Weakness in the Lorentz Force Equation for Moving Charged Particles in Magnetic Fields

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Abstract- All forces in the universe are created from changes in energy levels that result from changes in the separation of bodies, whether electromagnetic or otherwise. Even when particles are constrained so that they cannot expend the energy giving rise to those forces they still experience them. For the force to exist the energy system creating it must exist, even if only in potential. We examine the second part of the Lorentz Force Equation, which looks at the forces experienced by an electron travelling through a fixed magnetic field. Here there is a transverse force on the electron normal to the direction of travel, and the electron's path is deflected into a curve, with no expenditure of energy. However, the existence of the force still requires an energy mechanism and this paper sets out to identify it. There is no gradient in the electric field induced by the electron's motion through the magnetic field, so the energy is the same at all positions of the electron in the magnetic field, and hence there is no potential energy well to be tapped to create forces. There are enough clues to reach a sound conclusion, such as the fact that a neutron, with a bounded electric field, is not deflected, whereas an electron, with an infinite electric field, is deflected. With an energy mechanism clearly defined, we find that the Lorentz Force Equation fails to take an important aspect of geometry into account.

Keywords- *Lorentz Force, Electromagnetism, Field Theory*

I. Introduction (Heading 1)

In this paper we consider the second component of the Lorentz Force Equation, which describes the forces on a charge moving through a magnetic field. It affects how electrons move in magnetic fields and becomes important in more complex atoms. It also affects how we measure the strength of magnetic fields. It states that **F**=q(**v** x **B**) - force is equal to charge times the cross product of the magnetic field strength and the charge velocity through the field. How is this equation derived? - is it precise, or are there conditions under which it fails to describe the true situation? Although it states that the electron experiences forces when moving through a magnetic field, the electric field so induced has zero gradient, so generates no forces through its field gradient, as happens in the interaction between two electrons; the center of the electron will perceive no energy differentials wherever it is placed in the magnet so that cannot be the direct cause of the forces involved. We analyze the problem from first principles by using the integrated potential energy density, after working out the potential energy density at a point. We place ourselves in the rest frame of the moving electron, and use the electric field induced in that frame by the motion through the magnetic field.

We have two conundrums to solve.

A. The neutron is not deflected inside a magnetic field

First, we know that the neutron has an electric field even though it is bounded at a tiny radius, yet it is not deflected inside a magnetic field, so the magnetic component of the Lorentz Force Equation seems to work with electrons, but not with neutrons. Our theory needs to show why this anomalous behavior happens.

B. There is an apparent violation of the Conservation of Energy

Second, the Lorentz Force Equation suggests that if we replace the magnetic field with an electric field (in order that the electric field always pointed in the same direction rather than remaining normal to **B** x **v** and thus causing the electron to follow a curved path) an infinite number of electrons could traverse the field, increasing their kinetic energy inside the field, without any expenditure of energy anywhere in the system; when they left the field with their augmented kinetic energy they would have gained something for nothing; this violates the Principle of Conservation of Energy. Again, our theory must demonstrate there is no such violation. Whilst this is a thought experiment rather than a practical one it nevertheless highlights an important issue.

II. Identifying the Energy System

Energy systems drive every force in the universe. We know that the energy system involved here is not the transverse electric

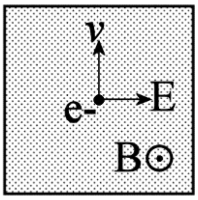

Figure 1. Electron moving through magnetic field in plan view.

