Mathematics as an interconnected Whole

Mark Burgin

University of California, Los Angeles
Los Angeles, CA 90095, USA

Abstract: Different thinkers suggested varied images or descriptions of mathematics. Platonists believe that mathematical objects exist as Platonic Ideas and mathematicians only discover them. Nominalists think that mathematics is the contents of mathematical manuscripts, books, papers and lectures, with the increasingly growing net of theorems, definitions, proofs, constructions, conjectures. Pragmatists assume that mathematics exists in mentality of people and when mathematicians introduce new objects they invent and then build them. An interesting peculiarity of the situation is that all these opinions and some others are true but … incomplete. The goal of this work is to explain this peculiarity presenting a complete vision of mathematics as an interconnected Whole.

1. Structure of the World

To understand information processes in mathematics, we need to answer (at least, informally) what mathematics is, while to obtain a grounded answer to this question, it is necessary to consider the structure of the world as a whole and the role of mathematics in it.

The majority of people think that the world in which they live is confined to the physical universe and only material things really exist. However, from ancient times, the best thinkers envisioned existence of other realities although they, as a rule, assumed that people could not rise above, go beyond and overcome the physical reality. In contrast to these views, reality in which people live is more sophisticated and unconventional.

To have a complete picture of reality, we come to the conclusion that all forms and kinds of existence are determined by the world stratification and structuration (Burgin, 2012). Taking the structuration determined by the Existential Triad of the world (Burgin, 1997; 2010), which stems from the long-standing tradition in philosophy and is presented in Figure 1, we come to the three *existential forms – material/physical existence*, *mental existence* and *structural existence*.

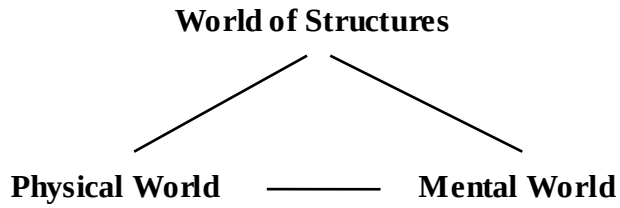

Figure 1. The Existential Triad of the World

In this stratification, the Physical (material) World is interpreted as the physical reality studied by natural sciences, the Mental World encompasses different levels of mentality, and the World of Structures consists of various forms and types of pure and abstract structures.

The Physical (material) World is the most familiar, accustomed and understandable of the three Worlds in the Existential Triad because people comprehend material things by their senses. Physicists study this World trying to reduce everything to physical reality (cf., for example, (Born, 1953)). However, many people encountered in their life illusions and misapprehension of physical things. This situation brought many thinkers to the conclusion about illusory character of the Physical (material) World or at least, about its imperfection and incomprehensibility. Such views are expressed in Indian Vedas (cf., for example, (Balasubramanian, 2000)) and teaching of Buddhism, as well as in works of Western philosophers such as Plato (427-347 B.C.E.), John Locke (1634–1704), George Berkeley (1685-1753) and David Hume (1711-1776).

The Mental World is even more complicated and evading than the Physical World. Usually people comprehend the Mental World as individual mentality. Science extended this picture exploring the Mental World on three levels, all of which are included in the Mental World of the Existential Triad:

- The first level treats mentality of separate individuals and is the subject of psychological studies. As in the case of physical reality, now psychology knows a lot about mentality/psyche of people. It is necessary also to remark that in the same way as physics does not study the physical reality as a whole but explores different parts, levels and aspects of it, psychology also separates and investigates different parts and aspects of the individual mental reality, such as intelligence, emotions, conscience or unconscious. However, there are components of individual mentality that yet lay beyond the studies of contemporary psychology.
- The second level deals with *group mentality* of various groups of people and is the subject of social psychology, which bridges sociology and conventional psychology. In

particular, this level includes group conscience, which incorporates collective memory (Durkheim, 1984), collective intelligence (Brown and Lauder, 2000; Suroweicki, 2004; Weiss, 2005; Nguen, 2008a) and is projected on the collective unconscious in the sense Jung (cf. (Jung, 1969)) by the process of internalization.

