SIX-DIMENSIONAL SPACETIME STRUCTURES OF MASSLESS PARTICLES

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Abstract: In this work, we will investigate possible spacetime structures of massless particles by formulating Dirac equation in a six-dimensional spatial-temporal continuum in both pseudo-Euclidean and Euclidean metrics. We show that the state of a massless particle flows as an irrotational and incompressible fluid in steady fluid dynamics in the sense that all spatial and temporal components of the wavefunction of a massless particle can be described by Laplace equation. Furthermore, we also show that the six components of the wavefunction are coupled in pairs to form two-dimensional subspaces and this result may suggest why quantum particles possess an intrinsic spin angular momentum that takes half-integral values.

1. Introduction

In classical physics, the dynamics of a physical object is formulated in terms of the rates of change of its position in space with respect to time in which the concept of momentum plays a central role via Newton's second law, simply, the rate of change of the momentum is determined by the force that influences the motion of the physical object. Even though classical dynamics is formulated mathematically in terms of spatial and temporal differential operators, the evolution of a physical system can only be conceived with respect to time but not to space. The momentum itself is defined as an evolution with time of the position, $\mathbf{p} = m\, d\mathbf{r}/dt$. The main reason for the exclusive temporal evolution is that in Newtonian physics time is perceived to be flowing while space is not. However, the perception of only static space has been changed since it was discovered that the observable universe is expanding and the expansion of the observable universe can be formulated using Einstein general theory of relativity [1]. On the other hand, in quantum mechanics, the momentum is defined as a differential operator without being itself defined as a rate of change of some other mathematical object, such as the position of a quantum particle [2]. Furthermore, via the canonical quantisation in which the position and the momentum of a quantum particle are replaced by spatial differential operators and the energy of the quantum particle is replaced by temporal differential operator, the roles played by the spatial operators and the temporal operator are almost identical, especially within the framework of Dirac's formulation of relativistic quantum mechanics [3]. The spatial and temporal operators play almost identical roles because there are still two important aspects that make them different. First, space is assumed to have three dimensions and time only one dimension. Second, we don't seem to be able to perceive an evolution with respect to space, even though Einstein general relativity has shown that space is expanding, because our perception and conception of such a spatial expansion also require the introduction of the concept of time. In this work we will discuss how these two conceptual problems can be overcome by introducing the three-dimensional temporal manifold that has been analysed in our previous works [4,5] and by assuming that, like time, space is also flowing by itself without the need to be perceived as being going through an expansion with respect to time. This assumption renders the formulation of the dynamics of a physical object in quantum mechanics more complete than that in classical mechanics in the sense that instead of the momentum, which is defined in terms of the rate of change of the position with respect to time, the dynamics of a quantum particle is determined directly by spatial differential operators without the requirement of rates with respect to time. With regards to these considerations, in this work, we will investigate these intricate problems and show that the identification of momentum in terms of spatial operators is a more natural way to construct spacetime as a unified six-dimensional continuum in which space and time are completely equivalent. In particular, we will show that the spacetime structure of a massless particle flows as an irrotational and incompressible fluid that can be described by Laplace equation.

2. Dirac equation in six-dimensional pseudo-Euclidean spacetime

In our previous investigations on the spacetime structures of quantum particles, we applied Dirac equation into various physical problems by modifying Dirac original formulation [6,7]. We showed that Dirac equation can be formulated within the framework of Euclidean relativity [8] and Dirac equation can be reformulated to a real-valued equation that shows that quantum particles may exist as three-dimensional differentiable manifolds embedded into a four-dimensional Euclidean space R^4. In this work we will extend our investigation into the more complete structure of spacetime, which is a six-dimensional continuum in which not only space but time also is considered to be a three-dimensional continuum. In this work, however, we will focus only on the formulation of possible spacetime structures of massless particles in the six-dimensional spacetime manifold. In the Minkowski spacetime of Einstein special relativity with pseudo-Euclidean metric, the energy-momentum relationship is given as follows

