Intuitive explanation of the Riemann hypothesis

I have decided not to try to state any proof of the hypothesis (for now), but only to explain what it is about, and why Riemann's zeroes tend to be on the line $Re(z) = 1/2$.

There is a unique one-form α on the upper half-plane $\mathbb{H}$, invariant under $\Gamma(2)$, which has a pole of residue -1 at $i\infty$ and a pole of residue $+1$ at 1. Under the embedding $\mathbb{H} \to \mathbb{C}$ with τ the coordinate in $\mathbb{C}$ we can cancel the cusp residue by adding $d\tau$.

Let, then, T be the connected component of the real multiplicative group, and consider the multiplication actions

$$\begin{cases} \mu_+ : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \sqrt{g}z \end{cases}$$

$$\begin{cases} \mu_- : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \frac{1}{\sqrt{g}}z \end{cases}$$

1. Theorem. For each unitary character ω of T and each real number c with $0 < c < 1$, the differential two-form

$$g^{2-2c}\omega(g)\mu_+^*(\alpha + d\tau) \wedge \mu_-^*(\alpha + d\tau)$$

is real and integrable (rapidly decreasing, that is, 'Schwartz') on $T \times [0, i\infty)$. Among rapidly decreasing forms, it is exact and only if $\zeta(c + i\omega_0)$ is zero where ζ is Riemann's zeta function.

Here ω_0 is the real number corresponding to ω under the rule $\omega(g) = g^{i\omega_0}$.

It is real because the factors besides $\omega(g)$ are anti-symmetric with respect to interchanging μ_+ and μ_- which matches the reversal of orientation of T.

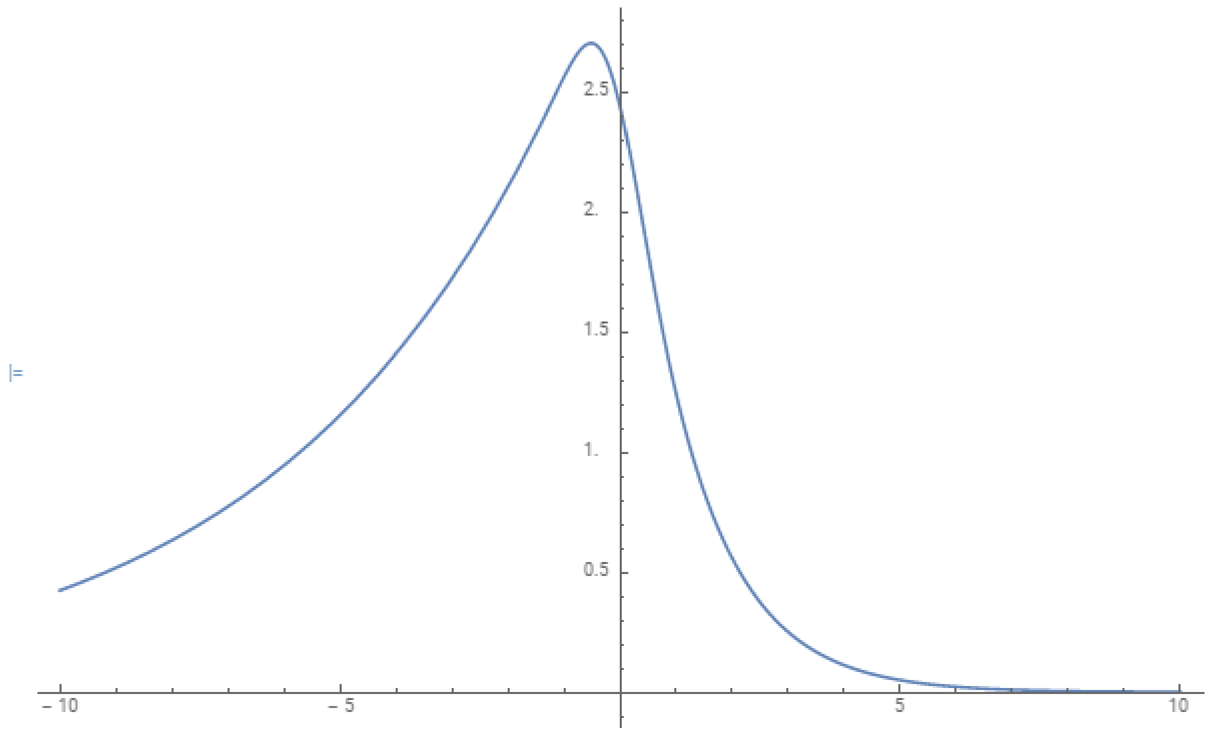
The integral of the two-form equals the squared absolute value of the holomorphic integral $\int g^{1-c}\omega(g)(\alpha + d\tau)$. It is easy to calculate the holomorphic *definite* integral; it is $iL(s, \chi)\Gamma(s)\pi^{-s}$ where i is the imaginary unit, and L is the L series for sums of four squares, and χ is the Dirichlet sign character. The rule $\omega(g_1 g_2^{-2}) = \omega(g_1)\omega(g_2)^{-1}$ is all that is needed.

