Turbulence, a controlled Route of Energy from Order into Chaos.

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Abstract.

One of the earliest attempts to understand the phenomenon of turbulence dated back to a theory of L. F. RICHARDSON which later became enhanced by A. N. KOLMOGOROV. By the contents of this theory turbulence is considered as an energy–transfer taking place in cascade–form between eddies of various orders of magnitude. The transfer becomes started by unspecified disturbance–signals acting on fibre–eddies and is further evaluated on account of the fibres stretching. This principally can explain, that eddies of various sizes will be created in generations while energy is distributed along these generations from the largest eddies down to the smallest. But it is impossible to determine the cascade–structure and its development more detailed.

In order to step forward this way, one exemplified will have to be provided the number of eddies for every cascade–generation, eddies sizes and their appropriate life–times, and a characteristic of the disturbance–signal as well. For this reason an eddy as fibre is replace by a more complicated model of an eddy as spinning sphere with surface–tension and with specified disturbance acting on it. Due this extended model the decay of an eddy can be explained as a partitioning into equal parts in a self–organizing balance between disturbance and a reaction forced by surface–tension. The eddy's life–time is then obtained as proportional to the square of the eddy's radius and the rotation–phase of the split–results as nearly doubled compared with the proper value before a split. The disturbance–signal is specified by the sum of resonance–terms each of them prepared for acting on an eddy of proper size. As final result of the extended eddy–model, the characterization of disturbance, the self–organizing balance of acting forces and derived quantities, turbulence can be declared similar to other dynamical systems as a controlled route of energy from order into chaos.

1. Introduction.

L. F. RICHARDSON [1] and A. N. KOLMOGOROV [2 , ..., 7] conceptually related dissipation with other macroscopic quantities of a turbulent flow. They started from the idea that a turbulent flow is fed with energy on large scales, which is transported by decay of eddies through an order of magnitudes to the smallest eddies where finally it is totally transformed into heat. This process is called energy–cascade and starts with the following proportionality:

1.1. $\epsilon \sim (u_\lambda)^3/\lambda \quad \leftarrow \quad \epsilon = \textit{Dissipation–rate},$
$u_\lambda = \textit{Tangential–speed of smallest eddy},$
$\lambda = \textit{KOLMOGOROV–length},$

For an appropriate REYNOLDS–number $/Re/$ it can be written:

1.2. $Re = (l/\lambda)^{4/3} \quad \leftarrow \quad l = \textit{Eddy–extend in a specific order of magnitude}$
$\lambda = (\nu^3/\epsilon)^{1/4} \quad \leftarrow \quad \nu = \textit{Viscosity of the turbulent medium}$

Thus the value of $/Re/$ measures the range of various length–scales within the turbulence. The KOLMOGOROV–length represents the extent of a smallest eddy in the turbulence.

According to L. F. RICHARDSON, turbulent flows show a hierarchy of eddies, where the larger ones are built in a preliminary creation–process of the turbulence. Afterward they decay successively in a sequence of instabilities down to a minimal magnitude $/\lambda/$ of eddies. Here they finally are disturbed and their energy is transformed into heat due to viscosity of the turbulent medium. During this hierarchical process, eddies submit most of their energy to their followers, only a small part each time is lost through dissipation. The hierarchy ends as soon as $/l/$ becomes comparable with $/\lambda/$ which results in $/Re \approx 1/$.

Tangential–speed and extend of an eddy from the hierarchy may be $/u/$ and $/l/$ respectively, then its energy is of the magnitude $/u^2/$ and its so–called eddy–turn–over–time $/\tau/$ is:

1.3. $\tau = l/u.$

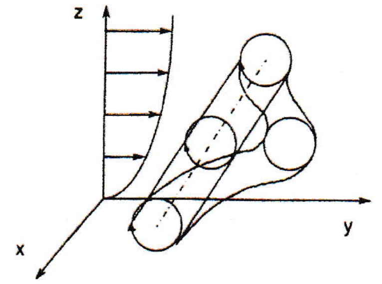
On base of these quantities the rate of energy–transport is given by:

1.4. $\Pi \sim u^2/\tau = u^3/l \sim \epsilon.$

Such an independence of transfer–rate $/\Pi/$ from viscosity $/\nu/$ can be explained – due to RICHARDSON – by the stretching of an eddy.