$$E^2 = (mc^2)^2 + (pc)^2 \tag{1}$$

In the following, we will treat space and time equivalently in the sense that the dynamics of other physical objects can be formulated in terms of differential operators with respect to both of them within the framework of Dirac's formulation of relativistic quantum mechanics. We will consider the energy $\mathbf{E}$ as a temporal vector quantity in the same status as the vector momentum $\mathbf{p}$. Therefore, if the momentum is defined as a three-dimensional differential operator then the energy should also be defined as a three-dimensional differential operator. Using the relationship between the energy and the momentum given in Equation (1) as a suggestive statement of a dynamical relationship between two physical objects, if we assume spacetime to be a six-dimensional continuum which consists of a three-dimensional spatial continuum and a three-dimensional temporal continuum, then we need to modify Dirac relativistic first order partial differential equation to be of the form

$$(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3)\Psi = (\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m)\Psi \tag{2}$$

where the unknown operators α_i and β are assumed to be independent of the momentum p and the mass m. From Equation (2), we obtain

$$(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3)^2\Psi = (\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m)^2\Psi \tag{3}$$

By expanding Equation (3), and due the fact that all energy and linear momentum operators commute mutually, in order to reduce to the form of the relationship given in Equation (1), the operators α_i and β must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{4}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{5}$$

$$\alpha_i^2 = 1 \tag{6}$$

$$\beta^2 = 1 \tag{7}$$

It is worth mentioning here that the statement that in order to reduce the equation given in Equation (3) to the relationship between the energy and the momentum given in Equation (1) is simply a suggestive statement that should not be considered as a necessary condition to formulate Dirac equation in relativistic quantum mechanics. Rather the conditions given in Equations (4-6) should be considered as necessary and this must be the case when we formulate Dirac equation in Euclidean relativity later on. As mentioned above, in this work we will consider the six-dimensional spacetime therefore wavefunctions that are solutions to the extended Dirac equation given in Equation (2) are assumed to be represented as vectors with six components which consist of three temporal components $(\omega_1, \omega_2, \omega_3)$ and three spatial components (ψ_1, ψ_2, ψ_3). Therefore, we will denote a wavefunction Ψ as a column vector $\Psi = (\omega_1, \omega_2, \omega_3, \psi_1, \psi_2, \psi_3)^T$. However, as shown in Appendix, there are only three $(6, 6)$-matrices that can be obtained to represent the α_i operators that satisfy the conditions given in Equations (4) and (5). Therefore in the following we will only consider massless particles that have six temporal-spatial components with $m = 0$. By applying the method of quantisation in quantum mechanics in which the energy **E**, the momentum **p** are replaced by operators

$$E_1 \to i\frac{\partial}{\partial t_1}, \quad E_2 \to i\frac{\partial}{\partial t_2}, \quad E_3 \to i\frac{\partial}{\partial t_3}, \quad p_x \to -i\frac{\partial}{\partial x}, \quad p_y \to -i\frac{\partial}{\partial y}, \quad p_z \to -i\frac{\partial}{\partial z} \tag{8}$$

Equation (2) is rewritten as

$$\left(\alpha_1\frac{\partial}{\partial t_1} + \alpha_2\frac{\partial}{\partial t_2} + \alpha_3\frac{\partial}{\partial t_3}\right)\Psi = -\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)\Psi \tag{9}$$

Equation (9) can also be written in the form

$$\left(\alpha_1\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}\right)+\alpha_2\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)+\alpha_3\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\right)\Psi \tag{10}$$

2a. Using the operators $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_3}$ in Appendix

If the operators α_1, α_1 and α_3 given in Equation (8) in Appendix are applied then Equation (10) can be written in full form as the following system of equations

$$\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_2+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}-i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_3=0 \tag{11}$$

$$-\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_3+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_2=0 \tag{12}$$

$$\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_3+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}-i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_1=0 \tag{13}$$

$$-\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_1+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_3=0 \tag{14}$$

$$\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_1+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}-i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_2=0 \tag{15}$$

$$-\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_2+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_1=0 \tag{16}$$

It is observed from the system of equations given in Equations (11-16) that the six components of the wavefunction Ψ are coupled in pairs, the space component ψ_2 is coupled with the space component ψ_3, the time component ω_1 is coupled with the time component ω_2, while the space component ψ_1 is coupled with the time component ω_3. The six components of the wavefunction are coupled in pairs to form two-dimensional subspaces may suggest why quantum particles possess an intrinsic spin angular momentum that takes half-integral values [9]. In the following we will consider three different cases for massless particles which are at rest in space, at rest in time and in a mixed spacetime motion.

