Sedeonic Duality-Invariant Field Equations for Dyons

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Abstract

We discuss the theoretical description of dyons having simultaneously both electric and magnetic charges on the basis of space-time algebra of sixteen-component sedeons. We show that the generalized sedeonic equations for electromagnetic field of dyons can be reformulated in equivalent canonical form as the equations for redefined field potentials, field strengths and sources. The relations for energy and momentum as well as the relations for Lorentz invariants of dyonic electromagnetic field are derived. Additionally, we discuss the sedeonic second-order Klein-Gordon and first-order Dirac wave equations describing the quantum behavior of dyons in an external dyonic electromagnetic field.

1 Introductoin

A dyon is a hypothetical point particle, which has simultaneously both elementary electric q_e and elementary magnetic q_m charges was proposed by J. Schwinger in 1969 [1]. In fact, the dyonic concept is a development of the idea of magnetic monopoles proposed previously by P.A.M. Dirac [2,3]. Magnetic monopoles as well as dyons increase the symmetry of the Maxwell equations. Taking into account the magnetic charges and corresponding magnetic currents the Maxwell equations for the electromagnetic field in a vacuum are represented in absolutely symmetric form [1]:

$$\begin{aligned}
&\nabla \cdot \mathbf{E} = 4\pi\rho_e, \\
&\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = -\frac{4\pi}{c}\mathbf{j}_e, \\
&\nabla \cdot \mathbf{H} = 4\pi\rho_m, \\
&\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = -\frac{4\pi}{c}\mathbf{j}_m.
\end{aligned} \tag{1}$$

Here ρ_e is a volume density of electric charge; $\mathbf{j}_e$ is a volume density of electric current; ρ_m is a volume density of magnetic charge and $\mathbf{j}_m$ is a volume density of magnetic current. These equations are invariant under the electromagnetic duality transformations for field strengths and sources [4]:

$$\begin{aligned}
&\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \\
&\mathbf{j}_e \to \mathbf{j}_m, \quad \mathbf{j}_m \to -\mathbf{j}_e, \\
&\rho_e \to \rho_m, \quad \rho_m \to -\rho_e.
\end{aligned} \tag{2}$$

In recent years, there have been a few publications devoted to the reformulation of equations for electromagnetic field in terms of hypercomplex field potentials. The first approach is based on four-component quaternions, which consist of scalar and vector parts that adequately describe the four-vector concept of special relativity [5-8]. In particular quaternions were applied for the description

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of dyons [9-11]. However, since the system of Maxwell equations consists of four equations for scalar, pseudoscalar, vector and pseudovector values, the application of eight-component algebras is more appropriate. Taking into account this spatial symmetry several approaches have been proposed to describe the electromagnetic field on the basis of eight-component octonions [12-16] and octons [17-19]. Particularly these algebras were used for the description of dyonic field [20-23]. However, a consistent relativistic consideration implies equally the space and time symmetries that require using the extended sixteen-component space-time algebras. Recently we proposed the space-time algebra of sixteen-component sedeons, which takes into account the symmetry of physical values with respect to the space-time inversion and realizes the scalar-vector representation of Poincare group [24]. In particular, we considered the equations for massive and massless fields based on sedeonic potentials and space-time operators [25-27]. In the present paper we consider the application of sedeonic algebra to the description of dyonic electromagnetic field and to the reformulation of relativistic quantum equations for dyons in an external electromagnetic field.

2 Algebra of space-time sedeons

To begin with we shortly recall the main properties of sedeons [24]. The algebra of sedeons encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars (A) and absolute vectors ($\vec{A}$) are not transformed under spatial and time inversion.
2. Time scalars ($B_{\mathbf{t}}$) and time vectors ($\vec{B}_{\mathbf{t}}$) change sign under time inversion and are not transformed under spatial inversion.
3. Space scalars ($C_{\mathbf{r}}$) and space vectors ($\vec{C}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
4. Space-time scalars ($D_{\mathbf{tr}}$) and space-time vectors ($\vec{D}_{\mathbf{tr}}$) change sign under spatial and time inversion.

The indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. The space-time sedeon $\tilde{\mathbf{S}}$ is defined by the following expression:

$$\tilde{\mathbf{S}} = A + \vec{A} + B_{\mathbf{t}} + \vec{B}_{\mathbf{t}} + C_{\mathbf{r}} + \vec{C}_{\mathbf{r}} + D_{\mathbf{tr}} + \vec{D}_{\mathbf{tr}}. \tag{3}$$