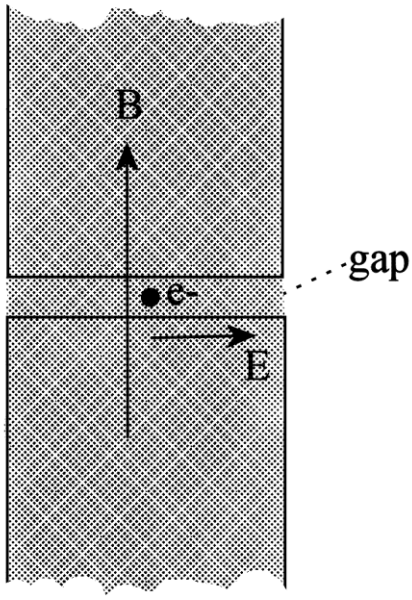

Figure 2. Side view of electron in magnet

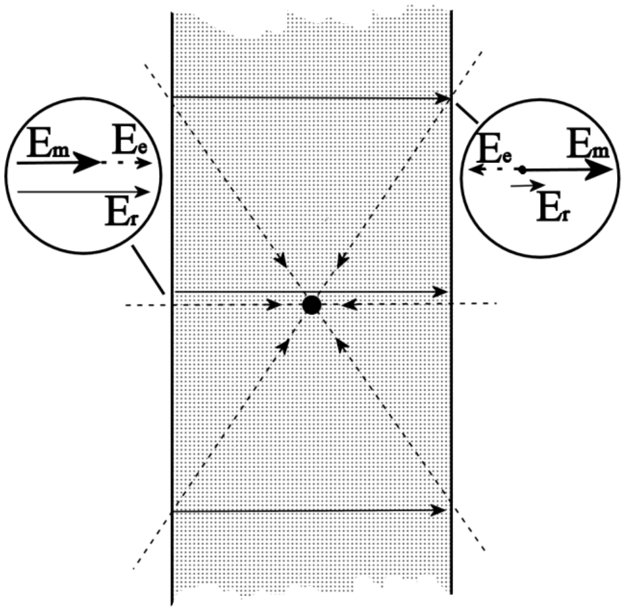

Figure 3. Electric field interaction at edges s of magnet

field curving the path of the electron as the induced electric field has no gradient. It is the gradient between two electric charges that gives rise to the forces between them. Since the gradient of the induced electric field is perfectly flat it cannot be the source of the deflecting force.

Let us start with a simple square magnetic field with the electron travelling in the field, parallel to one of the sides, as shown in Figure 1 in plan view. This shows a magnetic field ***B*** depicted in half-tone with the magnetic field vector pointing up out of the page. A test electron 'e-' is travelling through this field with a velocity v, and perceives an electric field **E** induced in its own frame of reference. The electron will perceive forces that direct it towards the left in this picture. These forces must be derived from a reduction in energy somewhere in the system that creates attractive forces to the left via the equation **F**=d**E**/dl where force **F** equals rate of change of Energy **E** over distance l, and/or repulsive forces from the right that will have the same effect. The fact that energy is not consumed but the electron merely deflected into a curved path by these forces does not change this argument. So where is the energy system that drives this deflection?

Although Figure 1 refers to a simple slice through the magnetic field, it is important not to make the mistake of analyzing only one slice. Looking at the magnet side-on in Figure 2 shows how the field extends in three dimensions. As before, the magnetic field is shown in halftone and its vector by **B**, and the induced electric field by **E**. The electron is shown travelling into the page inside the gap between the two poles of the magnet. As can be seen, the magnetic field, and therefore the induced electric field, continues through the poles of the magnet. The moving electron interacts with the whole field, including that inside the magnet poles. This is because the electron's electric field penetrates even into the atoms of the poles. Hence the computation must be over all interacting space, not just the gap between the poles.

In Figure 3 we show the interaction between the electron's field and the induced electric field. Two separate points are highlighted in inset pictures that show the electric vectors just within the edges of the magnet. The electron's field is shown in by dashed vectors $\mathbf{E}_e$ and the induced field is shown by solid vectors $\mathbf{E}_m$ describing the electric field induced by the magnetic field. In the right-hand inset the vertical component of the electron's field vector (a downward pointing vector) is not shown for clarity; it has no effect on the path of the electron, being balanced by an opposing vector below the electron. The cutaways show only the effect on the horizontal component and the resultant electric field vector $\mathbf{E}_r$ is shown in black.

The energy system that drives the deflection of the electron inside the magnet is in fact the interaction of the electron's fields with the edges of the magnet. If the electron moves a little to the left those parts of the fields of the electron that remain inside the magnetic field see no change in their energy density as the induced electric field is constant everywhere inside the magnet, so those parts of the field produce no force. Equally all those parts of the fields that remain outside the magnet see no change in their energy density and produce no force. However, those horizontal components of the electron's field that enter at the right edge of the magnet as a result of this movement will go from their normal field to a partial cancellation with the magnet's induced electric field and hence to a reduced energy density and a leftwards force. Those horizontal components of the electron's field leaving the magnet on the left will go from the increased energy density of interaction with the induced electric field to their normal independent energy density outside the magnet, leading to a drop in energy density and a leftwards force. Hence there is a drop in the energy densities at both edges leading to forces that tend to force the electron to the left.

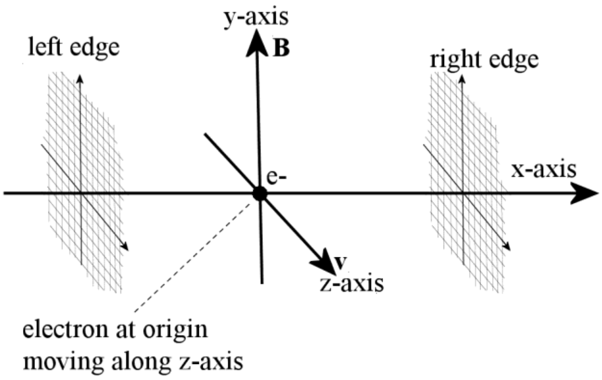

Figure 4. The interacting edges of the magnetic field

III. Comparison with the Lorentz Force Equation

How does this compare with the Lorentz Force Equation? Let us use the following co-ordinate system for Figure 4. The x-axis is parallel with the lines of the induced electric field, that is, left to right in Figures 2 and 3. The y-axis is parallel with the lines of the magnetic flux, that is, vertically in Figure 2. The z-axis is parallel with the electron's velocity vector, that is, vertically in Figure 3. In Figure 4 the z-axis points out of the page.