- Particles at rest in space, $\mathbf{p}=0$, and Equations (11-16) reduce to the following system of equations

$$\frac{\partial\psi_2}{\partial t_3}+\left(\frac{\partial}{\partial t_1}-i\frac{\partial}{\partial t_2}\right)\psi_3=0 \tag{17}$$

$$-\frac{\partial\psi_3}{\partial t_3}+\left(\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\psi_2=0 \tag{18}$$

$$\frac{\partial \omega_3}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\psi_1 = 0 \tag{19}$$

$$-\frac{\partial \psi_1}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_3 = 0 \tag{20}$$

$$\frac{\partial \omega_1}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\omega_2 = 0 \tag{21}$$

$$-\frac{\partial \omega_2}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_1 = 0 \tag{22}$$

By applying $(\partial/\partial t_1 + i\,\partial/\partial t_2\,)$ to Equation (17) we obtain

$$\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\frac{\partial \psi_2}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\psi_3 = 0 \tag{23}$$

Using Equation (18), we arrive at a temporal Laplace equation for the component ψ_3

$$\frac{\partial^2 \psi_3}{\partial t_1^2} + \frac{\partial^2 \psi_3}{\partial t_2^2} + \frac{\partial^2 \psi_3}{\partial t_3^2} = 0 \tag{24}$$

On the other hand, by applying $(\partial/\partial t_1 - i\,\partial/\partial t_2\,)$ to Equation (18) we obtain

$$-\left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\frac{\partial \psi_3}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\psi_2 = 0 \tag{25}$$

Using Equation (17), we also arrive at a temporal Laplace equation for the component ψ_2

$$\frac{\partial^2 \psi_2}{\partial t_1^2} + \frac{\partial^2 \psi_2}{\partial t_2^2} + \frac{\partial^2 \psi_2}{\partial t_3^2} = 0 \tag{26}$$

Similarly, it can be verified that all components of the wavefunction Ψ satisfy Laplace equation in the three-dimensional temporal continuum, $\nabla^2\Psi_\mu = 0$. The wavefunction Ψ behaves as an irrotational and incompressible temporal flow in the six-dimensional spatial-temporal spacetime similar to a steady fluid flow in fluid dynamics where the potential flow satisfies Laplace equation. If $\mathbf{v}$ is the velocity of an irrotational flow then it can be defined in terms of a potential flow φ as $\mathbf{v} = \boldsymbol{\nabla}\varphi$. If the flow is incompressible then $\boldsymbol{\nabla}.\mathbf{v} = 0$, therefore $\nabla^2\varphi = 0$ [10].

- Particles at rest in time

In spatial space, when we say a particle is at rest in space we mean $\mathbf{p} = 0$. Therefore, due to the assumed equivalence between space and time, when we say a particle is rest in time we will mean $\mathbf{E} = 0$. This is equivalent to setting $\partial/\partial t_i = 0,\ i = 1,2,3$. For a particle at rest in time, Equations (11-16) reduce to reduce to the following system of equations

$$\frac{\partial\psi_2}{\partial z}+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_3=0 \tag{27}$$

$$-\frac{\partial\psi_3}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_2=0 \tag{28}$$

$$\frac{\partial\omega_3}{\partial z}+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_1=0 \tag{29}$$

$$-\frac{\partial\psi_1}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_3=0 \tag{30}$$

$$\frac{\partial\omega_1}{\partial z}+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\omega_2=0 \tag{31}$$

$$-\frac{\partial\omega_2}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_1=0 \tag{32}$$

By applying $(\partial/\partial x+i\,\partial/\partial y)$ to Equation (27), we obtain

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\frac{\partial\psi_2}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_3=0 \tag{33}$$

Using Equation (28), we arrive at a Laplace equation for the component ψ_3

$$\frac{\partial^2\psi_3}{\partial x^2}+\frac{\partial^2\psi_3}{\partial y^2}+\frac{\partial^2\psi_3}{\partial z^2}=0 \tag{34}$$

Similarly, it can be shown that all components of the wavefunction Ψ satisfy Laplace equation in the spatial continuum, $\nabla^2\Psi_\mu=0$. When a particle is at rest in time, the wavefunction Ψ also behaves as an irrotational and incompressible spatial flow in the six-dimensional spatial-temporal spacetime similar to a steady fluid flow in fluid dynamics where the velocity potential satisfies Laplace equation.