The components of sedeon (3) can be written in the sedeonic space-time basis as

$$\begin{aligned}
&A = \mathbf{e_0} A \mathbf{a_0}, \\
&\vec{A} = \mathbf{e_0}\left(A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}\right), \\
&B_{\mathbf{t}} = \mathbf{e_t} B \mathbf{a_0}, \\
&\vec{B}_{\mathbf{t}} = \mathbf{e_t}\left(B_1\mathbf{a_1} + B_2\mathbf{a_2} + B_3\mathbf{a_3}\right), \\
&C_{\mathbf{r}} = \mathbf{e_r} C \mathbf{a_0}, \\
&\vec{C}_{\mathbf{r}} = \mathbf{e_r}\left(C_1\mathbf{a_1} + C_2\mathbf{a_2} + C_3\mathbf{a_3}\right), \\
&D_{\mathbf{tr}} = \mathbf{e_{tr}} D \mathbf{a_0}, \\
&\vec{D}_{\mathbf{tr}} = \mathbf{e_{tr}}\left(D_1\mathbf{a_1} + D_2\mathbf{a_2} + D_3\mathbf{a_3}\right),
\end{aligned} \tag{4}$$

where values $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are scalar-vector basis ($\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis) and values $\mathbf{e_0}$, $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are space-time basis ($\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_t}$ ($\mathbf{e_t} \equiv \mathbf{e_1}$) is a time unit; $\mathbf{e_r}$ ($\mathbf{e_r} \equiv \mathbf{e_2}$) is a space unit; $\mathbf{e_{tr}}$ ($\mathbf{e_{tr}} \equiv \mathbf{e_3}$) is a space-time unit). Going forward we will omit the units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for the sedeonic absolute unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and space-time units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are presented in the tables 1 and 2 respectively (i is the imaginary unit ($i^2 = -1$)). Note that sedeonic units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ commute with $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$.

Table 1: The rules of multiplication for absolute unit vectors

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2: The rules of multiplication for space-time units

	$\mathbf{e_t}$	$\mathbf{e_r}$	$\mathbf{e_{tr}}$
$\mathbf{e_t}$	1	$i\mathbf{e_{tr}}$	$-i\mathbf{e_r}$
$\mathbf{e_r}$	$-i\mathbf{e_{tr}}$	1	$i\mathbf{e_t}$
$\mathbf{e_{tr}}$	$i\mathbf{e_r}$	$-i\mathbf{e_t}$	1

Thus the sedeon $\tilde{\mathbf{S}}$ is understood as a compound space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

Further we assume the sedeonic multiplication of vectors. For example, the sedeonic product of two absolute vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B}\right) + \left[\vec{A} \times \vec{B}\right]. \tag{5}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A} \cdot \vec{B}\right) = A_1 B_1 + A_2 B_2 + A_3 B_3, \tag{6}$$

and sedeonic vector multiplication (external product) by symbol "$\times$" and square brackets

$$\left[\vec{A} \times \vec{B}\right] = i\left(A_2 B_3 - A_3 B_2\right)\mathbf{a_1} + i\left(A_3 B_1 - A_1 B_3\right)\mathbf{a_2} + i\left(A_1 B_2 - A_2 B_1\right)\mathbf{a_3}. \tag{7}$$

Note that in sedeonic algebra the definition of vector product is differed from the analogous expression in Gibbs-Heaviside vector algebra. For the transition to the common used vector algebra the change $i[\vec{\nabla} \times \vec{A}] \Rightarrow -[\nabla \times \mathbf{A}]$ should be made in all vector expressions.

3 The equations for electromagnetic field of dyons

The sedeonic wave equation for electromagnetic field of electric and magnetic charges can be written in the following form [26, 27]:

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}}. \tag{8}$$

Here $\tilde{\mathbf{W}}$ is the sedeon of electromagnetic field potential

$$\tilde{\mathbf{W}} = i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e, \tag{9}$$

where φ_e is electric scalar potential, φ_m is magnetic scalar potential, $\vec{A}_e$ is electric vector potential, $\vec{A}_m$ is magnetic vector potential. We use the following sedeonic definition of Hamilton nabla operator and vectors $\vec{A}_e$ and $\vec{A}_m$:

$$\vec{\nabla} = \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3,$$
$$\vec{A}_e = A_{e1}\mathbf{a}_1 + A_{e2}\mathbf{a}_2 + A_{e3}\mathbf{a}_3, \tag{10}$$
$$\vec{A}_m = A_{m1}\mathbf{a}_1 + A_{m2}\mathbf{a}_2 + A_{m3}\mathbf{a}_3.$$

The sedeonic source is

$$\tilde{\mathbf{J}} = -i\mathbf{e}_1 4\pi\rho_e - \mathbf{e}_2\frac{4\pi}{c}\vec{j}_e + i\mathbf{e}_2 4\pi\rho_m - \mathbf{e}_1\frac{4\pi}{c}\vec{j}_m, \tag{11}$$

where ρ_e is a volume density of electric charge, $\vec{j}_e$ is a density of electric current, ρ_m is a volume density of magnetic charge and $\vec{j}_m$ is a density of magnetic current. The electric and magnetic field strengths are defined as

$$\begin{aligned}\vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t} - \vec{\nabla}\varphi_e + i\left[\vec{\nabla}\times\vec{A}_m\right],\\ \vec{H} &= -\frac{1}{c}\frac{\partial\vec{A}_m}{\partial t} - \vec{\nabla}\varphi_m - i\left[\vec{\nabla}\times\vec{A}_e\right].\end{aligned} \tag{12}$$

