Diamond Operator as a Spinless Square Root of d' Alembertian

Hideki Mutoh
Link Research Corporation
Odawara, Kanagawa, Japan
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Abstract: **Dirac equation includes the 4 × 4 complex differential operator matrix, which is one of square roots of d' Alembertian with spin. We found another 4 × 4 complex differential matrix as a spinless square root of d' Alembertian, which we call diamond operator. The extended Maxwell's equations including charge creation-annihilation field and the linear gravitational field equations including energy creation-annihilation field can be simply written by using the diamond operator. It is shown that the linear gravitational field equations derive Klein-Gordon equation, time independent Schrödinger equation, and the principle of quantum mechanics.**

I. Introduction

Dirac found a relativistic wave equation for electrons with a 4 × 4 complex differential operator matrix as a square root of d' Alembertian.[1)] The equation is satisfied by Fermions with spin of half integer. Since elementary particles to mediate forces are Bosons, different relativistic wave equation is necessary to treat weak, strong, electromagnetic, and gravitational forces. We found a new 4 × 4 complex differential operator matrix, we call it diamond operator, as a spinless square root of d' Alembertian, which enable us to treat Bosons including four forces. Recently, we have reported that the extended Maxwell's equations including charge creation-annihilation scalar field can treat generation-recombination of electron-hole pairs in semiconductors and the similar equation for the linear gravitational field with energy creation-annihilation scalar field can derive Klein-Gordon and time independent Schrödinger equations.[2-8)] It is shown that the diamond operator successfully and simply describes the extended Maxwell's and linear gravitational field equations.

II. Dirac equation

Dirac equation is given by

$$i\hbar\gamma^{\mu}\partial_{\mu}\psi - mc\psi = 0\,, \tag{1}$$

where $\hbar$ is Dirac constant, ψ is a wave function, m is a mass, c is light speed in free space, and

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5}$$

σ_1, σ_2, and σ_3 are following Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Since ψ satisfies

$$\Box\psi + \frac{m^2c^2}{\hbar^2}\psi = 0\,, \tag{7}$$

$\gamma^{\mu}\partial_{\mu}$ satisfies

$$\left(\gamma^{\mu}\partial_{\mu}\right)^2 = \Box\,, \tag{8}$$

where $\Box$ denotes d' Alembertian defined by

$$\Box \equiv \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\,. \tag{9}$$

Therefore, $\gamma^{\mu}\partial_{\mu}$ is a square root of d' Alembertian with spin given by

$$\gamma^{\mu}\partial_{\mu}=\begin{pmatrix}\partial_0 & 0 & \partial_3 & \partial_1-i\partial_2\\ 0 & \partial_0 & \partial_1+i\partial_2 & -\partial_3\\ -\partial_3 & -\partial_1+i\partial_2 & -\partial_0 & 0\\ -\partial_1-i\partial_2 & \partial_3 & 0 & -\partial_0\end{pmatrix}. \quad (10)$$

It should be noticed that $\gamma^{\mu}\partial_{\mu}$ is not symmetrical for space axes for the purpose to include spin.

III. Diamond operator

We define the diamond operator $\Diamond$ as

$$\Diamond \equiv \delta^{\mu}\partial_{\mu}, \quad (11)$$

where

$$\delta^0=\begin{pmatrix}-I_2 & 0\\ 0 & -I_2\end{pmatrix}*=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}*, \quad (12)$$

$$\delta^1=\begin{pmatrix}0 & -i\sigma_1\\ i\sigma_1 & 0\end{pmatrix}*=\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix}*, \quad (13)$$

$$\delta^2=\begin{pmatrix}0 & i\sigma_3\\ -i\sigma_3 & 0\end{pmatrix}*=\begin{pmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{pmatrix}*, \quad (14)$$

$$\delta^3=\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}*=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}*. \quad (15)$$

In the above equation, * denotes the complex conjugate operator which satisfies

$$*A=A^*, \quad (16)$$

where A is a complex scalar, vector, or matrix, and A^* is the complex conjugate of A. The diamond operator satisfies

$$\Diamond^2=\Box. \quad (17)$$

(11) – (15) give

$$\Diamond=\begin{pmatrix}-\partial_0 & -i\partial_3 & i\partial_2 & -i\partial_1\\ i\partial_3 & -\partial_0 & -i\partial_1 & -i\partial_2\\ -i\partial_2 & i\partial_1 & -\partial_0 & -i\partial_3\\ i\partial_1 & i\partial_2 & i\partial_3 & -\partial_0\end{pmatrix}*. \quad (18)$$

Therefore, this operator is symmetric for three space axes and does not include spin.