- A mixed spacetime motion

It is observed that if we define mixed spacetime differential operators as follows

$$\frac{\partial}{\partial s_1}=\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x},\qquad \frac{\partial}{\partial s_2}=\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial s_3}=\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z} \tag{35}$$

then in terms of the mixed differential operators defined in Equation (35), Equations (11-16) are rewritten in the following form

$$\frac{\partial\psi_2}{\partial s_3}+\left(\frac{\partial}{\partial s_1}-i\frac{\partial}{\partial s_2}\right)\psi_3 \tag{36}$$

$$-\frac{\partial\psi_3}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_2 \tag{37}$$

$$\frac{\partial \omega_3}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\psi_1 \tag{38}$$

$$-\frac{\partial \psi_1}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\omega_3 \tag{39}$$

$$\frac{\partial \omega_1}{\partial s_3} + \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2}\right)\omega_2 \tag{40}$$

$$-\frac{\partial \omega_2}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\omega_1 \tag{41}$$

In this case, it can also be shown that in terms of the operators $\partial/\partial s_i$, $i = 1, 2, 3$, all components of the wavefunction Ψ satisfy Laplace equation of a potential flow $\nabla^2\varphi = 0$, where $\nabla^2 = \partial^2/\partial s_1^2 + \partial^2/\partial s_2^2 + \partial^2/\partial s_3^2$.

2b. Using the operators α_1, α_2 and the operator β as α_3 in Appendix

If the operator α_3 is replaced by the operator β given in Equation (11) in Appendix, together with the two operators α_1, α_2 then Equation (2) can be written in the full form as the following system of equations

$$\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\omega_1 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} - i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\psi_3 = 0 \tag{42}$$

$$\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\omega_2 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} + i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\psi_2 = 0 \tag{43}$$

$$\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\omega_3 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} - i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\psi_1 = 0 \tag{44}$$

$$-\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\psi_1 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} + i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\omega_3 = 0 \tag{45}$$

$$-\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\psi_2 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} - i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\omega_2 = 0 \tag{46}$$

$$-\left(\frac{\partial}{\partial t_3} + \frac{\partial}{\partial z}\right)\psi_3 + \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x} + i\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}\right)\right)\omega_1 = 0 \tag{47}$$

It is observed from the system of equations given in Equations (42-47) that even though the six components of the wavefunction Ψ are also coupled in pairs, each pair is a spacetime coupling. The space component ψ_3 is coupled with the time component ω_1, the space component ψ_2 is coupled with the time component ω_2, while the space component ψ_1 is coupled with the time component ω_3. As in the case in the previous section, in the following we will also consider three different cases for massless particles that are at rest in space, at rest in time and in a mixed spacetime motion.

- Particles at rest in space, $\mathbf{p} = 0$, and Equations (42-47) reduce to the following system of equations

$$\frac{\partial \omega_1}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\psi_3 = 0 \tag{48}$$

$$\frac{\partial \omega_2}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\psi_2 = 0 \tag{49}$$

$$\frac{\partial \omega_3}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\psi_1 = 0 \tag{50}$$

$$-\frac{\partial \psi_1}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_3 = 0 \tag{51}$$

$$-\frac{\partial \psi_2}{\partial t_3} + \left(\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_2}\right)\omega_2 = 0 \tag{52}$$

$$-\frac{\partial \psi_3}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_1 = 0 \tag{53}$$

It is seen from the system of equations given in Equations (48-53) that all components of the wavefunction Ψ also satisfy Laplace equation of a potential flow $\nabla^2\varphi = 0$.