The potentials satisfy the Lorentz gauge conditions

$$\frac{1}{c}\frac{\partial\varphi_e}{\partial t} + \left(\vec{\nabla}\cdot\vec{A}_e\right) = 0, \tag{13}$$

$$\frac{1}{c}\frac{\partial\varphi_m}{\partial t} + \left(\vec{\nabla}\cdot\vec{A}_m\right) = 0. \tag{14}$$

Then we have

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e\right) = \mathbf{e}_3\vec{E} - i\vec{H}, \tag{15}$$

and the wave equation (8) is rewritten as

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{E} - i\vec{H}\right) = -i\mathbf{e}_1 4\pi\rho_e - \mathbf{e}_2\frac{4\pi}{c}\vec{j}_e + i\mathbf{e}_2 4\pi\rho_m - \mathbf{e}_1\frac{4\pi}{c}\vec{j}_m. \tag{16}$$

Producing action of the operator on the left side of this equation and separating the values with different space-time properties, we obtain the system of Maxwell's equations for electric and magnetic charges

$$\begin{aligned}&\left(\nabla\cdot\vec{E}\right) = 4\pi\rho_e,\\ &\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + i\left[\nabla\times\vec{H}\right] = -\frac{4\pi}{c}\vec{j}_e,\\ &\left(\nabla\cdot\vec{H}\right) = 4\pi\rho_m,\\ &\frac{1}{c}\frac{\partial\vec{H}}{\partial t} - i\left[\nabla\times\vec{E}\right] = -\frac{4\pi}{c}\vec{j}_m.\end{aligned} \tag{17}$$

Let us further consider only dyonic fields and sources. Note that the electric and magnetic properties of dyons are not independent, since their electric and magnetic charges belong to the same point particle. The Coulomb force between two point bodies charged with dyons can be written as

$$\vec{F}_{1,2} = \frac{Q_{e1}Q_{e2}}{r^2} + \frac{Q_{m1}Q_{m2}}{r^2}, \tag{18}$$

where Q_e is dyon elictical charge, Q_m is dyon magnetic charge and r is the distatnce between bodies. Using elementary q_e and q_m charges the expression (18) can be rewritten as

$$\vec{F}_{1,2} = q_e^2\frac{N_1N_2}{r^2} + q_m^2\frac{N_1N_2}{r^2}, \tag{19}$$

or

$$\vec{F}_{1,2} = (q_e^2 + q_m^2)\frac{N_1 N_2}{r^2}, \tag{20}$$

where N_1 and N_2 are the numbers of dyons on first and second body. Then the Coulomb force (20) can be represented as the electrostatic interaction between two electrically charged bodies with elementary charge

$$q = \sqrt{(q_e^2 + q_m^2)}, \tag{21}$$

as

$$\vec{F}_{1,2} = q^2\frac{N_1 N_2}{r^2}. \tag{22}$$

Therefore, for dyons the following relations hold [28]:

$$\begin{aligned} \frac{1}{q_e}\rho_e &= \frac{1}{q_m}\rho_m, \\ \frac{1}{q_e}\vec{j}_e &= \frac{1}{q_m}\vec{j}_m, \\ \frac{1}{q_e}\varphi_e &= \frac{1}{q_m}\varphi_m, \\ \frac{1}{q_e}\vec{A}_e &= \frac{1}{q_m}\vec{A}_m. \end{aligned} \tag{23}$$

To describe the dyons it is convenient to introduce new sources:

$$\begin{aligned} \rho &= \frac{q}{q_e}\rho_e = \frac{q}{q_m}\rho_m, \\ \vec{j} &= \frac{q}{q_e}\vec{j}_e = \frac{q}{q_m}\vec{j}_m, \end{aligned} \tag{24}$$

and field potentials:

$$\begin{aligned} \varphi &= \frac{q}{q_e}\varphi_e = \frac{q}{q_m}\varphi_m, \\ \vec{A} &= \frac{q}{q_e}\vec{A}_e = \frac{q}{q_m}\vec{A}_m. \end{aligned} \tag{25}$$

Then taking into account (24) and (25) the field potential (9) and source (11) can be rewritten as

$$\begin{aligned} \tilde{\mathbf{W}} &= \left(i\mathbf{e}_1\frac{q_e}{q} - i\mathbf{e}_2\frac{q_m}{q}\right)\varphi + \left(\mathbf{e}_1\frac{q_m}{q} + \mathbf{e}_2\frac{q_e}{q}\right)\vec{A} \\ &= \left(i\mathbf{e}_1\varphi + \mathbf{e}_2\vec{A}\right)\left(\frac{q_e}{q} - i\mathbf{e}_3\frac{q_m}{q}\right), \end{aligned} \tag{26}$$

$$\begin{aligned} \tilde{\mathbf{J}} &= -4\pi\left(i\mathbf{e}_1\frac{q_e}{q} - i\mathbf{e}_2\frac{q_m}{q}\right)\rho - \frac{4\pi}{c}\left(\mathbf{e}_1\frac{q_m}{q} + \mathbf{e}_2\frac{q_e}{q}\right)\vec{j} \\ &= -4\pi\left(i\mathbf{e}_1\rho + \mathbf{e}_2\frac{1}{c}\vec{j}\right)\left(\frac{q_e}{q} - i\mathbf{e}_3\frac{q_m}{q}\right). \end{aligned} \tag{27}$$