- The third level encompasses mental issues of society as a whole. Social mentality includes social memory, social intelligence and social conscience. Social psychology also studies some features of this level.

However, these three levels do not exhaust the whole Mental World. In fact, the Mental World from the Existential Triad comprises higher (than the third) levels of mentality although they are not yet studied by science (Burgin, 1997; 2010). It is possible to relate higher levels of the Mental World to the spiritual mystical worlds described in many religious esoteric teachings.

Some thinkers, following Descartes, consider the individual mental world as independent of the physical world. Others assume that individual mentality is completely generated by physical systems of the organism, such as the nervous system and brain as its part. However, in any case, the mental world is different from the physical world and constitutes an important part of our reality.

Psychological experiments and theoretical considerations show that the Mental World is stratified into three spheres: *cognitive* or *intellectual sphere*, *affective* or *emotional sphere* and *effective* or *regulative sphere*. This stratification is based on the extended theory of triune brain and the concept of the *triadic mental information* (Burgin, 2010).

The Mental World has elements and components, which are similar to elements and components of the Physical World. In a natural way, the Mental World has its mental spaces, mental objects (structures) and mental representations (Burgin, 1998a).

It is also necessary to explain that the World of Structures directly corresponds to Plato's World of Ideas/Forms because ideas or forms might be associated with structures. Indeed, on the level of ideas, it is possible to link ideas or forms to structures in the same way as atoms of modern physics may be related to atoms of Democritus and Leucippus. Only recently, modern science came to a new understanding of Plato ideas, representing the global world structure as the *Existential Triad* of the world, in which the World of Structures is much more comprehensible, exact and explored in comparison with the World of Ideas/Forms. When Plato and other adherents of the World of Ideas/Forms were asked what idea or form was, they did not have a satisfactory answer. In contrast to this, many researchers have been analyzing and developing the concept of a structure (Ore, 1935; 1936; Lautman, 1938; Bourbaki, 1948; 1950; 1957; 1960; Bucur and Deleanu, 1968; Corry, 1996; Burgin, 1997; 2010; 2011; 2012; Landry,

1999; 2006). It is possible to find the most thorough analysis and the most advanced concept of a structure in (Burgin, 2012). As a result, in contrast to Plato, mathematics and science has been able to elaborate a sufficiently exact definition of a structure and to prove existence of the world of structures, demonstrating by means of observations and experiments, that this world constitutes the structural level of the world as the whole. Informally, a structure is a collection of symbolic (abstract) objects and relations between these objects. Each system, phenomenon or process in nature, technology or society has some structure. These structures exist like material things, such as tables, chairs, or buildings do, and form the *structural level* of the world. When it is necessary to learn or to create a system or to start a process, it is done, as a rule, by means of knowledge of the corresponding structure. Structures mold things in their being and comprehension.

If we declare that structures exist embodied in material things, then we have to admit that material things exist only in a structured form, i.e., matter (physical entities) cannot exist independently of structures. For instance, atomic structure influences how the atoms are bonded together, which in turn helps one to categorize materials as metals, ceramics, and polymers and permits us to draw some general conclusions concerning the mechanical properties and physical behavior of these three classes of materials. Even chaos has its structure and not a unique one.

The three worlds from the Existential Triad are complementary to one another shaping the totality of the world. Besides, they are not separate realities - they interact and intersect. Individual mentality is based on the brain, which is a material thing, while in the opinion of many physicists mentality influences physical world (see, for example, (Herbert, 1987)). At the same time, our knowledge of the physical world essentially depends on interaction between mental and material worlds (see, for example, (von Bayer, 2004)).

Moreover, our mentality influences the physical world and can change it. We can see how ideas change our planet, create many new things and destroy existing ones. Even physicists, who research the very foundation of the physical world, developed the, so-called, observer-created reality interpretation of quantum phenomena. A prominent physicist, Wheeler, suggests that in such a way it is possible to change even the past. He stresses that elementary phenomena are unreal until observed (Wheeler and Zurek, 1983).