- Particles at rest in time, $\mathbf{E} = 0$, and Equations (42-47) reduce to the following system of equations

$$\frac{\partial \omega_1}{\partial z} + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_3 = 0 \tag{54}$$

$$\frac{\partial \omega_2}{\partial z} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_2 = 0 \tag{55}$$

$$\frac{\partial \omega_3}{\partial z} + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_1 = 0 \tag{56}$$

$$-\frac{\partial \psi_1}{\partial z} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\omega_3 = 0 \tag{57}$$

$$-\frac{\partial \psi_2}{\partial z} + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\omega_2 = 0 \tag{58}$$

$$-\frac{\partial \psi_3}{\partial z} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\omega_1 = 0 \tag{59}$$

It is also seen from the system of equations given in Equations (48-53) that all components of the wavefunction Ψ also satisfy Laplace equation of a potential flow $\nabla^2\varphi = 0$.

- A mixed spacetime motion

As before, if we define mixed spacetime differential operators as

$$\frac{\partial}{\partial s_1} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial s_2} = \frac{\partial}{\partial t_2} + \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s_3} = \frac{\partial}{\partial t_3} + \frac{\partial}{\partial z} \tag{60}$$

then Equations (42-47) are rewritten in the form

$$\frac{\partial \omega_1}{\partial s_3} + \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2}\right)\psi_3 \tag{61}$$

$$\frac{\partial \omega_2}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\psi_2 \tag{62}$$

$$\frac{\partial \omega_3}{\partial s_3} + \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2}\right)\psi_1 \tag{63}$$

$$-\frac{\partial \psi_1}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\omega_3 \tag{64}$$

$$-\frac{\partial \psi_2}{\partial s_3} + \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2}\right)\omega_2 \tag{65}$$

$$-\frac{\partial \psi_3}{\partial s_3} + \left(\frac{\partial}{\partial s_1} + i\frac{\partial}{\partial s_2}\right)\omega_1 \tag{66}$$

In this case, it can also be shown that in terms of the operators $\partial/\partial s_i$, $i = 1,2,3$, all components of the wavefunction Ψ satisfy Laplace equation of a potential flow $\nabla^2\varphi = 0$, where $\nabla^2 = \partial^2/\partial s_1^2 + \partial^2/\partial s_2^2 + \partial^2/\partial s_3^2$.

3. Dirac equation in six-dimensional Euclidean spacetime

In the following we will extend and formulate Dirac equation in six-dimensional Euclidean spacetime, therefore the energy-momentum relationship is assumed to take the form.

$$E^2 = (mc^2)^2 - (pc)^2 \tag{67}$$

With the relationship between the energy and the momentum given in Equation (67), Dirac first order partial differential equation can also be formulated by proposing that it is of the forms given in Equations (2) and (3). However, in order to reduce to the form of the relationship given in Equation (67), the operators α_i and β must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{68}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{69}$$

$$\alpha_i^2 = -1 \tag{70}$$

$$\beta^2 = 1 \tag{71}$$

From the similarity between Dirac equation formulated in the pseudo-Euclidean spacetime and Dirac equation that will be formulated in Euclidean spacetime, it is anticipated that the following discussions will be similar to those that have been discussed above. Nonetheless, for clarity, we will list out below all systems of equations that involve. However, there is one particular feature that is worth noting here is that in order to reduce the expression given in Equation (3) to the energy-momentum relationship given in Equation (67) we require the operators α_i to satisfy the conditions given in Equations (68-71). However, in the case when $m = 0$, we obtain from Equation (67) the relation $E^2 = -(pc)^2$. This result clearly shows that references to the classical relativistic relationship between the energy and the momentum of a physical object are only suggestive when we formulate a relativistic Dirac equation and therefore Dirac's formulation of relativistic quantum mechanics should be considered as a postulated quantum dynamical law similar to Newton's second law in classical mechanics.