Substituting (26) and (27) in the wave equation (8) and multiplying on $\left(\frac{q_e}{q} + i\mathbf{e}_3\frac{q_m}{q}\right)$ from the right we obtain the following wave equation:

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi + \mathbf{e}_2\vec{A}\right) = -4\pi\left(i\mathbf{e}_1\rho + \mathbf{e}_2\frac{1}{c}\vec{j}\right). \tag{28}$$

Let us introduce new electric and magnetic field strengths of dyons

$$\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi, \tag{29}$$

$$\vec{\mathcal{H}} = -i[\vec{\nabla} \times \vec{A}]. \tag{30}$$

Since dyonic potentials satisfy the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \left(\vec{\nabla} \cdot \vec{A}\right) = 0, \tag{31}$$

we have

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla}\right)\left(i\mathbf{e}_1 \varphi + \mathbf{e}_2 \vec{A}\right) = \mathbf{e}_3 \vec{\mathcal{E}} - i\vec{\mathcal{H}}, \tag{32}$$

and the wave equation (28) is rewritten as

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla}\right)\left(\mathbf{e}_3 \vec{\mathcal{E}} - i\vec{\mathcal{H}}\right) = -4\pi i\mathbf{e}_1 \rho - \mathbf{e}_2 \frac{4\pi}{c}\vec{j}. \tag{33}$$

Producing action of the operator on the left side of this equation and separating the values with different space-time properties, we obtain the system of Maxwell's equations for dyons in the following canonical form:

$$\begin{aligned}
&\left(\nabla \cdot \vec{\mathcal{E}}\right) = 4\pi\rho, \\
&\frac{1}{c}\frac{\partial \vec{\mathcal{E}}}{\partial t} + i\left[\nabla \times \vec{\mathcal{H}}\right] = -\frac{4\pi}{c}\vec{j}, \\
&\left(\nabla \cdot \vec{\mathcal{H}}\right) = 0, \\
&\frac{1}{c}\frac{\partial \vec{\mathcal{H}}}{\partial t} - i\left[\nabla \times \vec{\mathcal{E}}\right] = 0.
\end{aligned} \tag{34}$$

There are the simple relations between $\vec{E}$, $\vec{H}$ and $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$ field strengths. Taking into account (25) the definitions (12) can be rewritten as

$$\begin{aligned}
\vec{E} &= -\frac{q_e}{q}\left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi\right) + i\frac{q_m}{q}\left[\vec{\nabla} \times \vec{A}\right], \\
\vec{H} &= -\frac{q_m}{q}\left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi\right) - i\frac{q_e}{q}\left[\vec{\nabla} \times \vec{A}\right].
\end{aligned} \tag{35}$$

Multiplying these relations respectively on q_e and q_m we have

$$q_e\vec{E} + q_m\vec{H} = -\frac{\left(q_e^2 + q_m^2\right)}{q}\left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi\right) = q\vec{\mathcal{E}} \tag{36}$$

and

$$-q_m\vec{E} + q_e\vec{H} = -\frac{\left(q_e^2 + q_m^2\right)}{q} i\left[\vec{\nabla} \times \vec{A}\right] = q\vec{\mathcal{H}}. \tag{37}$$

From these expressions we obtain the following relations between the field strengths:

$$\vec{\mathcal{E}} = \frac{1}{q}\left(q_e\vec{E} + q_m\vec{H}\right) \tag{38}$$

and

$$\vec{\mathcal{H}} = \frac{1}{q}\left(-q_m\vec{E} + q_e\vec{H}\right). \tag{39}$$

4 The energy and momentum of dyon electromagnetic field

Multiplying (33) on $\left(\mathbf{e}_3\vec{\mathcal{E}} - i\vec{\mathcal{H}}\right)$ from the left we have

$$\left(\mathbf{e}_3\vec{\mathcal{E}} - i\vec{\mathcal{H}}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{\mathcal{E}} - i\vec{\mathcal{H}}\right) = -\left(\mathbf{e}_3\vec{\mathcal{E}} - i\vec{\mathcal{H}}\right)\left(4\pi i\mathbf{e}_1\rho + \mathbf{e}_2\frac{4\pi}{c}\vec{j}\right). \tag{40}$$