In addition, there is a projection of the Mental World into the Physical World in the form of creations of human mentality (creativity), such as books, movies, magazines, newspapers, cars, planes, computers and computer networks. This projection determines the Extended Mental World, which consists of the Mental World and its projection. The Extended Mental World correlates with the World 3 from the General Popper Triad of the world (Popper, 1979).

Structural and material worlds are even more intertwined. Actually, no material thing exists without a structure. Even chaos has its chaotic structure. Structures make things what they are. For instance, it is possible to make a table from different material: wood, plastics, iron, aluminum, etc. What all these tables have in common is not their material; it is specific peculiarities of their structure. In a similar way, according to Poincaré, space is in reality amorphous, and it is only the things in space that give it a structure (form) (Poincaré, 1908). As some physicists argue, physics studies not physical systems as they are but structures of these systems, or physical structures. In some sciences, such as chemistry, and areas of practical activity, such as engineering, structures play the leading role. For instance, the spatial structure of atoms, chemical elements, and molecules determines many properties of these chemical systems.

While physical reality surrounds people in everyday life and mental actuality was clear to people for a long time, the World of Structures was always so far that it demanded a lot of mental efforts and intellectual courage to discover it. The discovery of the World of Structures has been full of adventures. It resembles in many ways the discovery of America.

To see this, let us take a journey through the history of philosophy, science, and mathematics, starting in ancient Greece. It will help us to explain how the World of Structures was discovered and what it is supposed to be.

The first who predicted existence this "continent" called the World of Structures were ancient Greek philosophers leaded by Plato in their intellectual vessels. The result of this intellectual journeys was one of the greatest ideas of humankind postulating existence of the world of Ideas (or Forms), which was introduced by Plato. According to Plato, there are two worlds: one of them is invisible eternal invariable unchanging actually existing world of Ideas or Forms, while the second one is visible ever changing world of material things and their motion (Plato, 1961). In spite of the attractive character of this idea, the majority of scientists and philosophers believe that the world of Ideas does not exist because nobody provided positive evidence in support of it. The crucial argument of physicists is that the main methods of verification in modern science are observations and experiments, and nobody has been able to find this world by means of observations and experiments. Nevertheless, some modern thinkers, including outstanding intellectuals such as philosopher Karl Popper, mathematicians Kurt Gödel, Alain Connes, David Mumford and Rene Thom, computer scientist Gregory Chaitin and physicist Roger Penrose, continued to believe in the world of Ideas giving different interpretations of this world but suggesting no ways for experimental validation of these interpretations. In contrast to this, by corresponding Plato Ideas to structures in the contemporary sense, we come to a new level of

understanding due to the following arguments. First, existence of structures is proved by observation and experiment (Burgin, 1997; 2011; 2017). Second, there is a theory, the general theory of structures, which allows one to study structures by methods of contemporary science (Burgin, 2012). Third, there is a mathematical apparatus that allows one to model structures and study their properties by theoretical and conceptual tools.

Another great philosopher Aristotle opposed and criticized the teaching of Plato about the world of ideas. However, in his philosophical system Aristotle introduced and used the concept of a form. He explained that forms are strictly connected to things and the form of a thing is just what makes this thing as it is. In a similar way, structures determine the essence of material things.

Although now we can relate both concepts of forms and ideas to structures, explanations of ancient philosophers and their followers was too vague and they did not bring experimental and observational evidence to support their claim that ideas or forms really exist. Consequently, scientific community as a whole did not believe in the existence of this huge "continent" of structures.

Much later "Vikings" from different regions discovered various "islands" of this "structural continent". Chemists found out and studied structures of chemical elements. Some of these discoveries were crucial for the corresponding branch of science. An example is the discovery of the structure of DNA, which was crucial for genetics. Physicists discovered their island of physical structures. Some mathematicians announced mathematics as a discipline that studies mathematical structures (Bourbaki, 1948; 1957; 1960).

Those, who, like Columbus with America, were very close to the discovery of the whole "continent of structures", were structuralists coming from different walks of theoretical research. *Structuralism* is defined as a theory that uses culturally interconnected signs to reconstruct systems of relationships rather than studying isolated, material things in themselves.