Turbulence in this sense starts with a picture about eddy–fibres in a shear–flow (see H. E. Fiedler [8]):

A.1.1.

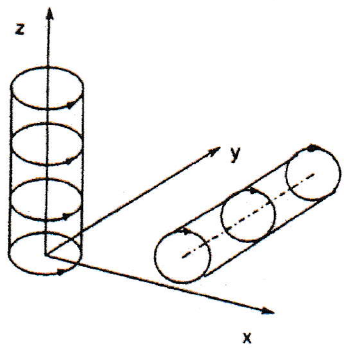


The smallest swelling–out of a fibre will stretch its length, strengthen its angular–speed $/\omega/$ and shrink its cross–section $/A/$ appropriately to HELMHOLTZ's law:

1.5. $\omega \cdot A = const.$

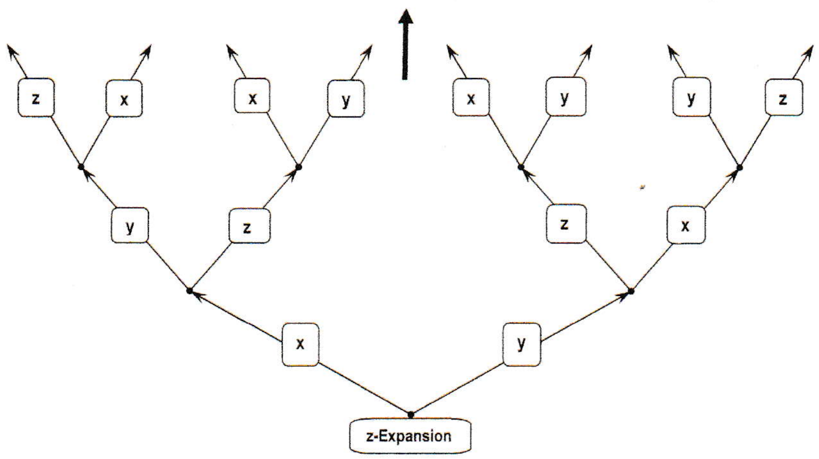
The stretching–mechanism by itself can be explained in a following way:

A.1.2.



Given a fibre in z–direction with a rotation in the (x,y)–plain. As soon as it becomes stretched in z–direction, its cross–section in (x,y)–plain and with it the appropriate rotation $/\nabla \times \underline{\omega}/$ and intensity $/\omega^2/$ will be enlarged. Such an increase of intensity – on its side – will cause further stretching of the fibre in the other space–directions. Thus stepping forward this way, the initial swelling–out of the fibre will finally have been resulted into an energy–cascade filling up the complete fluid–space. Such a procedure can be visualized qualitatively by the following graph:

A.1.3.



All this together will result in angular–speed $/\omega/$ and with it $/\omega^2/$ to such an extent that the energy–transfer–rate remains quasi constant. Therefore the energy–cascade is assumed to be quasi independent from the viscosity $/\nu/$ of the turbulent medium. For the energy–transfer–rate $/\epsilon/$ across the energy –cascade approximately the following proportionality can be obtained:

1.6. $\epsilon \sim \nu(u_\lambda/\lambda)^2$.

Compared to equation /1.4/ this will result in:

1.7. $\nu(u_\lambda/\lambda)^2 \sim u^3/l = u^2/\tau$.

During the extension of the energy–cascade each eddy partitions its energy $/{\sim}u^2/$ among the followers, the energy of the followers therefore must be less than that of their predecessor. The value of $/u^2/$ decreases permanently in a propagating energy–cascade and in a similar way $/l/$ does it too. Thus finally – in the case of smallest eddies – the product $/u{\cdot}l/$ will become comparable with $/\nu/$:

1.8. $Re = u_\lambda{\cdot}\lambda/\nu \approx 1$.

