1}(x+1) - B_{p+1}(0)}{p+1}$$

2010 Mathematics Subject Classification. Primary 11D41, 11D72.

Leo Moser proved that for another solution of EM equation two must divide p, see [1], what yields that $p+1$ must be odd and $B_{p+1}(0)$ with odd subscripts is equal zero. □

Lemma 3.2.

$$B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p \tag{3.2}$$

$$B_{p+1}(x+2) - B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.3}$$

Proof. Relation of Bernoulli polynomials given by Whittaker and Watson, see [2], what in general form is defined as $B_n(x+1) - B_n(x) = nx^{n-1}$. □

Lemma 3.3. *Lemma (3.1) in combination with rearranged Eq. (3.2) gives a relation*

$$\frac{B_{p+1}(x+1)}{B_{p+1}(x)} = \frac{(x+1)^p}{(x+1)^p - x^p} \tag{3.4}$$

Proof. Let us express $p+1$ from Eq. (3.2) as

$$\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} = p+1 \tag{3.5}$$

then by putting LHS of Eq. (3.5) in Eq. (3.1) we get

$$B_{p+1}(x+1) = (x+1)^p \left(\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} \right)$$

and after elementary rearrangements we can rearrange Eq. (3.1) to the form as is defined in Lemma (3.3). □

Theorem 3.4. *The EM equation has other solution than trivial if and only if holds the relation in Eq. (3.6) and therefore EM equation does not have any other solution than trivial.*

$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2 \tag{3.6}$$

for $x \in \mathbb{N}$.

Proof. Let us rearrange Eq. (3.1) as

$$B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.7}$$

the RHS of Eq. (3.3) and Eq. (3.7) are equal so we can define

$$B_{p+1}(x+2) - B_{p+1}(x+1) = B_{p+1}(x+1)$$
$$B_{p+1}(x+2) = 2B_{p+1}(x+1)$$
$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2$$

From Lemma (3.3) respectively by the expression on the LHS of Eq. (3.4) $\frac{B_{p+1}(x+1)}{B_{p+1}(x)}$ can be expressed the expression defined by the LHS of Eq. (3.6) $\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)}$ since the difference between $(x+2)-(x+1)$ is equal to the difference between $(x+1)-(x)$. Now let us focus on the integral solutions of the RHS of Eq. (3.4)

$$\frac{(x+1)^p}{(x+1)^p - x^p}$$

since we need only them as by expression on the LHS of Eq. (3.4) can be expressed the expression on the LHS of Eq. (3.6) and LHS of Eq. (3.6) must be equal to two, in other words Eq. (3.6) holds if and only if Eq. (3.4) has an integral solution (moreover equal to two). It is trivial to see that the expression $\frac{(x+1)^p}{(x+1)^p - x^p}$ has integral solutions for $x > 1$ if and only if $0 < p < 2$ (considering the EM equation, for this moment is important the exponent p not the variable x) since

$$\frac{(x+1)^p}{(x+1)^p - x^p} = \frac{x^p + px^{p-1} + ... + 1}{px^{p-1} + ... + 1} = \frac{x^p}{px^{p-1} + ... + 1} + 1$$

On the basis of this facts we can state that if there is an other solution, than the trivial, of the EM equation it is possible if and only if the exponent $p = 1$ what is impossible since there is only one solution - trivial when $p = 1$ as it follows from the basic formula of summation

$$\sum_{k=0}^{x} k^1 \equiv \frac{x * (x+1)}{2} = x + 1 \Rightarrow \frac{x}{2} = 1$$

where x must be equal to two. All of the above-mentioned facts unconditionally prove the Theorem (3.4) and at the same time the Erdös-Moser conjecture. $\square$

References

[1] L.Moser, On the Diophantine Equation $1^k + 2^k + ...(m-1)^k = m^k$, *Scripta Math. 19,* (1953), 84-88.

[2] E. T. Whittaker, G. N. Watson, A course of MODERN ANALYSIS, *Cambridge University Press 3rd edition,* (1920), 127.