Structuralism appeared in academia for the first time in the 19th century in the works of psychologists Wilhelm Maximilian Wundt (1832-1920) and Edward Bradford Titchener (1867-1927), who conjectured and tried to confirm experimentally that consciousness can be broken down into fundamental elements, relations between which give birth to complex mental structures. In the 20th century, structuralism reappeared as linguistic scientism in attempts to determine the general laws ("structures") by which language gains meaning as formulated by Ferdinand de Saussure (1857–1913). Later anthropologist Claude Lévi-Strauss (1908-2009) used structuralism to study the kinship systems of different societies. No single element in such a system has meaning except as an integral part of a set of structural connections. These interconnections are said to be binary in nature and are viewed as

the permanent, organizational categories of experience. Works of de Saussure had a great influence on other linguists, such as Leonard Bloomfield (1887-1949) in America, Antoine Meillet (1866-1936), André Martinet (1908-1999) and Émile Benveniste (1902-1976) in France, Louis Hjelmslev (1899–1965) and Rasmus Viggo Brendal (1887-1942) in Denmark and Alf Sommerfelt (1892-1965) in Norway, who further developed his approach.

Structuralism has been also influential in literary criticism and history, as with the work of Roland Barthes and Michel Foucault.

However, being encapsulated into humanities and social science, structuralism assumed that it investigated only unknown parts of ordinary domains of linguistics, anthropology, sociology, psychology, etc. In addition, structuralist understanding of the concept of structure was also very vague. For instance, anthropologists assume that our primal motivations are unconsciously mythological governed by structural codes.

Those codes reflect still deeper structures and these (rather than purely historical conditions and certainly not human freedom) regulate our thinking process at its basic level. One could even say that the *content* of thinking is what regulates thinking, apart that is from the ordinary confusions of everyday life (a kind of "imperfect participation" in Structure, which recalls the relation to Platonic Forms). Since any element of a story is replaceable without changing the story's content, the significance (content) of any story, including our own about the physical world, must be that governing Structure - and not its surface meaning.

As a result, this search for the deep structure of the mind was severely criticized by such "poststructuralists" as Jacques Derrida, who abandoned the goal of reconstructing social reality scientifically in favor of "deconstructing" the illusions of metaphysics

Only investigation of the regularities of the physical world made it possible at first to suggest and then to demonstrate that there exists the structural level that is more basic than the physical level of nature. That is why, mathematics, which is dealing with this structural level, is so powerful in cognition of the most fundamental physical objects. Moreover, the structural level provides our understanding of the phenomenon of knowledge and its most essential aspect, knowledge of structures.

This gives an exact answer to the following question of Albert Einstein, who discussed the significance of mathematics in the history of scientific thought in his address to the Prussian Academy of Science in Berlin on Jan 27, 1921.

"At this point an enigma present itself which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?"

The same issue was later addressed by Eugene Wigner, who asked why mathematics being so abstract and remote from nature is so efficient in science, explicating regularities of nature (Wigner, 1960).

Thus, existence of a new level of nature was discovered and its existence was grounded by experimental methods of modern science. In addition, it was demonstrated that any aspect of reality has its structural level. This level ties together nature, a human being, and society as their common existential and cognitive basis.

2. **Structure of mathematics**

It is natural that the structure of the world is reflected in the *existential structure* of mathematics, which describes how mathematics exists and mathematics as whole is projected onto its components, each which belongs to one of the three Worlds described by the Existential Triad.

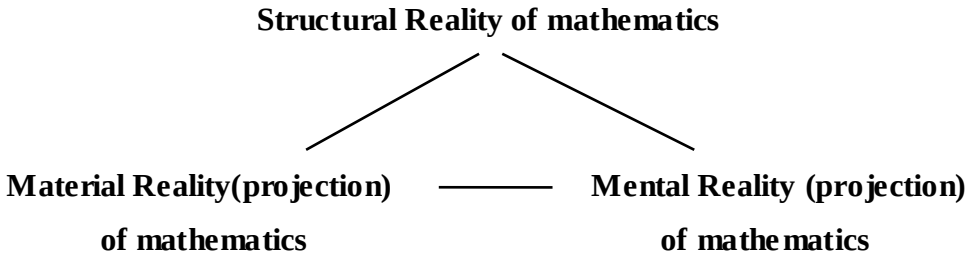

Figure 2. The existential structure of mathematics

Different approaches to the problem what mathematics is and how it functions reflect these realities (projections) of mathematics.

