The internal structure of natural numbers and one method for the definition of large prime numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: ''Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

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1 INTRODUCTION

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p=1\cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: ''Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

2 THE SEQUENCE $\mu(k,n)$

We consider the sequence of natural numbers

$$\mu(k,n) = k + (k+1) + (k+2) + \ldots + (k+n) = \frac{(n+1)(2k+n)}{2}$$
$$k \in \mathbb{N}^* = \{1,2,3,\ldots\} \qquad . \qquad (2.1)$$
$$n \in A = \{2,3,4,\ldots\}$$

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1.

'' For the sequence $\mu(k,n)$ the following hold:

1. $\mu(k,n) \in \mathbb{N}^*$.

2. No element of the sequence is a prime number.

3. No element of the sequence is a power of 2.

4. The range of the sequence is all natural numbers that are not primes and are not powers of 2.

Proof. 1. $\mu(k,n) \in \mathbb{N}^*$ as a sum of natural numbers.

2. $n \in A = \{2,3,4,\ldots\}$ and therefore it holds that

$$n \geq 2$$
$$n+1 \geq 3 .$$

Also we have that

$$2k + n \geq 4$$
$$\frac{2k+n}{2} \geq \frac{3}{2} > 1$$

Since $k \in \mathbb{N}^*$ and $n \in A = \{2,3,4,\ldots\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2} = \mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number $\mu(k,n)=\frac{(n+1)(2k+n)}{2}$ is a power of 2. Then, it exists $\lambda\in\mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2}=2^{\lambda}$$

$$(n+1)(2k+n)=2^{\lambda+1} \quad . \tag{2.2}$$

Equation (2.2) can hold if and only if there exist $\lambda_1,\lambda_2\in\mathbb{N}$ such as

$$n+1=2^{\lambda_1}\wedge 2k+n=2^{\lambda_2}$$

and equivalently

$$\left.\begin{array}{l} n=2^{\lambda_1}-1 \\ n=2^{\lambda_2}-2k \end{array}\right\}. \tag{2.3}$$

We eliminate n from equations (2.3) and we obtain

$$2^{\lambda_1}-1=2^{\lambda_2}-2k$$

and equivalently

$$2k-1=2^{\lambda_2}-2^{\lambda_1}$$

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form

$$N=\chi\psi$$

where at least one of the χ,ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3. We will prove that there are always exist $k\in\mathbb{N}$ and $n\in A=\{2,3,4,\ldots\}$ such as

$$N=\chi\cdot\psi=\mu(k,n).$$

We consider the following two pairs of k and n:

$$\chi \le 2\psi - 1, \chi, \psi \in \mathbb{N}$$
$$k = k_1 = \frac{2\psi + 1 - \chi}{2} \tag{2.4}$$
$$n = n_1 = \chi - 1$$

$$\chi \ge 2\psi + 1, \chi, \psi \in \mathbb{N}$$
$$k = k_2 = \frac{\chi + 1 - 2\psi}{2} \quad . \tag{2.5}$$
$$n = n_2 = 2\psi - 1$$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \le 2\psi - 1$ or the inequality $\chi \ge 2\psi + 1$. Thus, for each pair of naturals (χ, ψ), where χ is odd, at least one of the pairs (k_1, n_1), (k_2, n_2) of equations (2.4), (2.5) is defined. We now prove that "when the natural number k_1 of equation (2.4) is $k_1 = 0$ then the natural number k_2 of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$." For $k_1 = 0$ from equations (2.4) we take

$$\chi = 2\psi + 1$$

and from equations (2.5) we have that

$$k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1$$
$$n_2 = 2\psi - 1$$

and because $\psi \ge 2$ we obtain

$$k_2 = 1$$
$$n_2 = 2\psi - 1 \ge 3 > 2 .$$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$$\chi = 2\psi - 1$$

and from equations (2.4) we get

$$k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1$$
$$n_1 = \chi - 1 = 2\psi - 2 \ge 2 .$$

We now prove that at least one of the k_1 and k_2 is positive. Let

$k_1 < 0 \wedge k_2 < 0$.