The whole of the induced electric field from the magnet is constant (there being no gradient) and normal to the direction of motion. There is no component of the induced field in any other axis and the potential energy lies along only the x-axis component field of the electron. Hence all forces will be along the x-axis.

Next, if we consider that the induced electric field is constant within the magnet and zero elsewhere, then for any point on the electron's field the potential energy density will be constant inside the induced electric field and zero outside. There are changes in potential energy density only on the boundaries of the induced field. We need therefore consider only the potential energy associated with a plane of infinitesimal thickness at the induced-field boundary plane normal to the x-axis as shown in Figure 4.

The magnetic field B lies vertically along the y-axis and the motion v of the electron is instantaneously along the z-axis.

This potential energy density integrated over this boundary plane, being of infinitesimal thickness, is effectively the derivative dU/dx = **F**, where dU/dx is the derivative of the potential energy with respect to motion of the electron along the x-axis, giving force **F** directly and causing the electron to accelerate along the x-axis, thereby curving its path. There are two such planes, one on positive-x and one on negative-x with respect to the center of the magnet.

Using Figure 4 we can derive the equation for the energy density at a point [x,y,z] on one of these planes as the x-component of the electric field strength from the electron at the boundary, times the induced electric field strength from the magnet.

$$\frac{dU}{dx\,dy\,dz} = \frac{\varepsilon q}{4\pi\varepsilon(x^2+y^2+z^2)}\frac{x}{\sqrt{x^2+y^2+z^2}}(\boldsymbol{v}x\boldsymbol{B})$$

$$= \frac{\varepsilon q(\boldsymbol{v}x\boldsymbol{B})}{4\pi\varepsilon}\frac{x}{(x^2+\ y^2+\ z^2)^{3/2}}$$

Then to get the energy density of the sheet we have...

$$\frac{dU}{d\boldsymbol{x}} = \frac{\varepsilon q(\boldsymbol{v}x\boldsymbol{B})}{4\pi\varepsilon}\iint_{-\infty}^{\infty}\frac{x}{(x^2+\ y^2+\ z^2)^{\frac{3}{2}}}\,dy\,dz$$

We can simplify this integration by recognizing that the term $(y^2+z^2) = r^2$ is constant for a circle around the x-axis, where 'r' is the radius of the circle. Substitute semi-polar co-ordinates and multiply the function by 2πr, then integrate over radius r from r equals zero to r equals infinity.

$$\frac{dU}{d\boldsymbol{x}} = \frac{\varepsilon q(\boldsymbol{v}x\boldsymbol{B})}{4\pi\varepsilon}\int_0^{\infty}\frac{2\pi rx}{(x^2+\ r^2)^{3/2}}\,dr$$

$$= \frac{q(\boldsymbol{v}x\boldsymbol{B})}{2}\int_0^{\infty}\frac{rx}{(x^2+\ r^2)^{3/2}}\,dr$$

$$= -\frac{q(\boldsymbol{v}x\boldsymbol{B})x}{2\sqrt{(x^2+\ r^2)}}$$

Evaluate from r=0 to r=infinity.

$$\frac{dU}{d\boldsymbol{x}} = \boldsymbol{F} = \frac{q(\boldsymbol{v}x\boldsymbol{B})}{2}$$

The same result holds for the edge plane at the opposite side of the magnet. The force in both cases is in the same direction as there is repulsion on the electron from one edge and attraction from the other, so the total force is twice the above, namely $\mathbf{F}_{total}$ = q(**v** x **B**). This is simply the Lorentz result, and so it can be seen that the Lorentz force for an induced electric field is caused by the interaction of the electron's field with the edges of the induced field.

IV. An analysis of the discrepancies

However, the integration above is between plus and minus infinity – the left and right edges of the induce electric field parallel to the motion of the electron are assumed to be an infinite plane; anything less and we do not get the Lorentz result. It can therefore be seen that the result agrees with Lorentz only where the areas of the left and right sides of the magnet are infinite in extent and at a finite normal distance from the center of the electron, and the particle's electric field extends across the edges of the magnetic field (as it does for the electron, but not the neutron). Cutting the sides down in size to some finite area will reduce the integration sum and so one can expect significant discrepancies for short magnets between the actual magnetic field strength and that strength as measured by the deflecting force on a charged particle. Any measurement of a practical magnetic field made by looking at the deflective force on a moving charged particle will therefore be artificially low in value, although it will obviously be in full agreement with any experiment involving the deflection of moving charged particles.