Equating in (40) the components with different space-time properties we get

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2\right) - i\frac{c}{4\pi}\left(\vec{\nabla}\cdot\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right]\right) + \left(\vec{\mathcal{E}}\cdot\vec{j}\right) = 0, \tag{41}$$

$$\begin{gathered}\frac{1}{8\pi}\vec{\nabla}\left(\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2\right) - i\frac{1}{4\pi c}\frac{\partial}{\partial t}\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right] \\ -\frac{1}{4\pi}\left\{\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)\vec{\mathcal{E}} + \left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)\vec{\mathcal{H}}\right\} + \rho\vec{\mathcal{E}} + i\left[\vec{\mathcal{H}}\times\vec{j}\right] = 0,\end{gathered} \tag{42}$$

$$\begin{gathered}\frac{1}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\frac{\partial\vec{\mathcal{H}}}{\partial t}\right) - \left(\vec{\mathcal{H}}\cdot\frac{\partial\vec{\mathcal{E}}}{\partial t}\right)\right\} \\ -i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right) + \left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)\right\} - \left(\vec{\mathcal{H}}\cdot\vec{j}\right) = 0,\end{gathered} \tag{43}$$

$$\begin{gathered}-i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right] + \left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]\right\} + \frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right) - \vec{\mathcal{H}}\cdot\left(\vec{\nabla}\times\vec{\mathcal{E}}\right)\right\} \\ +\frac{c}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right] - \left[\vec{\mathcal{H}}\times\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right]\right\} + c\rho\vec{\mathcal{H}} - i\left[\vec{\mathcal{E}}\times\vec{j}\right] = 0.\end{gathered} \tag{44}$$

The expression(41) is the Poynting theorem for dyons. The value

$$\mathcal{W} = \frac{\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2}{8\pi} \tag{45}$$

is the volume density of field energy, while vector

$$\vec{\mathcal{P}} = -i\frac{c}{4\pi}\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right] \tag{46}$$

is the vector of energy flux density (Poynting vector). Using (38) and (39) one can see that

$$\mathcal{W} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{H}^2\right), \tag{47}$$

and

$$\vec{\mathcal{P}} = -i\frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]. \tag{48}$$

We see that these values are the volume density of energy and energy flux density of electromagnetic field.

5 Relations for Lorentz invariants of dyon electromagnetic field

Using sedeonic algebra it is easy to derive the relations for the values

$$\mathcal{I}_1 = \left(\vec{\mathcal{E}}^2 - \vec{\mathcal{H}}^2\right), \tag{49}$$

and

$$\mathcal{I}_2 = \left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right), \tag{50}$$

which are the Lorentz invariants of dyon electromagnetic field (the Lorentz transformations for sedeons are considered in [24, 29]). Indeed, substituting (38) and (39) we have

$$\mathcal{I}_1 = \frac{4q_e q_m}{(q_e^2+q_m^2)}\left(\vec{E}\cdot\vec{H}\right) + \frac{(q_e^2-q_m^2)}{(q_e^2+q_m^2)}\left(\vec{E}^2-\vec{H}^2\right), \tag{51}$$

and

$$\mathcal{I}_2 = \frac{(q_e^2-q_m^2)}{(q_e^2+q_m^2)}\left(\vec{E}\cdot\vec{H}\right) - \frac{q_e q_m}{(q_e^2+q_m^2)}\left(\vec{E}^2-\vec{H}^2\right). \tag{52}$$

Thus one can see that $\mathcal{I}_1$ and $\mathcal{I}_2$ are the combinations of Lorentz invariants of the electromagnetic field .

Multiplying both parts of equation (33) on sedeon $\left(\mathbf{e}_3\vec{\mathcal{E}}+i\vec{\mathcal{H}}\right)$ from the left we have:

$$\left(\mathbf{e}_3\vec{\mathcal{E}}+i\vec{\mathcal{H}}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right) = -\left(\mathbf{e}_3\vec{\mathcal{E}}+i\vec{\mathcal{H}}\right)\left(4\pi i\mathbf{e}_1\rho+\mathbf{e}_2\frac{4\pi}{c}\vec{j}\right), \tag{53}$$

Equating in (53) the components with different space-time properties we obtain the following relations for the Lorentz invariants of dyon electromagnetic field:

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{\mathcal{E}}^2-\vec{\mathcal{H}}^2\right)+i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)+\left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right)\right\}+\left(\vec{\mathcal{E}}\cdot\vec{j}\right)=0, \tag{54}$$

$$\begin{aligned}\frac{c}{8\pi}\vec{\nabla}\left(\vec{\mathcal{E}}^2-\vec{\mathcal{H}}^2\right) &= i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]+\left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right]\right\}\\ &+\frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)+\left(\vec{\mathcal{E}}\cdot\vec{\nabla}\right)\vec{\mathcal{E}}-\vec{\mathcal{H}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)-\left(\vec{\mathcal{H}}\cdot\vec{\nabla}\right)\vec{\mathcal{H}}\right\}\\ &-c\rho\vec{\mathcal{E}}+i\left[\vec{\mathcal{H}}\times\vec{j}\right],\end{aligned} \tag{55}$$

$$\frac{1}{4\pi}\frac{\partial}{\partial t}\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right)=i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right)-\left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)\right\}-\left(\vec{\mathcal{H}}\cdot\vec{j}\right), \tag{56}$$

$$\begin{aligned}\frac{c}{4\pi}\vec{\nabla}\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right) &= -i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right]-\left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]\right\}+i\left[\vec{\mathcal{E}}\times\vec{j}\right]\\ &+\frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)+\vec{\mathcal{H}}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)-(\vec{\mathcal{E}}\cdot\vec{\nabla})\vec{\mathcal{H}}+(\vec{\mathcal{H}}\cdot\vec{\nabla})\vec{\mathcal{E}}\right\}-c\rho\vec{\mathcal{H}}.\end{aligned} \tag{57}$$