The Platonic approach, which is called Platonism in mathematics (cf., for example, (Linnebo, 2017)), assumes that mathematical entities exist, that they are abstract, and that they are independent of all people's rational activities, while mathematics underlines the structure of the universe the more people understand this vast interplay of mathematical objects, the more they can understand nature itself. Naturally, Platonism in mathematics reflects the structural reality of mathematics.

Such directions in the foundations of mathematics as logicism and intuitionism hold that mathematics is a purely mental construct being and extension of human reasoning and logic in the case of logicism or of human operation and practice in the case of intuitionism. As a result, logicism and intuitionism reflect the mental reality of mathematics.

Note that some researchers ascribe mathematics to individual mentality, while others place it in social mentality. For instance, Lakoff and Núñez write that mathematics arises out of human brains and bodies (Lakoff and Núñez, 2000; Núñez and Lakoff, 2005). Similar opinions are expressed by Dehaene (Dehaene, 2002). An extreme reductionist approach treats mathematics as fully explainable in terms of brain mechanisms and the neuroscience of individual cognitive processing.

In contrast to this, formalism argues that mathematics is the people's activity of manipulation with physical man-made symbols in the material world or as Manin writes, mathematics is a human activity deeply rooted in [physical] reality, and permanently returning to this reality (Manin, 2007). The second approach represents pragmatism in mathematics, which suggests that mathematics is a science of quantitative and special forms of the physical world coming from people's practical activity (Kolmogorov, 1988). Consequently, formalism and related views represents the physical reality of mathematics.

A different view on mathematics is suggested by Reuben Hersh, who thinks that:

"Mathematics is neither physical nor mental, it's social. It's part of culture, it's part of history, it's like law, like religion, like money, like all those very real things which are real only as part of collective human consciousness. Being part of society and culture, it's both internal and external. Internal to society and culture as a whole, external to the individual, who has to learn it from books and in school. That's what math is." (cf. (Brockman, 1997))

However, according to the Existential Triad, this places mathematics in the Mental World as part of collective human consciousness, which in turn is a part of human mentality. It means that Hersh simply shifts mathematics from individual mentality to social mentality. In contrast to this, the existential structure of mathematics presumes existence of mathematics on both levels.

Note that it is possible to reasonably address mentality of artificial systems, for example, mentality of a computer or network (Burgin, 2012). This allows distinguishing mathematics of a computer as mathematics employed in the hardware and software of this computer or mathematics of a programming language. An important part of computer mentality is computational intelligence (Szuba, 2001)

Interestingly, all these considerations show that all views on the existential form of mathematics were correct but only partially correct. This reminds the famous parable "Blind men and an elephant" (cf., for example, (Burgin, 2010)).

In addition to the existential structure, mathematics has the *incidence stratification*, which consists of the following strata:

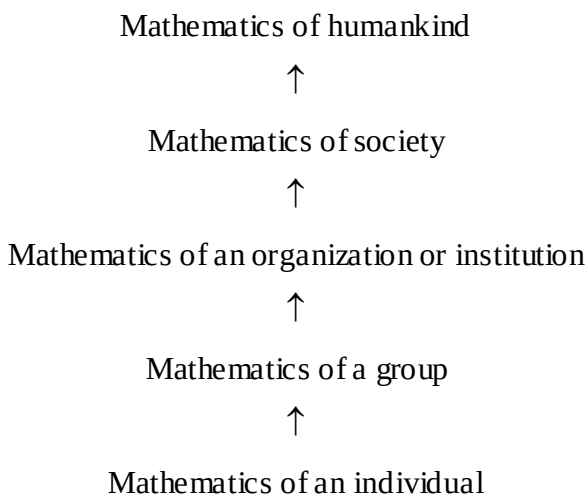

Figure 3. The incidence stratification of mathematics

Note that each of these strata also has the existential structure.