The values $/\lambda, u_\lambda, \tau_\lambda/$ of the smallest eddies in the turbulence are called KOLMOGOROV–scales, they can be summarized in the following way:

1.9. $\lambda \sim (\nu^3/\epsilon)^{1/4} \sim Re^{-3/4}{\cdot}l$

$u_\lambda \sim (\nu{\cdot}\epsilon)^{1/4} \sim Re^{-1/4}{\cdot}u_l$

$\tau_\lambda \sim (\nu/\epsilon)^{1/2} \sim Re^{-1/2}{\cdot}\tau_l$.

KOLMOGOROV completed the theory of the energy–cascade, which formally was initiated by RICHARDSON, with three additional hypotheses.
For eddies of $/\lambda < r \ll l/$ statistical isotropy can be assumed. In addition $/\tau_l/$ of large eddies will show in comparison with proper values of medium–eddies $/\tau_r \ll \tau_l/$, the letter will decay much faster. The smallest eddies are in a statistical equilibrium. Under these aspects he came to his hypothesis of the local isotropy:

H.1. *For large REYNOLDS–numbers turbulent motions on smallest scales are statistically isotropic and will expire in statistical equilibrium (universal equilibrium).*

By the next hypothesis KOLMOGOROV expressed his opinion, that:

H.2. *For large REYNOLDS–numbers and length–scales $/r \ll l/$ statistical quantities will only depend on three parameters – the length–scale $/r/$ itself, the energy–transfer–rate $/\Pi/$ and the viscosity $/\nu/$ of the turbulent medium.*

Eddies of length–scales $/l \gg r/$ – from the so–called Inertial–range – will remain nearly untouched by viscosity $/\nu/$. Those eddies will obtain their energy–influx nearly totally from their larger predecessors and will deliver it nearly completely to their smaller followers of universal equilibrium. Thus, for the statistics of these length–scales, energy–transfer is not decisive. In essence of this KOLMOGOROV formulated his final hypothesis:

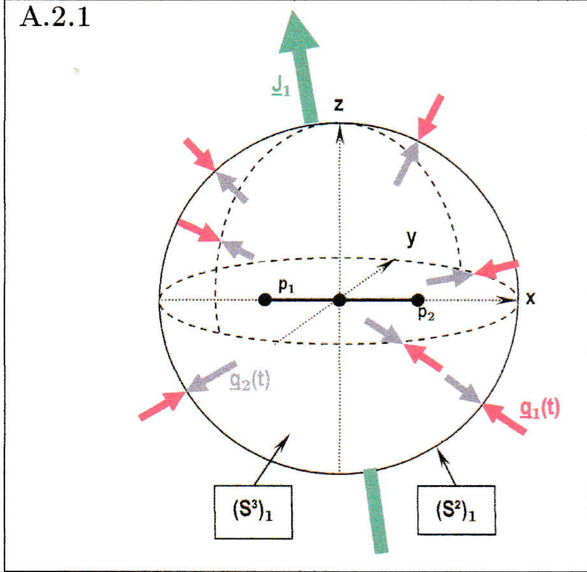
H.3. *For large REYNOLDS–numbers for scales $/l \gg r > \lambda/$ statistical quantities will have universal forms only depending on $/\epsilon/$ and $/r/$.*

2. Model of an eddy's decay due to a disturbance acting on it.

The theory of RICHARDSON enhanced by KOLMOGOROV apparently disclosed some deficiencies for instance with respect to the number and sizes of followers coming into existence as consequences of a predecessor's decay, the individual life–times of the various members in the cascade and last but least a characteristic of the disturbance–signal. This information however is needed in order to determine an appropriate cascade–structure, which then enables a statement about the roper development of the turbulence. The following discussions should be understood to appropriately enhance the former theory in this way.

Inside a fluid an eddy is now considered as sphere $/(S^3)_1/$ with a spin $/\underline{J}_1/$as united angular–momentum of its particle–set. The form of sphere is chosen because it possesses the smallest surface for an enclosed volume. A time–dependent force $/\underline{q}_1(t)/$ as disturbance may act on the eddy from outside trying to deform $/(S^3)_1/$ into another volume with increased surface. This will cause a reaction $/\underline{q}_2(t)/$ parallel to $/\underline{q}_1(t)/$ due to the surface –tension (consequence of the fluid–viscosity $/\nu/$).