A single holomorphic integral is just an example of a 'dynamical system,' it being zero just refers to a 'closed orbit' where the beginning point and endpoint agree. For this we just calculate the *indefinite* integral of $(\alpha + d\tau)d\tau$ 'synthetically,' which means, in terms of known functions. It is $\frac{1}{i\pi}log(\lambda(\tau)/q(\tau)$ where λ is the modular lambda function and $q(\tau) = e^{i\pi\tau}$.

We can model the effect on a point of the complex plane by imagining a desert, with a sun orbiting at infnite distance and a constant rate (as it would if the desert is near a pole of the earth). For some reason, a snail is attempting to crawl towards the light, and we watch this from a rotating reference frame with the starting point of the snail as the central axis point, and the sun stationary.

We are imagining a Fourier transform as a planar dynamical system. The grains of sand of the desert are rotating about this point clockwise but we ignore them. Let us draw this so the sun is at the top of the page, and the horizontal axis passes through our central point. If we fix the crawling speed of the snail, there is one point to our right, on the axis, where the snail would be stationary in our coordinates, crawling towards the sun at the same rate as the grains of sand go in the other direction. The circle which has our central point and this point as the ends of its diameter, has the property that if the snail is placed on this circle, it will appear to crawl radially (towards or away from our originally chosen center). Our horizontal axis has the property that the snail will appear to crawl perpendicular to it; if it is in the interior of the small circle, it will crawl counter-clockwise, otherwise it will crawl clockwise. There could be a limit cycle surrounding the fixed point, orbiting counter-clockwise with respect to the fixed point, entering and exiting the circle radially and crossing the axis perpendicularly, the segment in the circle moving counter-clockwise relative to our initially chosen center, and the segment outside moving clockwise relative to it. Regardless of whether there is a limit cycle or the fixed point is attracting, the general motion of a point near the fixed point is counter-clockwise about that fixed point, of a smaller radius than the distance from the fixed point to our central reference.

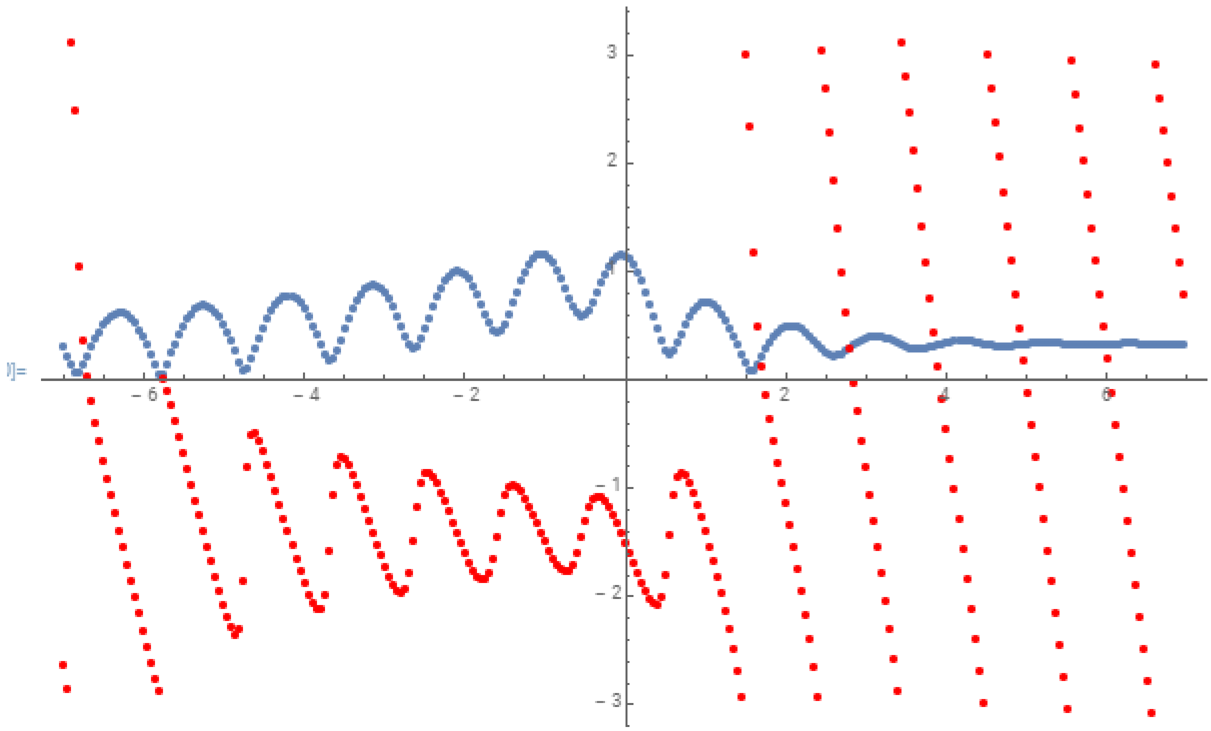
If we want to think of time ranging from $-\infty$ to ∞ we can let t be such that $\tau = ie^t$. Here is a graph of $e^{(c-1)t}log(\lambda/q)$ as a function of time, when $c = .2$