For electromagnetic and gravitational force, the four current C and the four field F satisfy

$$gC=\Diamond F. \quad (19)$$

In (19), g is a coupling constant and

$$C\equiv\begin{pmatrix}\mathbf{C}\\ iC_0\end{pmatrix}, \quad (20)$$

$$F\equiv\begin{pmatrix}\mathbf{D}+i\mathbf{R}\\ -iS\end{pmatrix}, \quad (21)$$

where $\mathbf{D}$, $\mathbf{R}$, and S are divergent, rotational, and scalar fields, respectively. (19)-(21) give

$$g\mathbf{C}=\nabla\times\mathbf{R}-\partial_0\mathbf{D}+\nabla S, \quad (22)$$

$$gC_0=\nabla\cdot\mathbf{D}-\partial_0 S, \quad (23)$$

$$\nabla\times\mathbf{D}+\partial_0\mathbf{R}=0, \quad (24)$$

$$\nabla\cdot\mathbf{R}=0. \quad (25)$$

The four field $\overline{F}$ with gauge parameter ξ and four potential A satisfy

$$\overline{F}=\Diamond A, \quad (26)$$

where

$$\overline{F}\equiv\begin{pmatrix}\mathbf{D}+i\mathbf{R}\\ -i\xi S\end{pmatrix}, \quad (27)$$

$$A\equiv\begin{pmatrix}\mathbf{A}\\ iA_0\end{pmatrix}. \quad (28)$$

(26)-(28) give

$$\mathbf{D}=-\partial_0\mathbf{A}-\nabla A_0, \quad (29)$$

$$\mathbf{R}=\nabla\times\mathbf{A}, \quad (30)$$

$$\xi S=-\nabla\cdot\mathbf{A}-\partial_0 A_0. \quad (31)$$

C, F, and A satisfy the following Lorentz transformation for the boost with velocity v along x-axis,

$$\begin{pmatrix}C_1'\\ C_2'\\ C_3'\\ iC_0'\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}}\\ 0&1&0&0\\ 0&0&1&0\\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}}\end{pmatrix}\begin{pmatrix}C_1\\ C_2\\ C_3\\ iC_0\end{pmatrix}, \quad (32)$$

$$\begin{pmatrix}F_1'\\ F_2'\\ F_3'\\ iF_0'\end{pmatrix}=\begin{pmatrix}1&0&0&0\\ 0 & \frac{1}{\sqrt{1-\beta^2}} & \frac{i\beta}{\sqrt{1-\beta^2}} & 0\\ 0 & \frac{-i\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0\\ 0&0&0&1\end{pmatrix}\begin{pmatrix}F_1\\ F_2\\ F_3\\ iF_0\end{pmatrix}, \quad (33)$$

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ iA_0' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ iA_0 \end{pmatrix}, \qquad (34)$$

where $\beta = v / c$.

IV. Extended Maxwell's equations

Maxwell's equations cannot treat generation-recombination of charge pairs by the charge conservation equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \qquad (35)$$

Because current $\mathbf{J}$ and charge concentration ρ should satisfy the following equation in semiconductors,[7-10)]

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = GR, \qquad (36)$$

where GR is charge generation-recombination rate. In order to obtain the extended Maxwell's equations, C and F are substituted by four current J, electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, and charge creation-annihilation field N as

$$C = J \equiv \begin{pmatrix} \mathbf{J} \\ ic\rho \end{pmatrix}, \qquad (37)$$

$$F = \begin{pmatrix} \mathbf{E}/c + i\mathbf{B} \\ -iN \end{pmatrix}. \qquad (38)$$

Since the coupling constant g is substituted by permeability μ, the extended Maxwell's equations are written by

$$\mathbf{J} = \frac{1}{\mu}\nabla \times \mathbf{B} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu}\nabla N, \qquad (39)$$

$$\rho = \varepsilon \nabla \cdot \mathbf{E} - \varepsilon \frac{\partial N}{\partial t}, \qquad (40)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad (41)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (42)$$

where ε is permittivity which satisfies $\varepsilon\,\mu = 1/c^2$. (39) and (40) give the following current continuity equation,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = -\frac{1}{\mu}\Box N. \qquad (43)$$

Therefore $-\Box N/\mu$ is charge creation-annihilation rate. By using (31), the charge creation-annihilation field N satisfies

$$\xi N = -\nabla \cdot \mathbf{A} - \partial_0 A_0. \qquad (44)$$

N is equivalent to Nakanishi-Lautrup field[11, 12)] except $\Box N \neq 0$ in the region where charges are created or annihilated.