A.2.1



The competing forces will influence each other and thus should be considered coupled together in a self–organizing system, appropriately in the following way:

2.1 $d\underline{q}_1/dt = -\gamma\cdot\underline{q}_1 - a\cdot\underline{q}_2\cdot\underline{q}_1 \;\rightarrow\; dq_1/dt = -\gamma\cdot q_1 - a\cdot q_2\cdot q_1 \;\leftarrow\; \gamma, \delta = damping\text{–}parameters$

2.2 $d\underline{q}_2/dt = -\delta\cdot\underline{q}_2 + b\cdot(\underline{q}_1)^2 \;\rightarrow\; dq_2/dt = -\delta\cdot q_2 + b\cdot(q_1)^2 \qquad a, b = coupling\text{–}parameters.$

With respect to $/\underline{q}_1(t)/$ and $/\underline{q}_2(t)/$ one should make use of the so–called adiabatic approximation (see H. HAKEN [10]):

2.3 $\delta \gg \gamma \;\rightarrow\; dq_2/dt \approx 0,$

Due to relation $/2.2/$ this will result in:

2.4 $q_2 = \delta^{-1}\cdot b\cdot(q_1)^2.$

Equation $/2.4/$ can be interpreted as: $/q_2/$ must follow $/q_1/$ immediately, $/q_2/$ has become enslaved by $/q_1/$ (H. HAKEN [10]). On the other hand $/q_2/$ will react on $/q_1/$ back again via equation $/2.1/$ with the consequence:

2.5 $dq_1/dt = -\gamma\cdot q_1 - \delta^{-1}\cdot a\cdot b\cdot(q_1)^3.$

By equation $/2.5/$ force $/q_1/$ is expressed by the dynamics of a so–called unharmonic oscillator – which depending on the conditions –:

2.6 $[(\gamma > 0)] \;\bigvee\; [(\gamma < 0) \bigwedge (\delta^{-1}\cdot a\cdot b) > 0]$

possesses two qualitatively distinct stability– modes:

2.7 $[p_0 = 0] \;\bigvee\; [p_{1/2} = \pm\langle|\gamma|\cdot \delta\, /(a\cdot b)\rangle^{1/2}].$

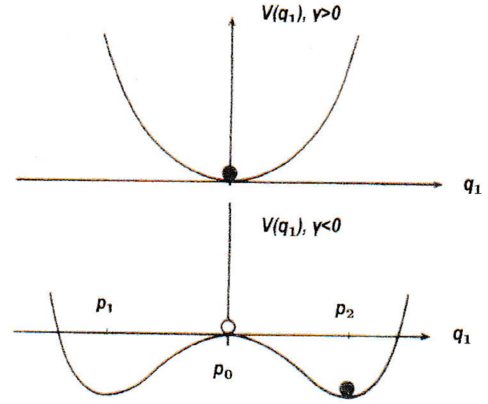
In the first case stable oscillations are performs with respect to the fix–point $/p_0/$, in the second case $/p_0/$ becomes instable and bifurcates symmetrically into new stable–points $/p_{1\wedge 2}\,/$, each becoming the centre for subsequent oscillations. Therefore by the bifurcation of $/p_0/$ is expressed, the sphere $/(S^3)_1/$ will be partitioned into smaller follower–spheres $/(S^3)_2 \bigwedge (S^3)_3/$, a process which can be explained as follows.

For $/q_1(t)/$ from outside $/(S^2)_1/$ – instead of $/\gamma = constant/$ – a damping of $/\gamma(t)/$ should be expected , presumably of the following kind:

2.8 $[\gamma(t_1) > 0] \quad \rightarrow \quad [\gamma(t_2) = 0] \quad \rightarrow \quad [\gamma(t_3) < 0].$

Potential–curves for $/V(q_1(t_1) \bigwedge V(q_1(t_3)/$ then qualitatively may be visualized in a following way:

A.2.2



Deforming the potential–curve on the way $/(\gamma < 0) \rightarrow (\gamma > 0)/$ will flatten the neighbourhood of $/p_0/$ steadily, stability accordingly will be reached more slowly and finally will be exchanged by instability at $/p_0/$. During this process, which takes a time $/\Delta t/$, force $/q_1(t)/$ will deform $/(S^2)_1/$ to a shape of surface with higher energy. As soon as the surface–energy has become high enough to build two times the surface of $/½\cdot(S^3)_1/$, this will occur at:

2.9

$(S^3)_1 = (4/3)\cdot\pi\cdot(r_1)^3$

$\rightarrow \quad r_{2\vee 3} = r_1/(2)^{1/3}$
$r_{2\vee 3} \approx 0.79\cdot r_1 \quad \rightarrow \quad (S^2)_{2\wedge 3} = 8\cdot\pi\cdot(r_{2\vee 3})^2 \approx 15.69\cdot(r_1)^2$

$\rightarrow \quad \Delta O \approx 3.12\cdot(r_1)^2$

$(S^2)_1 = 4\cdot\pi\cdot(r_1)^2 \approx 12.57\cdot(r_1)^2$

$(S^3)_{2\wedge 3} = (S^3)_1$

$(S^3)_{2\vee 3} = (4/3)\cdot\pi\cdot(r_{2\vee 3})^3$

Sphere $/(S^3)_1/$ will become partitioned symmetrically at $/p_1 \bigwedge p_2/$ into spheres $/(S^3)_2 \bigwedge (S^3)_3/$ each volume of $/½\cdot(S^3)_1/$.

After the split spin $/\underline{J}_1/$ will have been saved, thus for $/(S^3)_2 \bigwedge (S^3)_3/$ following condition must hold:

2.10 $\underline{J}_1 = \underline{J}_2 + \underline{J}_3 \quad \leftarrow \quad J_2 = J_3.$

Between the rotation–energies $/\epsilon_{2\vee 3}/$ of $/(S^3)_2 \bigvee (S^3)_3/$ and $/\epsilon_1/$ of $/(S^3)_1/$ the following relationship will exist:

2.11

$\epsilon_1 = ½\cdot\theta_1\cdot(\omega_1)^2 \quad \leftarrow \quad \theta_1 = \textit{momentum of inertia}$

$\rightarrow \quad \epsilon_1 = ½\cdot\kappa\cdot(r_1)^2\cdot(\omega_1)^2 \quad \leftarrow \quad \omega_1 = \textit{angular velocity}$

$\theta_1 = \kappa\cdot(r_1)^2 \quad \leftarrow \quad \kappa = constant$

	$\epsilon_{2\vee3} = \frac{1}{2}\cdot\theta_{2\vee3}\cdot(\omega_{2\vee3})^2$	$\leftarrow$	$\theta_{2\vee3} =$ *momentum of inertia*				
		$\rightarrow$	$\epsilon_{2\vee3} = \frac{1}{2}\cdot\kappa\cdot(r_{2\vee3})^2\cdot(\omega_{2\vee3})^2$	$\leftarrow$	$\omega_{2\vee3} =$ *angular velocity*		
				$\rightarrow$	$\epsilon_{2\vee3} \approx \kappa\cdot0.31\cdot(r_1)^2\cdot(\omega_{2\vee3})^2$		
						$\rightarrow$	$\kappa\cdot0.31\cdot(r_1)^2\cdot(\omega_{2\vee3})^2 \approx \kappa\cdot0.25\cdot(r_1)^2\cdot(\omega_1)^2$ $(\omega_{2\vee3})^2 \approx 0.8\cdot(\omega_1)^2$ $\omega_{2\vee3} \approx 0.9\cdot\omega_1$ $\omega_{2\wedge3} \approx 1.8\cdot\omega_1$
	$\theta_{2\vee3} = \kappa\cdot(r_{2\vee3})^2$		$(r_{2\vee3})^2 \approx 0.62\cdot(r_1)^2$ $\epsilon_{2\vee3} = \frac{1}{2}\cdot\epsilon_1$	$\rightarrow$	$\epsilon_{2\vee3} = \frac{1}{4}\cdot\kappa\cdot(r_1)^2\cdot(\omega_1)^2$		

This results in:

2.12 $J_{2\vee3} = \theta_{2\vee3}\cdot\omega_{2\vee3} \quad\rightarrow\quad J_{2\vee3} = \kappa\cdot(r_{2\vee3})^2\cdot\omega_{2\vee3} \quad\rightarrow\quad J_{2\vee3} \approx \kappa\cdot0.62\cdot(r_1)^2\cdot0.9\cdot\omega_1 \quad\rightarrow\quad J_{2\vee3} \approx 0.6\cdot J_1.$

Energy $/\epsilon_{\Delta O}/$ – which has to be transferred by $/q_1(t)/$ in order to enlarge $/(S^2)_1/$ up to an equivalent of $/(S^2)_{2\wedge3}/$ – is proportional to $/\Delta O/$ or $/(r_1)^2/$. Therefore a similar relationship must hold for the proper life–time $/\Gamma((S^3)_1)/$ (time of $(S^3)_1$–existence):

2.13 $\Gamma((S^3)_1) \sim (r_1)^2.$

3. <u>How a Disturbance–Signal acts on Splits in an Eddy's Decay–Process</u>.

How a disturbance–signal $/s(t)/$ acts on the various splits in an eddy's decay, may be understood in the following way.
The signal $/s(t)/$ might be decomposed as follows:

3.1 $s(t) = {}_{(j=1)}\sum^{\infty}[h_j(t)] = {}_{(j=1)}\sum^{\infty}[\langle\gamma_j(t)\cdot q_1(t)\rangle_j].$

As far as the functions $/h_j(t)/$ will obey the conditions:

3.2 ${}_0\int^{P}[h_j(t)]dt = 0 \quad\leftarrow\quad P = \Gamma\langle(S^3)_j\rangle$

FOURIER–series of the following kind are appropriate for $/h_j(t) = \gamma_j(t)\cdot\langle q_1(t)\rangle_j/$:

3.3 $h_j(t) = {}_{k=(-n)}\sum^{n}[\langle Z(f_{k+1}) - Z(f_k)\rangle\cdot exp\{2\pi i\cdot f'_k\cdot t\}].$

By the decomposition of an interval $/[-\Omega, \Omega]_j/$ on the frequency–axis:

3.4 $[-\Omega, \Omega]_j = [f_{-n} < f_{-n+1} < \ldots < f_n < f_{n+1}]_j \quad\rightarrow\quad f_k \leq f'_k \leq f_{k+1}$

a complex entity $/Z(f_{k+1}) - Z(f_k)/$ is associated with each interval $/[f_k, f_{k+1}]/$. This entity contains all information about amplitudes and phases from modes $/exp\{2\pi i\cdot f'_k\cdot t\}/$ in the frequency–range $/[f_k, f_{k+1}]/$. Via WIENER–CHINTSCHIN–Theorem /WIKIPEDIA/ all entities $/Z(f_{k+1}) - Z(f_k)/$ of the sum in equation /3.3/may be associated with auto–correlations in the splitting–process, which on their part will finally enable determinations of $/\gamma_j/$ and $/\langle q_1\rangle_j/$.

With this picture of $/s(t)/$ in mind, it can be understood how a follower $/(S^3)_j/$of $/(S^3)_1/$ in step $/j/$ of an eddy's decay will get the proper resonance–stimulation from $/s(t)/$ for its split.

4. The Route from Order into Chaos in an Eddy's Split–Cascade.

As soon as appropriate resonance–terms $/\gamma_j\cdot\langle q_1(t)\rangle_j/$ from disturbance–signal $/s(t)/$ will act on the followers $/(S^3)_{2\vee3}/$and the followers of the followers ..., equivalent conditions are provided for every split of the cascade. These splits will equivalently be performed as the one for $/(S^3)_1/$. Thus the life–time of a sphere in N_th generation of the cascade will be diminished by:

$$4.1 \quad \Gamma((S^3)_N) \sim (\langle 0.79\rangle^N\cdot r_1)^2 \quad \rightarrow \quad [\Gamma((S^3)_{N+1})/\,\Gamma((S^3)_N)] \approx (0.79)^2 \approx 0.62,$$

and the phase–speed will be nearly doubled:

4.2		
	$T_1 = 2\cdot\pi/\omega_1$	
		$\rightarrow \quad T_{2\vee3}\,/\,T_1 \approx 2.22 \quad \rightarrow \quad T_{N+1}/\,T_N \approx 2.22.$
	$T_{2\wedge3} = 4\cdot\pi/\omega_{2\wedge3}$ $T_{2\wedge3} \approx 4\cdot\pi/(0.9\cdot\omega_1)$	

All this together will draw a picture about an eddy's decay as controlled route from order into chaos similar to many other dynamical systems described for instance by O–H. PLEITGEN, H. JÜRGENS, D. SAUPE [11].