6 Sedeonic Klein-Gordon equation for dyons

The sedeonic wave equation for the quantum particle with electric charge q_e and magnetic charge q_m in an external electromagnetic field described by electric φ_e, $\vec{A}_e$ and magnetic φ_m, $\vec{A}_m$ potentials is obtained from the equation for free particle [30, 31] by the following replacements:

$$\begin{aligned}\frac{\partial}{\partial t} &\to \frac{\partial}{\partial t}+\frac{i}{\hbar}q_e\varphi_e+\frac{i}{\hbar}q_m\varphi_m,\\ \vec{\nabla} &\to \vec{\nabla}-\frac{i}{c\hbar}q_e\vec{A}_e-\frac{i}{c\hbar}q_m\vec{A}_m,\end{aligned} \tag{58}$$

and can be written as

$$\left\{i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q_e\varphi_e+\frac{i}{\hbar}q_m\varphi_m\right)-\mathbf{e}_2\left(\vec{\nabla}-\frac{i}{c\hbar}q_e\vec{A}_e-\frac{i}{c\hbar}q_m\vec{A}_m\right)-i\mathbf{e}_3\frac{mc}{\hbar}\right\}^2\tilde{\mathbf{V}}=0. \tag{59}$$

Here $\tilde{\mathbf{V}}$ is the sedeonic wave function and m is the mass of particle. For dyon in dyonic electromagnetic field this equation can be rewritten in simplified form. Taking into account the relations for dyonic potentials (25) we obtain

$$\left\{i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q\varphi\right)-\mathbf{e}_2\left(\vec{\nabla}-\frac{i}{c\hbar}q\vec{A}\right)-i\mathbf{e}_3\frac{mc}{\hbar}\right\}^2\tilde{\mathbf{V}}=0. \tag{60}$$

Producing the action of operators on the left side of equation (60) we obtain

$$\begin{aligned}&\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\triangle+\frac{m^2c^2}{\hbar^2}+\frac{i2q}{c\hbar}\left(\frac{\varphi}{c}\frac{\partial}{\partial t}+(\vec{A}\cdot\vec{\nabla})\right)+\frac{q^2}{c^2\hbar^2}(\vec{A}^2-\varphi^2)\right\}\tilde{\mathbf{V}}\\&+\mathbf{e}_3\frac{iq}{c\hbar}\vec{\mathcal{E}}\tilde{\mathbf{V}}-\frac{q}{c\hbar}\vec{\mathcal{H}}\tilde{\mathbf{V}}=0.\end{aligned} \tag{61}$$

In this equation the term

$$\mathbf{e}_3\frac{iq}{c\hbar}\vec{\mathcal{E}}\tilde{\mathbf{V}} \tag{62}$$

describes the interaction of dyon with dyonic electric field, while the term

$$\frac{q}{c\hbar}\vec{\mathcal{H}}\tilde{\mathbf{V}} \tag{63}$$

describes the interaction of dyon with dyonic magnetic field.

The sedeonic wave equation (60) can be rewritten as the system of Maxwell-like equations for some quantum field [24]. Let us introduce for brevity the following new operators:

$$\begin{aligned}\partial_0&=\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q\varphi\right),\\\vec{\nabla}_0&=\left(\vec{\nabla}-\frac{i}{c\hbar}q\vec{A}\right),\\m_0&=\frac{c}{\hbar}m.\end{aligned} \tag{64}$$

Then the wave equation (60) is rewritten in the following compact form:

$$(i\mathbf{e}_1\partial_0-\mathbf{e}_2\nabla_0-i\mathbf{e}_3m_0)\,(i\mathbf{e}_1\partial_0-\mathbf{e}_2\nabla_0-i\mathbf{e}_3m_0)\,\tilde{\mathbf{V}}=0. \tag{65}$$

On the other hand, let us define a field strength $\tilde{\mathbf{G}}$ according to

$$\tilde{\mathbf{G}}=(i\mathbf{e}_1\partial_0-\mathbf{e}_2\nabla_0-i\mathbf{e}_3m_0)\,\tilde{\mathbf{V}}. \tag{66}$$

Then the wave equation (65) takes the following equivalent form:

$$(i\mathbf{e}_1\partial_0-\mathbf{e}_2\nabla_0-i\mathbf{e}_3m_0)\,\tilde{\mathbf{G}}=0. \tag{67}$$

Let us choose the wave function in the following form [26]:

$$\tilde{\mathbf{V}}=iV_1\mathbf{e}_1-iV_2\mathbf{e}_2+V_3-iV_4\mathbf{e}_3+\vec{V}_1\mathbf{e}_2+\vec{V}_2\mathbf{e}_1-\vec{V}_3\mathbf{e}_3+i\vec{V}_4, \tag{68}$$

and the field strength $\tilde{\mathbf{G}}$ as

$$\tilde{\mathbf{G}}=-g_1+ig_2\mathbf{e}_3+ig_3\mathbf{e}_1-ig_4\mathbf{e}_2+\vec{G}_1\mathbf{e}_3-i\vec{G}_2+\vec{G}_3\mathbf{e}_2+\vec{G}_4\mathbf{e}_1. \tag{69}$$