V. Linear gravitational field and quantum mechanics

Einstein's gravitational equation is given by

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad (45)$$

where $G_{\mu\nu}$ is Einstein tensor and κ is Einstein's gravitational constant. $T_{\mu\nu}$ is momentum density tensor written by

$$T_{\mu\nu} = -\rho v_\mu v_\nu, \qquad (46)$$

where v_μ and v_ν are μ and ν component of the velocity. When the momentum density is enough small, metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad (47)$$

where $\eta_{\mu\nu}$ and $h_{\mu\nu}$ are tensors which satisfy

$$\eta_{\mu\nu} \equiv \begin{cases} 1 & (\mu = \nu = 1,2,3) \\ 0 & (\mu \neq \nu) \\ -1 & (\mu = \nu = 0) \end{cases}, \qquad (48)$$

$$|h_{\mu\nu}| \ll 1. \qquad (49)$$

Here we define $\bar{h}_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda}. \qquad (50)$$

In Lorentz gauge condition of $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$, we obtain

$$-\Box \bar{h}_{\mu\nu} = 2\kappa T_{\mu\nu}. \qquad (51)$$

The above equation is regarded as the wave equation for linear gravitational field.[13)] In order to obtain Lorentz vector, we assume small volume Ω. Then the gravitational vector potential A_g and gravitational current C_g are given by

$$A_{g\mu} \equiv \frac{1}{2\kappa c}\bar{h}_{\mu 0}\Omega, \qquad (52)$$

$$C_{g\mu} \equiv -\frac{1}{c}T_{\mu 0}\Omega = \rho v_\mu \Omega. \qquad (53)$$

Therefore

$$C_{g\mu} = \Box A_{g\mu}. \qquad (54)$$

Then the gravitational fields $F_g = (\mathbf{D}_g + i\mathbf{R}_g, -iS_g)$ and $\bar{F}_g = (\mathbf{D}_g + i\mathbf{R}_g, -i\xi S_g)$ satisfy

$$C_g = \Diamond F_g, \qquad (55)$$

$$\bar{F}_g = \Diamond A_g, \qquad (56)$$

where $\mathbf{D}_g$, $\mathbf{R}_g$, and S_g are the divergent, rotational, and scalar fields of the linear gravitational field.

Since the four current vector C_g is equivalent to the four momentum vector $P \equiv (\mathbf{P}, iP_0)$, where $\mathbf{P}$ and cP_0 denote 3D momentum and energy,

$$\mathbf{P} = \nabla \times \mathbf{R}_g - \partial_0 \mathbf{D}_g + \nabla S_g , \tag{57}$$

$$P_0 = \nabla \cdot \mathbf{D}_g - \partial_0 S_g . \tag{58}$$

If we assume existence of the four potential $V \equiv (\mathbf{V}, i\psi/c)$, the total four momentum π is given by

$$\pi \equiv \begin{pmatrix} \boldsymbol{\pi} \\ i\dfrac{E}{c} \end{pmatrix} = \begin{pmatrix} \mathbf{P} + \mathbf{V} \\ i\left(P_0 + \dfrac{\psi}{c}\right) \end{pmatrix} . \tag{59}$$

3D total momentum $\boldsymbol{\pi}$ and total energy E satisfy

$$\boldsymbol{\pi} = \nabla \times \mathbf{R}_{total} - \partial_0 \mathbf{D}_{total} + \nabla S_{total} , \tag{60}$$

$$\frac{E}{c} = \nabla \cdot \mathbf{D}_{total} - \partial_0 S_{total} , \tag{61}$$

where $\mathbf{D}_{total}$, $\mathbf{R}_{total}$, and S_{total} are the total divergent, rotational, and scalar fields considering the four potential, respectively. If the four potential is appropriate and the motion is stable, the wave sources of the total divergent and rotational fields should be zero as

$$\Box \mathbf{D}_{total} = \Box \mathbf{R}_{total} = 0 . \tag{62}$$

Therefore

$$c\partial_0 \boldsymbol{\pi} + \nabla E = 0 , \tag{63}$$

$$\nabla \times \boldsymbol{\pi} = 0 . \tag{64}$$

The above conditions seem to be equivalent to Newton's second law. Because (59), (63), and (64) give

