ON SURFACE MEASURES ON CONVEX BODIES AND GENERALIZATIONS OF KNOWN TANGENTIAL IDENTITIES

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Abstract. One theme of this paper is to extend known results from polygons and balls to the general convex bodies in $n-$ dimensions. An another theme stems from approximating a convex surface with a polytope surface. Our result gives a sufficient and necessary condition for an natural approximation method to succeed (in principle) in the case of surfaces of convex bodies. Thus, Schwartz's paradox does not affect our method. This allows us to define certain surface measures on surfaces of convex bodies in a novel and simple way.

1. Introduction

Archimedes discovered the 2-dimensional formula for a disk $D(x, r)$

$$\frac{|A|}{|\partial A|} = \frac{1}{2}R.$$

In the early millenia Apostol and Mnatsakanian extended above equality to the class of polygons that they called "tangential" [1]. We will prove that if we extend the definition on tangentiality we obtain in n - dimensions (for that class of polytopes)

$$\frac{|A|}{|\partial A|} = \frac{1}{n}R,$$

where again R is the radius of the inscribed maximal ball (John's ball [2]).

2. Some definitions and known results

A convex body is a compact and convex set. We expect always that $0 \in K$. For any measurable set $K \in \mathbb{R}^n$ we use $|K|$ as sign of the Lebesgue measure of that set. For sets $A, B \in \mathbb{R}^n$ we define the Minkowski addition as

$$A + B = \{a + b : a \in A \quad b \in B\}.$$

We define the set dilation as

$$\lambda A = \{\lambda a : a \in A\}.$$

We always use B_2^n an unit ball in n dimensions. The surface area of an convex body is defined as

$$|\partial(K)| := \lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}$$

We use H_θ for the $n-1$ dimensional hyperplane going through the origo and orthogonal to $\theta \in S^n$. In our sense a polyhedra is an intersection of finite number

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of closed half-spaces containing origo. A polytype is an polyhedra with a finite Lebesque measure. See for example [4] and references therein. For convex bodies in John's position, the inradius R is defined as the radius of the John's ball [2].

3. On polytopes and normal points

A convex body can be defined as a compact set

$$K = \bigcap_{i \in I} H',$$

where each H' is the half space containing origo that is bounded by the corresponding hyperplane H [4]. It can be proved that in each $n-1$-dimensional hyperplane H there exists an unique point x with the vector $\mathbf{x}$ orthogonal to the hyperplane H [4]. These points x_i are the unique solutions to the minimization problems

$$\min_{y \in H} ||y|| = ||x_i||, \tag{1}$$

where for each hyperplane $y \in H$, and H is the boundary of a half space used to define K. We call these points normal points. They depend on the position of K. They exists as long as the origo belongs to the interior of K. We prove that the convex body inherits the normal points from the half spaces used to define it.

Theorem 1. *Let K be a convex set containing origo. If the halfspace H is used in the definition of K, then the normal point of H belongs $K \cap H$.*

Proof. If x_i is defined as the solution to

$$< y, x_i >= 0,$$

then a closed origo centered ball of radius $||\mathbf{x}_i||$ contains only that point from the hyperplane H, and because the ball has that common supporting hyperplane with K, we have the claim. □

In light of the theorem the next corollary is clear.

Corollary 1. *Let P be a polytope. Then each facet of P contains an unique normal point in it's interior.*

We define minimal normal points as follows. They are the normal points of minimal value $||\cdot||_2$ norm. We will use the following definition.

Definition 1. *A convex body is tangential iff every normal point is minimal.*

With respect to any normal point x_i we denote F_i as the corresponding facet. For each K we define the Borel space $(K, \mathcal{X})$. We consider the finite intersection of half spaces used to define K. For each intersection $\bigcap_{i=1}^m H'$ of half spaces forming a polytope and used to define K we define a probability measure

$$\mu_j := \frac{\sum_{i=1}^m |F_i| \delta_{x_i}(x)}{\sum_{i=1}^m |F_i|}, \tag{2}$$

where every $\delta_{x_i}(x)$ is a Dirac measure.

Theorem 2. *Let $x_1, ..., x_m$ be the normal points of a polytope P, then*

$$\frac{1}{n}\mathbb{E}(||x_i||) = \frac{|A|}{|\partial A|}. \tag{3}$$

Proof. We know that a volume of an n-dimensional cone A_i is given by

$$|A_i| = \frac{|\text{base}| * \text{height}}{n}.$$

The theorem 1 implies that the polytope P's volume is

$$|P| = \frac{1}{n}\sum_{i=1}^{m} ||x_i|||F_i|,$$

where F_i is the facet corresponding to the normal point x_i. Thus,

$$\frac{|P|}{|\partial P|} = \frac{\sum_{i=1}^{m} ||x_i|||F_i|}{n\sum_{i=1}^{m} |F_i|} = \frac{1}{n}\mathbb{E}(||x_i||).$$

□

Corollary 2. *Let a polytope P be tangential, then*

$$\frac{1}{n}R = \frac{|A|}{|\partial A|}, \tag{4}$$

where R is the inradius of K.