3a. Using the operators α_1, α_1 and α_3 in Appendix

If the operators α_1, α_1 and α_3 given in Equation (16) in Appendix are applied then Equation (10) can be written in the full form as the following system of equations

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_2+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_3=0 \tag{72}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_3+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_2=0 \tag{73}$$

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_3+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_1=0 \tag{74}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_1+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_3=0 \tag{75}$$

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_1+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_2=0 \tag{76}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_2+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_1=0 \tag{77}$$

It is seen from the system of equations given in Equations (72-77) that the six components of the wavefunction Ψ are also coupled in pairs, the space component ψ_2 is coupled with the space component ψ_3, the time component ω_1 is coupled with the time component ω_2, while the space component ψ_1 is coupled with the time component ω_3. In the following we will also consider three different cases for massless particles that at rest in space, at rest in time and in a mixed spacetime motion.

- Particles at rest in space, $\mathbf{p} = 0$, and Equations (72-77) reduce to the following system of equations

$$i\frac{\partial\psi_2}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\psi_3 = 0 \tag{78}$$

$$-i\frac{\partial\psi_3}{\partial t_3} + \left(-\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\psi_2 = 0 \tag{79}$$

$$i\frac{\partial\omega_3}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\psi_1 = 0 \tag{80}$$

$$-i\frac{\partial\psi_1}{\partial t_3} + \left(-\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_3 = 0 \tag{81}$$

$$i\frac{\partial\omega_1}{\partial t_3} + \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_2 = 0 \tag{82}$$

$$-i\frac{\partial\omega_2}{\partial t_3} + \left(-\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)\omega_1 = 0 \tag{83}$$

It can be verified that all components of the wavefunction Ψ satisfy Laplace equation in the three-dimensional temporal continuum, $\nabla^2\Psi_\mu = 0$. The wavefunction Ψ behaves as an irrotational and incompressible temporal flow in the six-dimensional spatial-temporal spacetime similar to a steady fluid flow in fluid dynamics where the velocity potential satisfies Laplace equation.

- Particles at rest in time, $\mathbf{E} = 0$

For a particle at rest in time, $\mathbf{E} = 0$, Equations (72-77) reduce to

$$i\frac{\partial\psi_2}{\partial z} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 = 0 \tag{84}$$

$$-i\frac{\partial\psi_3}{\partial z} + \left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_2 = 0 \tag{85}$$

$$i\frac{\partial\omega_3}{\partial z} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_1 = 0 \tag{86}$$

$$-i\frac{\partial\psi_1}{\partial z}+\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_3=0 \tag{87}$$

$$i\frac{\partial\omega_1}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_2=0 \tag{88}$$

$$-i\frac{\partial\omega_2}{\partial z}+\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_1=0 \tag{89}$$

All components of the wavefunction Ψ also satisfy Laplace equation of a potential flow $\nabla^2\varphi=0$.

- A unified spacetime motion

If we also define mixed spacetime differential operators as

$$\frac{\partial}{\partial s_1}=\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x},\quad \frac{\partial}{\partial s_2}=\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y},\quad \frac{\partial}{\partial s_3}=\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z} \tag{90}$$

then Equations (72-77) are reduced to the forms

$$i\frac{\partial\psi_2}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_3 \tag{91}$$

$$-i\frac{\partial\psi_3}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_2 \tag{92}$$

$$i\frac{\partial\omega_3}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_1 \tag{93}$$

$$-i\frac{\partial\psi_1}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_3 \tag{94}$$

$$i\frac{\partial\omega_1}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_2 \tag{95}$$

$$-i\frac{\partial\omega_2}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_1 \tag{96}$$

In this case, all components of the wavefunction Ψ satisfy Laplace equation of a potential flow $\nabla^2\varphi=0$, where $\nabla^2=\partial^2/\partial s_1^2+\partial^2/\partial s_2^2+\partial^2/\partial s_3^2$.