We can see that as the snail's speed increases, it stops orbiting clockwise as the grains of sand do, and becomes attracted to the limit cycle or fixed point instead. It becomes 'stuck' on an endless treadmill, and the large-scale picture would be that it is just stuck at the stationary point.

When the graph suddenly decays, it becomes tired and stops. Another intuition might be someone being rescued on a raft, from a whirlpool, with the force being what rescues them, but when the force becomes weak they fall out of the boat and are swept away.

Here is a graph of the magnitude and argument of the integral of $e^{(c-1)v}log(\lambda/q)d\tau$ divided by $(1-c)$ to match the integral of $e^{c-1}d\ log(\lambda/q)$.



The way the argument and magnitude behave does suggest that the sailor 'stays in the boat' for five or six cycles before falling out and being swept into the larger whirlpool, when he becomes too tired to swim towards the sun. It is obvious that he has not ended up in his original position. He is close to the reference origin at time 3/4 but it is the average magnitude that determines the limiting magnitude. The squared magnitude of the limiting distance is the integral of $g^{2-2c}\omega(g)\mu_+^*\beta \wedge \mu_-^*\beta$ and is the 'obstruction' to exactness of that two-form among Schwartz forms.

It is only when $c = 1/2$ that the limiting behaviour as $t \to \infty$ and $t \to -\infty$ are matching exponential rates. Only then does it seem feasable that when the sailor falls out of the boat, he happens to land at the center of the maelstrom.

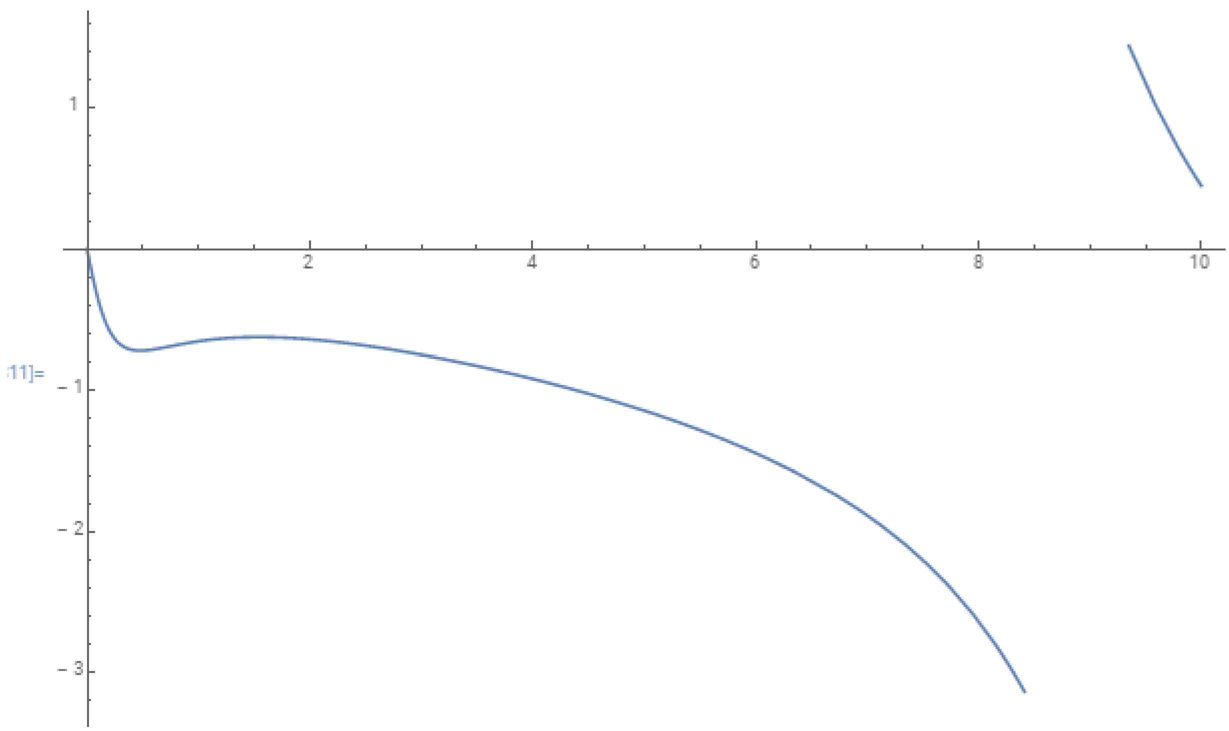
The picture of a dynamical system and limit cycle etc are fictitious, depending in essential ways on our choice of central point for our rotating reference frame, which we chose to be the origin of the boat. The number of 'missed cycles' when the sailor is in the spinning boat is exactly the phase shift of the L series times the other factors. Here for $s = c + i\omega$ is a graph of the argument of