Mathematics as pure knowledge has structural embodiment in the World of Structures as its constituent (part).

Mathematics as pure knowledge has physical embodiment in mathematical books, journals, databases, computer files, etc.

Mathematics as pure knowledge has mental embodiment in heads of people.

Mathematics of society (a group or institution) has physical embodiment in different mathematical organizations such as the American mathematical society, departments of mathematics at colleges and universities, research mathematical institutes, etc.

Mathematics of society (a group or institution) has mental embodiment in social (organizational or group) mentality.

Mathematics of society (a group or institution) has structural embodiment in the World of Structures as its constituent (part).

Mathematics of an individual has mental embodiment in the mentality of this person.

Mathematics of an individual has structural embodiment in the World of Structures as its constituent (part).

Mathematics of an individual has physical embodiment in the brain of this person, her/his mathematical books, journals and writings.

Note that there are other structures of mathematics, such as the *field structure* of mathematics. Different authors and organizations construct this structure in distinctive ways. For instance, the American Mathematical Society elaborated the Mathematics Subject Classification (MSC), which gives a field structure of mathematics dividing it into over 60 fields, such as mathematical logic and foundations, combinatorics, general algebraic systems or measure and integration, with further subdivisions within each field.

The Dewey Decimal Classification structures mathematics into seven fields - algebra & number theory, arithmetic, topology, analysis, geometry, numerical analysis, and probabilities & applied mathematics.

An interesting peculiarity of the existential structure is its fractality, when parts have the same structure as the whole. Taking the existential structure of mathematics, we see that each its field, such as arithmetic, topology, analysis or geometry, has the same existential structure, in which, for example, arithmetic has the structural, material and mental components. Numbers belong to the structural component, numerals and digits belong to the material component and mental representations of numbers (Dehaene, 1997) belong to the mental component.

At the same time, material symbols, such as numerals and digits, have their structures, i.e., they are represented in the structural reality, and possess mental reflections being present, in such a way, in the mental reality.

It is possible to continue this structuration on and on demonstrating intrinsic fractality of the existential structure. Moreover, this fractality persists in language and other global phenomena of the human civilization.

The existential structure of mathematics persuasively demonstrates that in spite of distinctive images of mathematics suggested by different thinkers, mathematics is an interconnected Whole, while those images reflect only one side (aspect) of the magnificent mountain (building) of mathematics.

To conclude, we give additional explanation of why mathematics is an interconnected Whole although some noted researchers argued for a distributed multicomponet image of mathematics.

The prominent physicist Paul Benioff developed a theory of local mathematics arguing that there are different mathematics at different regions of the physical space-time manifold (Benioff, 2012; 2016). In a similar, but less formalized way, the distinguished mathematician Philip Davis

challenges the idea of a unified universal mathematics and argues that there are different mathematics at different geographical and regions (Davis, 2003). On the one hand, it is possible to interpret these phenomena stating that mathematics is created by people, who invented mathematical entities based on their culture and, may be, on other factors. Nevertheless, on the other hand, we can assume that there is a unified universal mathematical knowledge, which is an indispensible part of the World of Structures but people even living at one place and time can discover, represent and observe only different parts of the same "continent" or "galaxy" of the mathematical veracity. Even more, discovering the same component of this "galaxy", different people can see it, represent it and understand it differently based on their previous knowledge, cultural peculiarities and social biases. For instance, in ancient Babylonia, natural numbers were represented using the positional numerical system with the base 60. Maya Indians in America utilized the positional numerical system with the base 20, ancient Egyptians employed the positional numerical system with the base 10 to represent the same natural numbers, while Roman numerical system was not even positional. Moreover, notation for representing numbers was even more diverse. Many nations used letters of their alphabets while others invented special symbols for this purpose.