3b. Using the operators $\boldsymbol{\alpha_1,\alpha_2}$ and the operator $\boldsymbol{\beta}$ as $\boldsymbol{\alpha_3}$ in Appendix

If the operator α_3 given in Equation (16) in Appendix is replaced by the operator β given in Equation (11) in Appendix, together with the two operators α_1,α_2 then Equation (10) can be written in the full form as the following system of equations

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_1+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_3=0 \tag{97}$$

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_2+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_2=0 \tag{98}$$

$$i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\omega_3+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\psi_1=0 \tag{99}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_1+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_3=0 \tag{100}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_2+\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_2=0 \tag{101}$$

$$-i\left(\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z}\right)\psi_3+\left(-\frac{\partial}{\partial t_1}-\frac{\partial}{\partial x}+i\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y}\right)\right)\omega_1=0 \tag{102}$$

It is also observed from the system of equations given in Equations (97-102) that even though the six components of the wavefunction Ψ are also coupled in pairs, each pair is a spacetime coupling. The space component ψ_3 is coupled with the time component ω_1, the space component ψ_2 is coupled with the time component ω_2, while the space component ψ_1 is coupled with the time component ω_3. As in the case in the previous section, in the following we will consider three different cases for massless particles which are at rest in space, at rest in time and in a mixed spacetime motion.

- Particles at rest in space, $\mathbf{p}=0$, and Equations (97-102) reduce to the following system of equations

$$i\frac{\partial\omega_1}{\partial t_3}+\left(\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\psi_3=0 \tag{103}$$

$$i\frac{\partial\omega_2}{\partial t_3}+\left(-\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\psi_2=0 \tag{104}$$

$$i\frac{\partial\omega_3}{\partial t_3}+\left(\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\psi_1=0 \tag{105}$$

$$-i\frac{\partial\psi_1}{\partial t_3}+\left(-\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\omega_3=0 \tag{106}$$

$$-i\frac{\partial\psi_2}{\partial t_3}+\left(\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\omega_2=0 \tag{107}$$

$$-i\frac{\partial\psi_3}{\partial t_3}+\left(-\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}\right)\omega_1=0 \tag{108}$$

It is seen that all components of the wavefunction Ψ satisfy Laplace equation of a potential flow $\nabla^2\varphi=0$.

- Particles at rest in time, $\mathbf{E}=0$, and Equations (97-102) reduce to the following system of equations

$$i\frac{\partial\omega_1}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3=0 \tag{109}$$

$$i\frac{\partial\omega_2}{\partial z}+\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_2=0 \tag{110}$$

$$i\frac{\partial\omega_3}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1=0 \tag{111}$$

$$-i\frac{\partial\psi_1}{\partial z}+\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_3=0 \tag{112}$$

$$-i\frac{\partial\psi_2}{\partial z}+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_2=0 \tag{113}$$

$$-i\frac{\partial\psi_3}{\partial z}+\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\omega_1=0 \tag{114}$$

All components of the wavefunction Ψ satisfy Laplace equation of a potential flow $\nabla^2\varphi=0$.

- A mixed spacetime motion

If we also define mixed spacetime differential operators as

$$\frac{\partial}{\partial s_1}=\frac{\partial}{\partial t_1}+\frac{\partial}{\partial x},\qquad \frac{\partial}{\partial s_2}=\frac{\partial}{\partial t_2}+\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial s_3}=\frac{\partial}{\partial t_3}+\frac{\partial}{\partial z} \tag{115}$$

then Equations (97-102) is rewritten in the form

$$i\frac{\partial\omega_1}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_3=0 \tag{116}$$

$$i\frac{\partial\omega_2}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_2=0 \tag{117}$$

$$i\frac{\partial\omega_3}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\psi_1=0 \tag{118}$$

$$-i\frac{\partial\psi_1}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_3=0 \tag{119}$$

$$-i\frac{\partial\psi_2}{\partial s_3}+\left(\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_2=0 \tag{120}$$

$$-i\frac{\partial\psi_3}{\partial s_3}+\left(-\frac{\partial}{\partial s_1}+i\frac{\partial}{\partial s_2}\right)\omega_1=0 \tag{121}$$

In this case, all components of the wavefunction Ψ also satisfy Laplace equation of a potential flow $\nabla^2\varphi=0$, where $\nabla^2=\partial^2/\partial s_1^2+\partial^2/\partial s_2^2+\partial^2/\partial s_3^2$.