Then taking into account (66) we have the following definitions of $\tilde{\mathbf{G}}$ components through the components of the wave function $\tilde{\mathbf{V}}$:

$$\begin{aligned}g_1&=\partial_0V_1+\left(\vec{\nabla}_0\cdot\vec{V}_1\right)+m_0V_4,\\g_2&=\partial_0V_2+\left(\vec{\nabla}_0\cdot\vec{V}_2\right)-m_0V_3,\\g_3&=\partial_0V_3+\left(\vec{\nabla}_0\cdot\vec{V}_3\right)+m_0V_2,\\g_4&=\partial_0V_4+\left(\vec{\nabla}_0\cdot\vec{V}_4\right)-m_0V_1,\\\vec{G}_1&=-\partial_0\vec{V}_1-\vec{\nabla}_0V_1+i\left[\vec{\nabla}_0\times\vec{V}_2\right]+m_0\vec{V}_4,\\\vec{G}_2&=-\partial_0\vec{V}_2-\vec{\nabla}_0V_2-i\left[\vec{\nabla}_0\times\vec{V}_1\right]-m_0\vec{V}_3,\\\vec{G}_3&=-\partial_0\vec{V}_3-\vec{\nabla}_0V_3-i\left[\vec{\nabla}_0\times\vec{V}_4\right]+m_0\vec{V}_2,\\\vec{G}_4&=-\partial_0\vec{V}_4-\vec{\nabla}_0V_4+i\left[\vec{\nabla}_0\times\vec{V}_3\right]-m_0\vec{V}_1.\end{aligned} \tag{70}$$

Separating in wave equation (67) the values with different space-time properties we obtain the system of Maxwell-like equations for quantum field $\tilde{\mathbf{G}}$:

$$\begin{aligned}
&\partial_0 g_1 + \left(\vec{\nabla}_0 \cdot \vec{G}_1\right) - m_0 g_4 = 0,\\
&\partial_0 g_2 + \left(\vec{\nabla}_0 \cdot \vec{G}_2\right) + m_0 g_3 = 0,\\
&\partial_0 g_3 + \left(\vec{\nabla}_0 \cdot \vec{G}_3\right) - m_0 g_2 = 0,\\
&\partial_0 g_4 + \left(\vec{\nabla}_0 \cdot \vec{G}_4\right) + m_0 g_1 = 0,\\
&\partial_0 \vec{G}_1 + \vec{\nabla}_0 g_1 + i\left[\vec{\nabla}_0 \times \vec{G}_2\right] + m_0 \vec{G}_4 = 0,\\
&\partial_0 \vec{G}_2 + \vec{\nabla}_0 g_2 - i\left[\vec{\nabla}_0 \times \vec{G}_1\right] - m_0 \vec{G}_3 = 0,\\
&\partial_0 \vec{G}_3 + \vec{\nabla}_0 g_3 - i\left[\vec{\nabla}_0 \times \vec{G}_4\right] + m_0 \vec{G}_2 = 0,\\
&\partial_0 \vec{G}_4 + \vec{\nabla}_0 g_4 + i\left[\vec{\nabla}_0 \times \vec{G}_3\right] - m_0 \vec{G}_1 = 0.
\end{aligned} \tag{71}$$

Multiplying each of the equations (71) to the corresponding field strength and adding these equations to each other, we obtain that

$$\partial_0 W + \left(\vec{\nabla}_0 \cdot \vec{P}\right) = 0, \tag{72}$$

where

$$W = g_1^2 + g_2^2 + g_3^2 + g_4^2 + \vec{G}_1^2 + \vec{G}_2^2 + \vec{G}_3^2 + \vec{G}_4^2 \tag{73}$$

and

$$\vec{P} = g_1\vec{G}_1 + g_2\vec{G}_2 + g_3\vec{G}_3 + g_4\vec{G}_4 - i\left[\vec{G}_1 \times \vec{G}_2\right] + i\left[\vec{G}_3 \times \vec{G}_4\right] \tag{74}$$

are analogues of volume density of energy and flux energy respectively [25]. The expression (72) is the analog of Poynting theorem for quantum field $\tilde{\mathbf{G}}$ describing the quantum dyon in an external dyonic electromagnetic field.