Although it looks that the collection of local mathematics or of cultural mathematics exist as separate systems, it is possible to unite all local mathematics and cultural mathematics in a complete whole using logical varieties, which provide powerful and efficient tools for knowledge unification and integration (Burgin, 1997a; 2004a; Burgin and de Vey Mestdagh, 2015). Existence of unified foundations of mathematics gives additional supportive evidence for unity of mathematics (Burgin, 2004).

References

1. Balasubramanian, R. (2000) *Introduction*, in *History of Science, Philosophy and Culture in Indian Civilization.* V. II Part 2: *Advaita Vedanta* (Chattopadhyana, Ed.), Centre for Studies in Civilizations, Delhi
2. Benioff, P. (2012) Local availability of mathematics and number scaling: effects on quantum physics, in *Quantum Information and Computation* X (Donkor, E.; Pirich, A.; Brandt, H., Eds.) Proceedings of SPIE, v. 8400; SPIE: Bellingham, WA, 84000T
3. Benioff, P. (2016) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities, *Quantum Information Processing*, v. 15, No. 3, pp. 1081-1102 (also arXiv:1508.01732)
4. Bourbaki, N. (1948) L'architecture des mathématiques, Legrands courants de la pensée mathématiques, *Cahiers Sud*, pp.35-47

5. Bourbaki, N. (1950) The architecture of mathematics, *American Mathematical Monthly*, v. 57, pp. 221-232

6. Bourbaki, N. *Elements of the History of Mathematics*, New York, Springer-Verlag, 1993

7. Bourbaki, N. *Structures*, Hermann, Paris, 1957

8. Bourbaki, N. *Theorie des Ensembles*, Hermann, Paris, 1960 (English translation: Bourbaki, N. *Theory of Sets*, Hermann, Paris, 1968)

9. Brockman, J. (1997) *What Kind of Thing is A Number*? A Talk with Reuben Hersh, *Edge* (https://www.edge.org/conversation/reuben_hersh-what-kind-of-thing-is-a-number)

10. Brown, P. and Lauder, H. (2000) Collective intelligence, in *Social Capital: Critical Perspectives*, Oxford University Press, New York

11. Bucur, I. and Deleanu, A. *Introduction to the theory of categories and functors*, John Wiley, London, 1968

12. Burgin, M. *Fundamental Structures of Knowledge and Information*, Ukrainian Academy of Information Sciences, Kiev, 1997 (in Russian)

13. Burgin, M. (1997a) *Logical Varieties and Covarieties*, in Methodological and Theoretical Problems of Mathematics and Information and Computer Sciences, Kiev, pp. 18-34 (in Russian)

14. Burgin, M. *On the Nature and Essence of Mathematics*, Ukrainian Academy of Information Sciences, Kiev, 1998 (in Russian)

15. Burgin, M. *Unified Foundations of Mathematics*, Preprint Mathematics LO/0403186, 2004, 39 p. (electronic edition: http://arXiv.org)

16. Burgin, M. *Logical Tools for Program Integration and Interoperability,* in Proceedings of the IASTED International Conference on Software Engineering and Applications, MIT, Cambridge, 2004a, pp. 743-748

17. Burgin, M. *Theory of Information*: *Fundamentality, Diversity and Unification*, World Scientific, New York/London/Singapore, 2010

18. Burgin, M. *Theory of Named Sets*, Nova Science Publishers, New York, 2011

19. Burgin, M. *Structural Reality*, Nova Science Publishers, New York, 2012

20. Burgin, M. *Theory of Knowledge*: *Structures and Processes*, World Scientific, New York/London/Singapore, 2016

21. Burgin, M. Ideas of Plato in the context of contemporary science and mathematics, *Athens Journal of Humanities and Arts*, July 2017, pp. 161 - 182

22. Burgin, M. *Mathematical Knowledge and the Role of an Observer*: *Ontological and epistemological aspects*, Preprint in Mathematics History and Overview (math.HO), 1709.06884, 2017a, 15 p. (electronic edition: http://arXiv.org)

23. Burgin, M. and de Vey Mestdagh, C.N.J. *Consistent structuring of inconsistent knowledge*, Journal of Intelligent Information Systems, v. 45, No. 1, 2015, pp. 5-28