$$\frac{d\mathbf{P}}{dt} = \mathbf{v} \times \left(\nabla \times \mathbf{V}\right) - \frac{\partial \mathbf{V}}{\partial t} - \nabla \psi , \tag{65}$$

where $\mathbf{v}$ denotes 3D velocity. In electromagnetic field case, the right side of (65) is equivalent to the sum of Lorentz and Coulomb forces. By using (63), (64), and a appropriate scalar field S_c, $\boldsymbol{\pi}$ and E are written as

$$\boldsymbol{\pi} = \nabla S_c , \tag{66}$$

$$E = -\frac{\partial S_c}{\partial t} . \tag{67}$$

Here we call S_c energy creation-annihilation field, because energy creation-annihilation rate σ is defined by

$$\sigma \equiv c^2 \nabla \cdot \boldsymbol{\pi} + \frac{\partial E}{\partial t} = -c^2 \Box S_c . \tag{68}$$

When we define the wave function ϕ as

$$\phi \equiv \exp\left(\frac{iS_c}{\hbar}\right), \tag{69}$$

we obtain

$$\Box \phi = \left(\frac{-E^2 + \pi^2 c^2}{c^2 \hbar^2} + \frac{i}{\hbar} \Box S_c\right)\phi . \tag{70}$$

In the case of $\Box S_c = 0$, we obtain Klein-Gordon equation of

$$\Box \phi + \frac{m^2 c^2}{\hbar^2} \phi = 0 . \tag{71}$$

If we assume existence of the potential U, the above equation is rewritten as[14)]

$$\Box \phi + \frac{m^2 c^2}{\hbar^2} \phi = -\frac{U}{\hbar^2 c^2} \phi . \tag{72}$$

Since $\Box S_c \neq 0$ in the above case, we obtain

$$E^2 = \pi^2 c^2 + m^2 c^4 + U - i\hbar\sigma . \tag{73}$$

The above equation suggests the principle of quantum mechanics, that it is equivalent to classical mechanics when the absolute value of $\hbar\sigma$ is much smaller than $\pi^2 c^2$, otherwise the imaginary part of energy creation-annihilation field creates or annihilates quantized interactive energy depending on the potential U.

If $\pi^2 c^2$ and the absolute value of $\hbar\sigma$ are much smaller than $m^2 c^4$, we obtain

$$E = \sqrt{\pi^2 c^2 + m^2 c^4 + U - i\hbar\sigma} \approx mc^2 + \frac{\pi^2 + i\hbar \Box S_c}{2m} + \frac{U}{2mc^2} . \tag{74}$$

When we assume E and $\partial_0 S_c$ do not depend on time, and redefine the total energy $E' \equiv E - mc^2$ and the potential $V \equiv U/2mc^2$, we obtain

$$E' = \frac{\pi^2 - i\hbar \nabla^2 S_c}{2m} + V . \tag{75}$$

Since $\nabla^2 \phi$ is given by

$$\nabla^2 \phi = \left(-\frac{\pi^2}{\hbar^2} + \frac{i}{\hbar} \nabla^2 S_c\right)\phi , \tag{76}$$

the following time independent Schrödinger equation is obtained

$$E' \phi = -\frac{\hbar^2}{2m} \nabla^2 \phi + V\phi . \tag{77}$$

VI. Conclusion

We found the diamond operator, which is a 4 × 4 complex differential operator matrix as a square root of d'Alembertian without spin, although Dirac equation's operator matrix includes spin. The extended Maxwell's and the linear gravitational field equations are simply written by using the diamond operator. The linear gravitational field equations derive Klein-Gordon equation, time independent Schrödinger equation, and the principle of quantum mechanics. It was found that imaginary part of the energy creation-annihilation field creates or annihilates quantized interactive energy depending on the potential.

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Appendix

Since products of complex numbers $a + ib$ and $c + id$ is written by

$$\left(a+ib\right)\left(c+id\right)=ac-bd+i\left(bc+ad\right), \tag{A1}$$

$(a + ib)$ and $(c + id)$ can be regarded as matrix and vector, respectively as

$$a+ib=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{A2}$$

$$c+id\equiv\begin{pmatrix} c \\ d \end{pmatrix}. \tag{A3}$$

Because

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} ac-bd \\ bc+ad \end{pmatrix}. \tag{A4}$$

