The Irrationality and Transcendence of e Connected

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Using the techniques of a proof of e's transcendence given in Herstein's Topics in Algebra [2], Beatty and Jones gave a proof of the irrationality of e^n, n a positive integer [1]. The mean value theorem is used in both proofs. In this article we show how the mean value theorem can be avoided in favor of a simpler approach that yields a nice path from the irrationality of e^n to e's transcendence.

In what follows, x is a real number, all polynomials are integer polynomials, and p is a prime.

Definition 1. Given a polynomial $f(x)$, lowercase, the sum of all its derivatives is designated with $F(x)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(x)$ denote the infinite series

$$\frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \cdots + \frac{x^j}{(n+1)(n+2)\ldots(n+j)} + \ldots.$$

Lemma 1. *If $f(x) = cx^n$, then*

$$F(0)e^x = F(x) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Proof. As $F(x) = c(x^n + nx^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^x &= cn!(1 + x/1 + x^2/2! + \cdots + x^n/n! + \ldots) \\ &= cx^n + cnx^{(n-1)} + \cdots + cn! + cx^{n+1}/(n+1)! + \ldots \\ &= F(x) + cx^n(x/(n+1) + x^2/(n+1)(n+2) + \ldots) \\ &= F(x) + f(x)\epsilon_n(x). \end{aligned}$$

Now $f(x)$ has polynomial growth in n and $\epsilon_n(x) \leq e^x$, so the product has polynomial growth in n. $\square$

Lemma 2. *If F is the sum of the derivatives of the polynomial $f(x) = c_0 + c_1x + \cdots + c_nx^n$ of degree n, then*

$$e^x F(0) = F(x) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Proof. Let $f_j(x) = c_jx^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n} (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^x F_k(0) = F_k(x) + f_k(x)\epsilon_k(x) \tag{3}$$

and summing (3) from $k = 0$ to n, gives

$$e^x F(0) = F(x) + \sum_{k=0}^{n} f_k(x)\epsilon_k(x).$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n, we arrive at (2). $\square$

Lemma 3. *Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer root r of multiplicity p, then $p!|F(r)$.*

Proof. Suppose $r = 0$, then $f(x) = x^p(b_jx^j + \cdots + b_0)$ and the 0 through $p-1$ derivatives of $f(x)$ will have $r = 0$ as a root. The pth derivative through the (n-p)th derivative will have $p!$ in each coefficient. This shows $p!|F(r)$ when $r = 0$.

If $r \neq 0$, then $f(x) = (x - r)^pQ(x)$, where $Q(x)$ is a polynomial. Define $g(x) = f(x + r) = x^pQ(x + r)$. Then $g^{(k)}(0) = f^{(k)}(r)$ for all $k \geq 0$, where k superscripts give derivatives. The same argument used for the $r = 0$ case applies. $\square$

Lemma 4. *Let polynomial $f(x)$ have root $r = 0$ of multiplicity $p - 1$ then, for large enough p, $p \nmid F(0)$.*

Proof. We can write $f(x) = x^{p-1}(b_j x^j + \cdots + b_0)$ then the $p-1$ derivative is $(p-1)!b_0$ and all subsequent derivatives have $p!$ in all their terms. Now if $p > b_0$, then $p \nmid F(0)$. $\square$

Lemma 5. *If a and b are integers and p is a prime, $p > a$, then $a(p-1)! + bp!$ is a non-zero integer divisible by $(p-1)!$.*

Proof. Suppose, to obtain a contradiction, that $a(p-1)! + bp! = 0$. Then $p|a$ or $p|(p-1)!$, a contradiction. Clearly, $(p-1)!|p!$. $\square$

Theorem 1. *For positive, non-zero rational r, e^r is irrational.*

Proof. It is sufficient to prove that e^n, n a natural number is irrational. Suppose not, suppose $e^n = a/b$ with a, b natural numbers $a > b$. Define $f(x) = x^{p-1}(x-n)^p$. Then, using Lemmas 2, $e^n F(0) = F(n) + \epsilon$ and this implies $aF(0) - bF(n) = b\epsilon$. Dividing by $(p-1)!$ gives

$$\frac{aF(0) - bF(n)}{(p-1)!} = \frac{b\epsilon}{(p-1)!}. \tag{4}$$

If p is sufficiently large, (4), using Lemmas 3, 4, and 5, gives an absolute value of the left hand side that is at least 1 while the absolute value of the right hand side is less than 1, a contradiction. $\square$

Theorem 2. *e is transcendental.*

Proof. A number is transcendental if it doesn't solve an integer polynomial. Suppose e solves an nth degree integer polynomial, then

$$0 = c_n e^n + c_{n-1}e^{n-1} + \cdots + c_0.$$

Define $f_n(x) = x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p$; and, using the above lemmas,

$$0 = F(0)(c_n e^n + c_{n-1}e^{n-1} + \cdots + c_0) = c_0 F(0) + \sum_{k=1}^{n} c_k F(k) + \epsilon,$$

giving a contradiction for large enough p. $\square$

References

[1] T. Beatty and T.W. Jones, A Simple Proof that $e^{p/q}$ is Irrational, *Math. Magazine*, **87**, (2014) 50–51.

[2] I. N. Herstein, *Topics in Algebra*, 2nd ed., John Wiley, New York, 1975.