24. Corry, L. (1996) Modern Algebra and the Rise Mathematical Structures, Birkhäuser, Basel/Boston/Berlin

25. Davis, P. J. (2003) Is Mathematics a Unified Whole, *SIAM News*, v. 36, No. 3, p. 6

26. Dehaene, S. *The number sense*: *How the mind creates mathematics*, Oxford University Press, New York, 1997

27. Dehaene, S. (2002) Single-neuron arithmetic, *Science*, v. 297, pp. 1652–1653

28. Durkheim, E. *The Division of Labor in Society*, The Free Press, New York, 1984

29. Herbert, N. *Quantum Reality*: *Beyond the New Physics*, Anchor Books, New York, 1987

30. Jung, C.G. (1969) *The Structure and Dynamics of the Psyche*, Princeton University Press, Princeton

31. Kolmogorov, A.N. (1988) Mathematics, *Mathematical Encyclopedic Dictionary*, Soviet Encyclopedia, Moscow, pp. 7-38

32. Lakoff, G. and Núñez, R. Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being, Basic Books, New York, 2000

33. Landry, E. (1999) Category Theory as a Framework for Mathematical Structuralism, *The 1998 Annual Proceedings of the Canadian Society for the History and Philosophy of Mathematics*, pp. 133-142

34. Landry, E. (2006) Category Theory as a Framework for an *in re* Interpretation of Mathematical Structuralism, in *The Age of Alternative Logics*: *Assessing Philosophy of Logic and Mathematics Today*, Springer Netherlands, pp. 163-179

35. Landry, E.M. (2007) Shared Structure need not be Set-Structure, *Synthese*, v. 58, pp. 1–17.

36. Lautman, A. *Essai sur les notions de structure et d' existence en mathématique*, Hermann, Paris, 1938

37. Linnebo, Ø. Platonism in the Philosophy of Mathematics, *The Stanford Encyclopedia of Philosophy* (Summer 2017 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/sum2017/entries/platonism-mathematics/>.

38. Manin, Y. *Mathematical knowledge: internal, social and cultural aspects*, preprint in Mathematics History and Overview, 2007 (arXiv:math/0703427 [math.HO])

39. Nguen, N. T. (2008) Inconsistency of knowledge and collective intelligence, *Cybernetics and Systems*, v. 39, No. 6, pp. 542-562

40. Núñez, R. and Lakoff, G. (2005) The cognitive foundations of mathematics: The role of conceptual metaphor, in *Handbook of Mathematical Cognition* (Campbell, J., Ed), Psychology Press, New York, pp. 109–124

41. Ore, O. (1935) On the foundation of abstract algebra, I, *Ann. of Math*., v. 36, pp. 406-437

42. Ore, O. (1936) On the foundation of abstract algebra, II, *Ann. of Math*., v. 37, pp. 265-292

43. Plato, *The Collected Dialogues of Plato*, Princeton University Press, Princeton, 1961

44. Poincaré, H. *Science et Méthode*, Flamarion, Paris, 1908

45. Popper, K. R. *Objective knowledge*: *An evolutionary approach*, Oxford University Press, New York, 1979

46. Suroweicki, J. *The Wisdom of Crowds: Why the Many Are Smarter Than the Few and How Collective Wisdom Shapes Business, Economies, Societies and Nations*, Little & Brown, Boston, 2004

47. Szuba, T. *Computational Collective Intelligence*, Wiley, NY, 2001

48. von Bayer, H.C. *Information*: *The New Language of Science*, Harvard University Press, Harvard, 2004

49. Weiss, A. (2005) The Power of Collective Intelligence, netWorker - Beyond file-sharing, *Collective Intelligence*, v. 9, No. 3, pp. 16-24

50. Wheeler, A. and Zurek, W.H. (Eds) *Quantum Theory and Measurement*, Princeton University Press, 1983

51. Wigner, E. (1960) The Unreasonable Effectiveness of Mathematics, *Communications in Pure and Applied Mathematics*, v. 13, No. 1, pp. 1-14