A Simple Proof that $\zeta(2)$ is Irrational

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January 14, 2018

Abstract

We prove that a partial sum of $\zeta(2) - 1 = z_2$ is not given by any single decimal in a number base given by a denominator of its terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational values. The limit of the partials is z_2 and the limit of the exclusions leaves only irrational numbers. This is a set theoretical proof. We also give a topological proof using nested intervals and Cantor's intersection theorem.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [3]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. Here we give a simpler proof that uses just basic number theory.

We use the following notation: for $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n}.$$

2 Decimal intervals

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

The first lemma is a little more difficult than an exercise in Apostol's Introduction to Analytic Number Theory [1, p. 23, problem 30], its inspiration. We prove the general case.

Lemma 1. *The reduced fraction, r/s giving*

$$s_k^m = \sum_{j=2}^{k} \frac{1}{j^m} = \frac{r}{s} \tag{1}$$

is such that 2^m divides s.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^m, 3^m, \ldots, k^m\}$ will have a greatest power of 2, ma. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^m$ will have a greatest power of 2 exponent of mb. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + (k!)^m/3^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{2}$$

The term $(k!)^m/2^{ma}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $mb - ma$ for 2. As all other terms but this term will have more than an exponent of 2^{mb-ma} in their prime factorization, we have the numerator of (2) has the form

$$2^{mb-ma}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^m/2^{ma}$. The denominator, meanwhile, has the factored form

$$2^{mb}C,$$

where $2 \nmid C$. This leaves 2^{ma} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If p is a prime such that $k > p > k/2$, then p^m divides s in* (1).

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^m}{(k!)^m}\sum_{j=2}^{k}\frac{1}{j^m}=\frac{(k!)^m/2^m+\cdots+(k!)^m/p^m+\cdots+(k!)^m/k^m}{(k!)^m}. \tag{3}$$

As $(k,p)=1$, only the term $(k!)^m/p^m$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^m/p^m$. As $p<k$, p^m divides $(k!)^m$, the denominator of r/s, as needed. □

Theorem 1. *If*

$$s_k^m=\frac{1}{2^m}+\frac{1}{3^m}+\cdots+\frac{1}{k^m}=\frac{r}{s}, \tag{4}$$

with r/s reduced, then $s>k^m$.

Proof. Bertrand's postulate states that for any $k\geq 2$, there exists a prime p such that $k<p<2k$ [4]. If k of (4) is even we are assured that there exists a prime p such that $k>p>k/2$. If k is odd $k-1$ is even and we are assured of the existence of prime p such that $k-1>p>(k-1)/2$. As $k-1$ is even, $p\neq k-1$ and $p>(k-1)/2$ assures us that $2p>k$, as $2p=k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have 2^mp^m divides the denominator of (4) and as $2^mp^m>k^m$, the proof is completed. □

So, for z_2, we have the following.

Definition 1.

$$D_{k^2}=\{0,1/k^2,\ldots,(k^2-1)/k^2\}=\{0,.1,\ldots,.(k^2-1)\}\ \textit{base}\ k^2$$

Corollary 1.

$$s_n^2\notin\bigcup_{k=2}^{n}D_{k^2}$$

Proof. This is an immediate consequence of Theorem 1. □

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1. The table shows that adding the numbers above each D_{k^2}, for all $k\geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example,

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper right of the partial.

$1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says. Note that every rational $a/b \in (0,1)$ is included in at least one D_{k^2}. For example, $ab/b^2 = a/b$, $a < b$ and so $a/b \in D_{b^2}$.

3 Set theoretical proof

In this and the next section

$$\Xi_n = \bigcup_{j=2}^{n} D_{j^2}.$$

and $S_n = s_n^2$.

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$. We will designate the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$.

Theorem 2. *z_2 is irrational.*

Proof. Theorem 1 implies the following

$$s_n^2 \in \mathbb{R} \setminus \Xi_n.$$

As

$$\lim_{n\to\infty} s_n^2 = z_2$$

and

$$\lim_{n\to\infty} \Xi_n = \mathbb{Q}(0,1),$$

$$z_n \in \mathbb{R} \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1).$$

That is z_2 is irrational. $\square$

4 Topological proof

In consideration of Table 1, all partials from some point on are in an interval that partitions $[0,1]$.

4.1 Lower bound

Lemma 3. *For every natural number k greater than 1, there exists a first natural number N_k such that*

$$s_n^2 \in ((x-1)/k^2, x/k^2) \tag{5}$$

for all $n > N_k$.

Proof. We know $0 < z_2 < 1$. For a given $k > 1$, we can partition the interval $[0,1]$:

$$\bigcup_{j=1}^{k} \left[\frac{j-1}{k^2}, \frac{j}{k^2}\right] = [0,1].$$

Also, as no partial equals an endpoint and s_n^2 is a strictly increasing, convergent sequence, there will be an endpoint that separates those intervals with a finite number of partials in it from the one with an infinite number, a tail of the series. The lemma is thus established. $\square$

4.2 Upper bounds

Lemma 4. *For S_n and $k < n$ there exists a minimum x/k^2 such that $S_n < x/k^2$.*

Proof. Using Theorem 1, $S_n \notin \Xi_n$ and the result follows. $\square$

Lemma 5. *For every k there exists an x/k^2 such that for all $n > \max\{N_k, k\}$ $[S_n, x/k^2]$ is an interval.*

Proof. This follows from Table 1 and Theorem 1. $\square$

4.3 z_2 is irrational

Theorem 3. *z_2 is irrational.*

Proof. The following is a nested sequence of intervals:

$$[S_2, x_4/4] \supset [S_3, x_9/9] \supset \cdots \supset [S_n, x_{n^2}/n^2] \supset \ldots,$$

where the right endpoints represent the best approximations in Ξ_n as given by Lemma 5.

The intersection of these intervals gives z_2 [2]. As all right endpoints are excluded, z_2 must be irrational. $\square$

5 Conclusion

This result for the irrationality of z_2 can be generalized; Theorem 1 gives a result for the general case; Corollary 1 and Table 1 and the subsequent lemmas can be easily modified for any $n > 2$.

References

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