Naturally we wan't to extend the theorem and it's corollary to the general convex bodies. We will first do that for the corollary. For that we need a lemma.

Lemma 1. *For any point $x \in \partial K$, there exists a normal point x' s.t $x \in H_{x'}$.*

Proof. Now, x belongs to some supporting hyperplane. To this supporting hyperplane there corresponds at least one normal point and via theorem 1 the normal point belongs to ∂K. □

Next we proof that the space of normal points N is separable as a metric compact space.

Theorem 3. *The space of normal points is closed in the subspace topology and the space is separable.*

Proof. Suppose that there exists a Cauchy sequence of normal points (x_j) s.t

$$\lim_{j=1} x_j = x. \tag{5}$$

It holds that $x \in \partial K$ because the surface is closed. Thus, via lemma 1 it holds that $x \in H_{x'}$, where x' is a normal point. Now, if $y \in H_{x'}$ it holds that

$$\lim_{j\to\infty} < x_j, x' - y >=< x, x' - y >= 0 \tag{6}$$

because of the continuity of the inner product. Thus, $H_{x'}$ is orthogonal to $\mathbf{x}$ and $x' = x$. In addition it's well known that every compact metric space is separable [9]. □

For any sequence of normal points (x_j) we denote the corresponding sequence of nested polyhedras to be the sequence

$$(\cap_{j=1}^{m} H_{x_j}) := (P_m).$$

The following theorem is a key theorem.

Theorem 4. *Let (x_j) be any sequence of normal points. Then for the corresponding nested sequence of polyhedras (P_m), it holds that*

$$\bigcap_{m=1}^{\infty} P_m = K, \tag{7}$$

iff the set of the corresponding normal points is dense in the space of normal points of K.

Proof. Let the set of normal points corresponding to the sequence (x_j) be dense in N. Because $\bigcap_{m=1}^{\infty} P_m$ is some convex body, the set of it‘s normal points is closed. Thus, it follows from the definition of K and from theorem 1 that $K = \bigcap_{m=1}^{\infty} P_m$. On the other hand, if the sequence of normal points is not dense in N, then there exist a normal point of K and a halfspace used in the definition of K that has a boundary that is not a supporting hyperplane to $\bigcap_{m=1}^{\infty} P_m$. Thus,

$$\bigcap_{i=1}^{\infty} P_m \neq K.$$

□

Corollary 3. *It holds that*

$$\lim_{m\to\infty} |P_m| = |K| \tag{8}$$

iff the set of normal points corresponding to the polyhedras is dense in N.

Proof. We need only to prove the following. If the set of normal points is not dense then

$$|K| < |\bigcap_{i=1}^{\infty} P_m|.$$

Via the theorem 4 the sets are not the same. So there exists $x \in \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. Now,

$$\min_{y_i \in K} ||x - y_i|| = ||x - y|| > 0, \tag{9}$$

because there exists a hyperplane separating K and x. It follows that

$$|K| < |\mathrm{conv}(x, K)| \leq |\bigcap_{i=1}^{\infty} P_m|,$$

because there exists a small ball $B(x, \epsilon) \subset \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. □

4. On normal measures on convex bodies

In this section we will define a surface measure on convex surfaces in $n-$ dimensions. In general the measure will not be continuous, because it can be discrete in the segments of an convex body (for example on the ends of an cylinder). First we will prove the existence of the measure. We use the sequences of measures defined in the last section.

Lemma 2. *For any borel set $A \subset K$ the sequence $(\mu_m(A))$ is monotonic.*

Proof. We need to prove that

$$\mu_{m+1}(A) \le \mu_m(A).$$

Let us denote

$$\mu_m(A) = \frac{|\partial P'_m|}{|\partial P_m|}$$

and

$$\mu_{m+1}(A) = \frac{|\partial P'_{m+1}|}{|\partial P_{m+1}|},$$

where $\partial P'_m$ and ∂P_{m+1} are the unions of those facets that contain the normal points of $A \cap P_j$ and $A \cap P_{m+1}$, respectively. Furthermore, let us denote

$$P_{m+1} := P_m - B.$$

Now,

$$|\partial P'_{m+1}| \le |\partial P'_m| \tag{10}$$

via the normal point condition. Let us denote

$$\partial B' := \partial P'_m - \partial P'_{m+1}.$$

Thus, we have

$$\frac{|\partial P'_{m+1}|}{|\partial P_{m+1}|} = \frac{|\partial P'_m| - |\partial B'|}{|\partial P_m| - |\partial B| + |F_{m+1}|}.$$