The term in 'x' disappears for the infinite-area integration. It does not matter what finite width the magnet is in 'x' if the length of the sides is infinite. For any specific solid angle projected from the electron onto the edge of the magnetic field, there will be a constant potential energy density across that the projection surface. Hence 'x' disappears from the equation when measuring potential energy in terms of solid angles (not the way derived here). Since the solid angle projected from a finite distance onto an infinite surface must always be a hemisphere, it follows that the length of the normal projection to that surface, passing through the center of the electron, is immaterial when the edges are infinite in extent.

Therefore, if we make both the magnet's y-axis and z-axis very much larger than its x-axis we find that the resultant computed force on the electron approaches the value given by the Lorentz Force Equation. Conversely, as the y-axis and/or the z-axis length of the magnet reduce in size, the resultant force drops further below the Lorentz value. It should be clear that in such circumstances the deflection of an electron by the field is reduced. Since many measurements of magnetic field strength are made by measuring the effect of the field on the motion of an electron they often report a lower magnetic field than they should; however this is not usually a problem as the field is then generally used to provide a deflective force to electron motion, whether in an electric motor or a particle accelerator.

The treatment here has been simplified. The electron in Figure 4 interacts with all of the magnetic field, not just that inside the magnet. In actuality the field diverges beyond the end of the magnetic poles and then loops around to meet up with the flux lines from the opposite pole in a return loop outside the magnet. The initial diverging flux lines near a pole acts to extend the effective length of the magnet, but we can ignore the region where they have looped back to meet the flux from the opposite pole as every electric flux line from the electron intersects both the inside and outside edges of the return flux and the effects at the two edges therefore cancel out. The return flux path edges are often not well defined in magnets without a yoke where the path is through air rather than soft iron, as the field intensity in free space generally drops slowly as we move away from the magnet but this does not affect the cancellation. These calculations are best done using Finite Element Analysis because of the complexity of applying limits near the poles.

V. THE SOLUTIONS TO OUR CONUNDRUMS

The solutions to our two conundrums are then:-

A. *The neutron is not deflected inside a magnetic field*

A neutron will not be subject to any forces inside the magnetic field despite having a strong electric field, because its electric field is wholly contained within the magnet where the induced electric field gradient is flat, and it has no interaction with the edges of the magnetic field. There is therefore no change in potential energy density as it moves through the magnetic field, and hence no forces exist.

B. *There is an apparent violation of the Conservation of Energy*

What if we replace the magnetic field with an electric field so that the electric field always pointed in the same direction rather than remaining normal to **B** x **v** and thus following a curved path? A theoretical electron falling from infinity is indeed accelerated across the field. But it is decelerated before entering the field by the interaction of the leading edge of the field with the electron's field, its effect being reversed by the center of the electron being on the other side of the boundary, and likewise decelerated after exit from the field, the sum of the two decelerations matching the single acceleration. The overall effect is of no net change in the kinetic energy. The Principle of Conservation of Energy stands.

VI. TESTING THE CONCEPT

This concept makes predictions at odds with the Lorentz Force Equation which can be tested. For example, consider an electron **e-** circling inside a magnet, where the plan view of the magnetic field gap is rectangular, as shown in Figure 5 (i.e. the magnetic field lines lie vertically up out of the page). Here the Lorentz Force Equation predicts a perfect circle. However, our analysis predicts that the forces on the electron are greater when the electron is travelling parallel to the longer sides, causing the electron to follow an elliptical rather than a circular path. The major axis of the ellipse would therefore be parallel to the shorter sides.

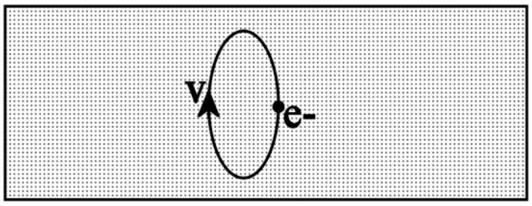

Figure 5. Electron circulating in magnet gap in plan view

Another prediction is that a short wide magnet and a long narrow magnet that have the same measurement of magnetic field strength when measured by electron deflection will have different measurements of magnetic field strength when measured by Paramagnetic resonance.

VII. CONCLUSION

The magnetic-deflection component of the Lorentz Force Equation fails to take into account the geometry of the magnetic field. Where the magnetic field strength is measured by the deflection of an electron inside the field in equipment such as Hall-effect devices and applied involving the deflection of electrons in the field, there is no conflict as identical errors appear in both cases. However, discrepancies may appear when a magnetic field is calibrated by electron deflection and then used to measure a magnetic dipole, as in paramagnetic resonance. For most geometries electron deflection measurements will underestimate the magnetic field.