$$\int_{-\infty}^{\infty} e^{i\omega t} e^{(c-1)t} dlog(\lambda/q) = i\pi \int_{\infty}^{\infty} e^{i\omega t} e^{(c-1)t}(\alpha + d\tau)$$

$$= i\pi \cdot iL(s,\chi)\Gamma(s)\pi^{-s}$$

$$= -L(s,\chi)\Gamma(s)\pi^{s-1}$$

while ω ranges from 0 to 10.



The argument orbiting through one cycle means that for the angular rate of rotation changing from 0 to 10 radians per second (if t is time in seconds) the number of cycles that the sailor stays in the boat changes by just one. We can state this as a theorem.

2. Theorem. The angular cycling of the argument of the L function as a function of ω matches the missed cycles of orbiting around the fictional center due to being in the 'boat' and orbiting the smaller limit cycle (or fixed point).

Also

3. Theorem. The equivalent condition in Theorem 1, of Riemann's function having a nontrivial zero of real part c less than $1/2$, that $g^{2c-2}\omega(g)\mu_+^*\beta \wedge \mu_-^*\beta$ is exact for some unitary character ω, is equivalent to an orbit (and also every orbit) of the corresponding dynamical system being closed in a situation when the 'dropping off' exponential rate is larger than the 'picking up' rate.

If one wants to choose a precise definition of the moment when the sailor 'falls out of the boat,' there is nothing mysterious about the location of the fixed point at that moment or any other. The distance of the fixed point from our central reference point is just the function value at that time t divided by the rotation rate ω. The sailor will seem to us to be swept away by the tide, orbiting clockwise, and he will be moving away from our reference point during all the time he is in the upper half of our picture, and moving towards it during all the time he is in the lower half of the picture. This will superimpose on the large clockwise circular motion a smaller counter-clockwise motion. When the decaying exponential rate matches the original one, he can be near enough the center when he falls out that this motion takes him back home. [1]

[1] Also, for $0 < c < 1/2$, the range where the drop-off slope is the faster one, the squared magnitude of the distance (the value of the integral of the two-form) should strictly increase as a function of the drop-off slope magnitude, that is to say, the value of the integral of the two-form or equivalently the squared magnitude of $L(s,\chi)\Gamma(s)\pi^{-s}$ should increase monatonically as c is decreased. Since it is a positive-valued real analytic function it cannot be zero except on the boundary of that region, if so.

Standard techniques show that the contour integral of $\frac{1}{2\pi i} d\ log\ \zeta(s)$ counts zeroes with multiplicity inside the contour. Because $L(s\chi)\Gamma(s)\pi^{-s}$ is a Fourier transform value with same zeroes for $0 < s < 1$ as ζ has, the same contour integral in that region also counts, with multiplicity, the closed orbits of the holomorphic one-parameter family of one-dimensional holomorphic dynamical systems. This is of the most trivial type of two-dimensional holomorphic dynamical system.

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