The above leads to

$$|\partial P'_{m+1}||\partial P_m| = |\partial P'_m||\partial P_{m+1}| - |\partial B'||\partial P_{m+1}| + |\partial P'_{m+1}||\partial B| - |\partial P'_{m+1}||F_{m+1}|, \tag{11}$$

which is the same as

$$\begin{aligned}|\partial P'_{m+1}||\partial P_m| =& |\partial P'_m||\partial P_{m+1}| - |\partial P'_m||\partial P_{m+1}| + |\partial P'_{m+1}||\partial P_{m+1}| + \\ & |\partial P'_{m+1}||\partial B| - |\partial P'_{m+1}||F_{m+1}|.\end{aligned}$$

Moreover it follows that

$$\begin{aligned}|\partial P'_{m+1}||\partial P_m| \le & |\partial P'_m||\partial P_{m+1}| - |\partial P'_{m+1}||\partial P_m| + |\partial P'_{m+1}||\partial P_{m+1}| - |\partial P'_{m+1}||\partial B| + \\ & |\partial P'_{m+1}||\partial B| + |\partial P'_{m+1}||F_{m+1}| - |\partial P'_{m+1}||F_{m+1}|.\end{aligned}$$

Thus, via above and inequality (10) we have

$$\mu_{m+1}(A) = \frac{|\partial P'_{m+1}|}{|\partial P_{m+1}|} \le \frac{|\partial P'_m|}{|\partial P_m|} = \mu_m(A).$$

□

The total variation distance of probabily measures is defined as

$$||\mu - v||_{TV} := 2 \sup_A |\mu(A) - v(A)|.$$

Theorem 5. *The set function defined by* $\lim_{j\to\infty} u_j = \mu$ *is a radon measure on* $(K, \mathcal{X})$*, where the metric is understood in the total variation sense.*

Proof. By lemma 2 and Riesz representation theorem K $\lim_{j\to\infty} u_j = \mu$ uniformly. Thus u_j is a radon measure. See for example [8] and [6]. □

We need the following lemma.

Lemma 3. *Suppose that every normal point of K' is a normal point of K. Then for normal measures μ and v of $K \subset K'$ and K', respectively, it holds that*

$$\int_K |f|\mu \leq \int_{K'} |f|v. \tag{12}$$

Proof. It's enough to prove the theorem for charasteristic functions and use a standard limit process. Now,

$$\mu(P_m \cap A) \leq v(P_m \cap A) = v_m(A),$$

by a limit process and lemma 2. Thus,

$$\mu(A) = \lim_{m\to\infty} \mu(P_m \cap A) \leq \lim_{m\to\infty} v(P_m \cap A) = v(A),$$

by continuity of measures for nested sequence of polyhedras. □

We need to prove that the measure does not depend on the sequence of polytopes, if the sequence of normal points is dense.

Theorem 6. *If A is Borel set it holds that*

$$\lim_{j\to\infty} \int_K 1_A \mu_j = \int_K 1_A \mu = \lim_{l\to\infty} \int_K 1_A v_l = \int_K 1_A v. \tag{13}$$

Proof. If $v(A) < \mu(A)$, then there exists P_l s.t

$$v_l(A) < \mu(A),$$

thus via previous lemma 3 K does not contain the normal points of P_l which is a contradiction.

□

For each convex body K we call the measure just defined the normal measure of K.

Next we generalize the result 2 from polytopes to general convex bodies.

Theorem 7. *For any convex body it holds that*

$$\frac{1}{n}\mathbb{E}||x|| = \frac{|K|}{|\partial K|}, \tag{14}$$

where $x \in X$ are the normal points of K and the expected value is understood with respect to the measure given by the theorem 5.

Proof. Now, via lemma 3

$$\int 1_A \mu_{j+1} \leq \int 1_A \mu_j$$

for all open sets. It follows that

$$\mathbb{E}(|f|)\mu_j \to \mathbb{E}(|f|)\mu \tag{15}$$

for all bounded continuous functions.

□

The tangential case is an easy corollary.

Corollary 4. *For tangential convex body K, it holds that*

$$\frac{R}{n} = \frac{|K|}{|\partial K|},$$

where R is the inradius of K.

The next corollary is an analogue of the corollary 3 from the last section.

Corollary 5.

$$\lim_{m\to\infty} |\partial P_m| = |\partial K| \tag{16}$$

iff the set of normal points corresponding to the polyhedras is dense in X.

Proof. Via the theorem 4 we need only to prove that if

$$|\partial K| = |\partial \bigcap_{i=1}^{\infty} P_m|,$$

then the set of normal points is dense. Now, if it holds that

$$|K| < |\bigcap_{i=1}^{\infty} P_m|. \tag{17}$$

It follows that

$$|\partial(\bigcap_{i=1}^{\infty} P_m)| - |\partial K| = \lim_{\epsilon\to 0} \frac{|\bigcap_{i=1}^{\infty} P_m + \epsilon B_2^n| - |\bigcap_{i=1}^{\infty} P_m| - |K + \epsilon B_2^n| + |K|}{\epsilon} < 0$$

which is a contradiction.

□

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