5. Conclusion.

Due to the theory of L. F. RICHARDSON – successively enhanced by A. N. KOLMOGOROV – turbulence is characterized as production of eddies in a hierarchical order. Only the largest of them are created during initialization of the process, but all will successively decay by series of instabilities into followers on decreasing orders of magnitude. This process transports energy along a cascade nearly without dissipation. Only eddies on lowest hierarchical level will distinctively be influenced by viscous dissipation and finally be destroyed by transformation of their energy into heat.

According to this theory the energy–cascade will be started by disturbances on eddies of highest level which cause them to stretch in all directions. The initial stretching will continually create series of followers on lower hierarchal levels with decreasing portions of energy. This is the concept of cascading so far, but this picture lacks on details about decays, like numbers and sizes of followers coming into existence and individual life–times in any specific case, and on characteristics of appropriate disturbance–signals as well. This information however is needed in order to determine an appropriate cascade–structure, which then enables a statement about the proper development of the turbulence.

In order to enhance the former process with respect to these deficiencies, the model of an eddy in a fluid will be changed. The existing picture of an eddy is replaced by a spinning sphere whose surface is exposed to a self–organizing balance between an outer disturbance–signal and a reaction–force of the sphere due to its surface–tension (on account to the fluid's viscosity). An adiabatic approximation on the forces damping–parameters makes the disturbance to become the leading–force of the system while the reaction–force is enslaved and must follow the disturbance immediately. Due to these facts the behaviour of the self–organizing system at variations of the disturbance could be best described by the dynamics of an unharmonic oscillator, which is characterized by two different stability– modes. Depending on the value of the disturbance damping–parameter oscillations with respect to a stable fix–point bifurcate into a mode where the former fix–point loses its stability and becomes replaced by two other symmetrically positioned stable fix– points. The bifurcation of the initial stability mode with one fix–point into another one with two fix–points has to be interpreted by a split of the initial sphere into two follower–spheres.

During this splitting–process disturbance transfers energy to the surface of the initial sphere – and thereby deforms it – up to an energy–levels equal to the surface–energies of two follower–spheres each with a half of the predecessor's volume. The split will save the initial spin and the follower–spins of equal lengths will sum–up for the predecessor's spin. The rotation–energy of the initial sphere will be equally partitioned among the followers which obtain individual spin–lengths of about 60% and phase–velocities of about 111% relative to the proper values of the predecessor. Thus after any split rotation has nearly doubled its phase–speed. Split–energy of a sphere (due to enlargement of its surface) and its life–time are supposed to be proportional to the square of sphere–radius.

A disturbance signal within the model's frame is appropriately be assumed as sum of resonance–terms, each suitable for a split of a proper sphere. A term shall be product of a periodic function and an associated time–dependent damping–parameter and may vanish by integration over an appropriate time–period (life–time of a proper sphere). Due to the letter quality it can be decomposed into a FOURIER–series with complex coefficients. Every coefficient will be derived on base of an individual set of frequency–modes. If the coefficients – which now principally contain all information about amplitudes and phases of their proper modes – are formulated in a suitable way, the FOURIER–series – which they belong to – will result in a periodic function and an associated damping–parameter appropriate for the proper resonance.

Because each follower–sphere will now find its proper resonance–term for a split, it can and will go through the same split–procedure with equivalent conditions as the predecessor did. This means, starting from the split of the initial sphere, one obtains a series of subsequent follower–splits, each nearly doubles phase–speed and shortens life–time by about a third compared to the proper predecessor. This way a picture about an eddy's decay can be drawn as a controlled route of energy from order into chaos, similar to those of many other dynamical systems too.

6. References.

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