Therefore, imaginary unit i can be regarded as the 2×2 matrix of

$$i=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A5}$$

The complex conjugate operator $*$ can be also regarded as the 2×2 matrix of

$$*=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A6}$$

because

$$*\left(a+ib\right)=a-ib=\begin{pmatrix} a \\ -b \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \tag{A7}$$

Then product of i and $*$ is also regarded as the following 2×2 matrix

$$i*=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A8}$$

Here

$$\left(*\right)^2=\left(i*\right)^2=-i^2=1. \tag{A9}$$

When we define the following bracket operator

$$\left[A,B\right]=\begin{cases} 1 & \left(AB=BA\right) \\ 0 & \left(AB\neq\pm BA\right) \\ -1 & \left(AB=-BA\right) \end{cases}, \tag{A10}$$

we obtain

$$\left[i,*\right]=\left[*,i*\right]=\left[i*,i\right]=-1. \tag{A11}$$

Here, we define $n \times n$ real basis matrices b_n^μ (μ = 0, 1,…, n^2-1) as the real matrices whose linear combination can give all of $n \times n$ real matrices and square are equal to $\pm I_n$. Then, 1, i, $*$, and $i*$ are equivalent to the 2×2 real basis matrices b_2^μ given by $b_2^0 = I_2$ and

$$b_2^1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ b_2^2=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ b_2^3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A12}$$

where

$$\left(b_2^0\right)^2=\left(b_2^1\right)^2=-\left(b_2^2\right)^2=\left(b_2^3\right)^2=1, \tag{A13}$$

and for μ = 0, 1, 2, 3 and j, k = 1, 2, 3,

$$\left[b_2^0,b_2^\mu\right]=1,\quad \left[b_2^j,b_2^k\right]_{j\neq k}=-1. \tag{A14}$$

Now we define partial product of matrices A and B as

$$A\times B\equiv\begin{pmatrix} b_{11}A & . & . & . & b_{1n}A \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ b_{m1}A & . & . & . & b_{mn}A \end{pmatrix}, \tag{A15}$$

where

$$B \equiv \begin{pmatrix} b_{11} & . & . & . & b_{1n} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ b_{m1} & . & . & . & b_{mn} \end{pmatrix}. \tag{A16}$$

Then 4 × 4 real basis matrices can be written by

$$b_4^{\lambda} = b_2^{\mu} \times b_2^{\nu} \cdot \tag{A17}$$

If another 4 × 4 real matrix is written by

$$b_4^{\alpha} = b_2^{\beta} \times b_2^{\gamma} , \tag{A18}$$

the following relation is satisfied

$$b_4^{\lambda} \cdot b_4^{\alpha} = \left(b_2^{\mu} \cdot b_2^{\beta} \right) \times \left(b_2^{\nu} \cdot b_2^{\gamma} \right) \cdot \tag{A19}$$

Then

$$\left[b_4^{\lambda} \cdot b_4^{\alpha} \right] = \left[b_2^{\mu} \cdot b_2^{\beta} \right] \cdot \left[b_2^{\nu} \cdot b_2^{\gamma} \right] \cdot \tag{A20}$$

Since γ^{μ} and δ^{μ} are given by

$$\gamma^0 = b_2^0 \times b_2^3 , \tag{A21}$$

$$\gamma^1 = -b_2^1 \times b_2^2 , \tag{A22}$$

$$\gamma^2 = -ib_2^2 \times b_2^2 , \tag{A23}$$

$$\gamma^3 = -b_2^3 \times b_2^2 , \tag{A24}$$

and

$$\delta^0 = -b_2^0 \times b_2^0 * , \tag{A25}$$

$$\delta^1 = ib_2^1 \times b_2^2 * , \tag{A26}$$

$$\delta^2 = -ib_2^3 \times b_2^2 * , \tag{A27}$$

$$\delta^3 = ib_2^2 \times b_2^0 * , \tag{A28}$$

we obtain

$$\left(\gamma^0 \right)^2 = \left(\delta^0 \right)^2 = -\left(\gamma^{\mu} \right)^2_{\mu \neq 0} = -\left(\delta^{\mu} \right)^2_{\mu \neq 0} = 1 , \tag{A29}$$

$$\left[\gamma^{\mu}, \gamma^{\nu} \right]_{\mu \neq \nu} = \left[\delta^{\mu}, \delta^{\nu} \right]_{\mu \neq \nu} = -1 \cdot \tag{A30}$$

The diamond operator matrix without spin needs the additional commutation relation of * and *i* * compared with Dirac operator matrix with spin. It seems to be a kind of spontaneous symmetry breaking.