Appendix

Assume the operators α_i are represented in terms of the operators σ_i in the forms

$$\alpha_i=\begin{pmatrix}\sigma_i & 0 & 0\\ 0 & \sigma_i & 0\\ 0 & 0 & \sigma_i\end{pmatrix},\ \begin{pmatrix}0 & 0 & \sigma_i\\ 0 & \sigma_i & 0\\ \sigma_i & 0 & 0\end{pmatrix},\ \begin{pmatrix}\sigma_i & 0 & 0\\ 0 & \sigma_i & 0\\ 0 & 0 & -\sigma_i\end{pmatrix}\text{ or }\begin{pmatrix}\sigma_i & 0 & 0\\ 0 & -\sigma_i & 0\\ 0 & 0 & -\sigma_i\end{pmatrix} \tag{1}$$

then we obtain

$$\alpha_i^2=\begin{pmatrix}\sigma_i^2 & 0 & 0\\ 0 & \sigma_i^2 & 0\\ 0 & 0 & \sigma_i^2\end{pmatrix} \tag{2}$$

If $\sigma_i^2=1$ then $\alpha_i^2=1$. On the other hand, if $\sigma_i^2=-1$ then $\alpha_i^2=-1$.

Now, if we write the operator σ_i as a two by two matrix in the form

$$\sigma_i=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \tag{3}$$

then from the requirement $\sigma_i^2=1$, we arrive at the following system of equations for the unknown quantities a,b,c and d

$$a^2+bc=1 \tag{4}$$

$$b(a+d)=0 \tag{5}$$

$$c(a+d)=0 \tag{6}$$

$$d^2+bc=1 \tag{7}$$

From Equations (4) and (7) we require $d=\pm a$. If $d=a\neq 0$ then $b=c=0$ and the operator σ_i can take the values $\sigma_i=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. If $d=-a$ and if $b=c=0$, then the operator σ_i can be as $\sigma_i=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. If $d=-a$ but $b\neq 0$ and $c\neq 0$, then the operator σ_i can be written in the form $\sigma_i=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. These are only a few standard representations of the operators σ_i. In terms of Pauli matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the operators α_i defined as $\alpha_i = \begin{pmatrix} 0 & 0 & \sigma_i \\ 0 & \sigma_i & 0 \\ \sigma_i & 0 & 0 \end{pmatrix}$ are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

It can be verified that the representations of the operators α_i given in Equations (8) satisfy the following relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \text{for } i \neq j \tag{9}$$

$$\alpha_i^2 = 1 \tag{10}$$

However, if we define the operator β as

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

then we have $\beta^2 = 1$, $\beta\alpha_1 + \alpha_1\beta = 0$ and $\beta\alpha_2 + \alpha_2\beta = 0$ but $\beta\alpha_3 + \alpha_3\beta \neq 0$. In this case if we replace α_3 by β then together the three operators α_1, α_2 and β will form a system that satisfy the relations given in Equations (9) and (10).

Now from the requirement $\sigma_i^2 = -1$, we arrive at the following system of equations for the unknown quantities a, b, c and d

$$a^2 + bc = -1 \tag{12}$$

$$b(a + d) = 0 \tag{13}$$

$$c(a + d) = 0 \tag{14}$$

$$d^2 + bc = -1 \tag{15}$$

From Equations (12) and (15) we require $d = \pm a$. If $d = a \neq 0$ then $b = c = 0$ and the operator σ_i can take the values $\sigma_i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. If $d = -a$ and if $b = c = 0$, then the operator σ_i can be as $\sigma_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. If $d = -a$ but $b \neq 0$ and $c \neq 0$, then the operator σ_i can be written in the form $\sigma_i = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These are only a few standard representations of the operators σ_i. Using the matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, the operators α_i defined as $\alpha_i = \begin{pmatrix} 0 & 0 & \sigma_i \\ 0 & \sigma_i & 0 \\ \sigma_i & 0 & 0 \end{pmatrix}$ are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

However, if we define the operator β as

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{17}$$

then we have $\beta^2 = 1$, $\beta\alpha_1 + \alpha_1\beta = 0$ and $\beta\alpha_2 + \alpha_2\beta = 0$ but $\beta\alpha_3 + \alpha_3\beta \neq 0$. In this case if we replace α_3 by β then together the three operators α_1, α_2 and β will form a system that satisfy the relations given in Equations (9) and (10).

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