7 Sedeonic Dirac equation for dyons

The equation (65) admits a special class of solutions that are described by the sedeonic first-order wave equation [25]:

$$\left(i\mathbf{e}_1\partial_0 - \mathbf{e}_2\nabla_0 - i\mathbf{e}_3 m_0\right)\tilde{\mathbf{V}} = 0. \tag{75}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\partial_0 V_1 + \left(\vec{\nabla}_0 \cdot \vec{V}_1\right) + m_0 V_4 = 0,\\
&\partial_0 V_2 + \left(\vec{\nabla}_0 \cdot \vec{V}_2\right) - m_0 V_3 = 0,\\
&\partial_0 V_3 + \left(\vec{\nabla}_0 \cdot \vec{V}_3\right) + m_0 V_2 = 0,\\
&\partial_0 V_4 + \left(\vec{\nabla}_0 \cdot \vec{V}_4\right) - m_0 V_1 = 0,\\
&-\partial_0 \vec{V}_1 - \vec{\nabla}_0 V_1 + i\left[\vec{A}_0 \times \vec{V}_2\right] + m_0 \vec{V}_4 = 0,\\
&-\partial_0 \vec{V}_2 - \vec{\nabla}_0 V_2 - i\left[\vec{A}_0 \times \vec{V}_1\right] - m_0 \vec{V}_3 = 0,\\
&-\partial_0 \vec{V}_3 - \vec{\nabla}_0 V_3 - i\left[\vec{A}_0 \times \vec{V}_4\right] + m_0 \vec{V}_2 = 0,\\
&-\partial_0 \vec{V}_4 - \vec{\nabla}_0 V_4 + i\left[\vec{A}_0 \times \vec{V}_3\right] - m_0 \vec{V}_1 = 0.
\end{aligned} \tag{76}$$

In fact, the dyons described by equation (75) do not create a quantum field $\tilde{\mathbf{G}}$, since

$$\tilde{\mathbf{G}} = (i\mathbf{e}_1\partial_0 - \mathbf{e}_2\nabla_0 - i\mathbf{e}_3 m_0)\,\tilde{\mathbf{V}}. \tag{77}$$

It is clear that increasing the order of the equation (75), namely acting on it by the operator $(i\mathbf{e}_1\partial_0 - \mathbf{e}_2\nabla_0 - i\mathbf{e}_3 m_0)$ from the left, we arrive at the second-order wave equation (65).

8 Discussion

The hypothesis of dyons, which have simultaneously both elementary electric q_e and elementary magnetic q_m charges enables considering the electric and magnetic phenomena from the same unified positions. But as we have shown the introduction of unified sources ρ, $\vec{j}$ and unified field strengths $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ leads us to the classical Maxwell equations (34) for the dyonic electromagnetic field. Besides it is amazing that the expressions for volume density of electromagnetic field energy and Poynting vector for the vectors $\vec{E}, \vec{H}$ and $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$ are the same. Therefore one can suppose that electron is dyon and elementary electric charge e is q and can be represented as

$$e = \sqrt{{q_e}^2 + {q_m}^2}. \tag{78}$$

Then the Coulomb force between two electrically charged point bodies is

$$\vec{F}_{1,2} = e^2 \frac{N_1 N_2}{r^2} \tag{79}$$

and can be rewritten as

$$\vec{F}_{1,2} = q_e^2 \frac{N_1 N_2}{r^2} + q_m^2 \frac{N_1 N_2}{r^2}. \tag{80}$$

On the other side, according (42) the volume density of force $\vec{f}$ is equal to

$$\vec{f} = \rho\vec{\mathcal{E}} + i\left[\vec{\mathcal{H}} \times \vec{j}\right] = 0, \tag{81}$$

The value ρ plays the role of volume density of dyonic charge:

$$\rho = qn, \tag{82}$$

where n is the volume density of dyons. The value $\vec{j}$ is the volume density of dyonic charge current:

$$\vec{j} = q\vec{i}, \tag{83}$$

where $\vec{i}$ is the density of flow of dyons. Substituting in (81) the expressions for $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ from (47) and (48) we have

$$\vec{f} = \rho_e\vec{E} + \rho_m\vec{H} + i\left[\vec{H} \times \vec{j}_e\right] - i\left[\vec{E} \times \vec{j}_m\right] = 0. \tag{84}$$

This is generalized Lorentz force for dyon in dyonic electromagnetic field. The same role the charge q plays in the quantum equations (61) and (75) Thus from the experimental point of view the assumptions of only electrical or dyonic nature of electron have the same consequences. The hypothesis that electrons and protons are dyons should be additionally investigated.

9 Conclusion

The advantage of sedeons is that the sedeonic space-time algebra provides the Lorentz invariance of the equations and can be correctly applied for the description of classical and quantum fields. We have shown that the sedeonic wave equation for electromagnetic field of dyons can be reformulated in equivalent form as the system of classical Maxwell equations for the field strengths $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$ and sources ρ, $\vec{j}$. The relations for energy and momentum of dyonic electromagnetic field (Poynting theorem) as well as the relations for Lorentz invariants have been derived. Also we have shown that the sedeonic second-order Klein-Gordon equation describing the quantum behavior of dyons in an external dyonic electromagnetic field can be reformulated in the form of Maxwell-like equations for the quantum field $\tilde{\mathbf{G}}$ and the analogue of Poynting theorem for this field have been derived. Additionally we shown that the sedeonic first-order Dirac wave equation describes the dyons, which do not create the quantum field $\tilde{\mathbf{G}}$.

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