Then from equations (2.4) and (2.5) we have that

$$2\psi + 1 - \chi < 0 \wedge \chi + 1 - 2\psi < 0. \tag{2.6}$$

Taking into account that $\chi > 1$ is odd, that is $\chi = 2\rho + 1, \rho \in \mathbb{N}$, we obtain from inequalities (2.6)

$$\begin{aligned}
&2\psi + 1 - (2\rho - 1) < 0 \wedge (2\rho + 1) + 1 - 2\psi < 0 \\
&2\psi - 2\rho < 0 \wedge 2\rho - 2\psi + 2 > 0 \\
&\psi < \rho \wedge \psi > \rho + 1
\end{aligned}$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1, n_1) = \frac{(n_1 + 1)(2k_1 + n_1)}{2}$$
$$= \frac{(\chi - 1 + 1)\left(2\frac{2\psi + 1 - \chi}{2} + \chi - 1\right)}{2} = \frac{\chi(2)\psi}{2} = \chi\psi = N.$$

For equations (2.5) we obtain

$$\mu(k_2, n_2) = \frac{(n_2 + 1)(2k_2 + n_2)}{2}$$
$$= \frac{(2\psi - 1 + 1)\left(2\frac{\chi + 1 - 2\psi}{2} + 2\psi - 1\right)}{2} = \frac{2\psi\chi}{2} = \chi\psi = N.$$

Thus, there are always exist $k \in \mathbb{N}^*$ and $n \in A = \{2, 3, 4, \ldots\}$ such as

$N = \chi\psi = \mu(k, n)$ for every N which is not a prime number and is not a power of 2. □

Example 2.1. For the natural number $N = 40$ we have

$$N = 40 = 5 \cdot 8$$
$$\chi = 5$$
$$\psi = 8$$

and from equations (2.4) we get

$$k = k_1 = \frac{16+1-5}{2} = 6$$
$$n = n_1 = 5-1 = 4$$

thus, we obtain

$$40 = \mu(6,4)$$.

Example 2.2. For the natural number $N = 51$,

$$N = 51 = 3 \cdot 17 = 17 \cdot 3$$

there are two cases. First case:

$$N = 51 = 3 \cdot 17$$
$$\chi = 3$$
$$\psi = 17$$

and from equations (2.4) we obtain

$$k = k_1 = \frac{34+1-3}{2} = 16$$
$$n = n_1 = 3-1 = 2$$

thus,

$$51 = \mu(16,2)$$.

Second case:

$$N = 51 = 17 \cdot 3$$
$$\chi = 17$$
$$\psi = 3$$

and from equations (2.5) we obtain

$$k = k_2 = \frac{17+1-6}{2} = 6$$
$$n = n_2 = 6-1 = 5$$

thus,

$$51=\mu(6,5).$$

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.

$$135=15\cdot 9=27\cdot 5=9\cdot 15=45\cdot 3=5\cdot 27=3\cdot 45$$
$$135=\mu(2,14)=\mu(9,9)=\mu(11,8)=\mu(20,5)=\mu(25,4)=\mu(44,2)$$

a. $135=9\cdot 15=\mu(2,14)=\mu(11,8)$

$$135=2+3+4+......+15+16=11+12+13.....+18+19.$$

b. $135=5\cdot 27=\mu(9,9)=\mu(25,4)$

$$135=9+10+11+.....+17+18=25+26+27+28+29.$$

c. $135=3\cdot 45=\mu(20,5)=\mu(44,2)$

$$135=20+21+22+23+24+25=44+45+46.$$

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers $\chi,\psi\in\mathbb{N}$:

$$\Phi=\chi\cdot\psi=\psi\cdot\chi.$$

On the contrary, the natural number Φ can be written in the form $\Phi=\mu(k,n)$ using different natural numbers $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,...\}$, through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

$$135=3\cdot 45=45\cdot 3$$
$$135=44+45+46=20+21+22+23+24+25.$$

From Theorem 2.1 the following corollary is derived:

Corollary 2.1. "1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more *consecutive* natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more *consecutive* natural numbers."

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.□

3 THE CONCEPT OF REARRANGEMENT

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

Definition. ''We say that the sequence $\mu(k,n), k \in \mathbb{N}^*, n \in A = \{2,3,4,...\}$ is rearranged if there exist natural numbers $k_1 \in \mathbb{N}^*, n_1 \in A$, $(k_1,n_1) \neq (k,n)$ such as

$$\mu(k,n) = \mu(k_1,n_1). \text{''} \quad (3.1)$$

From equation (2.1) written in the form of

$$\mu(k,n) = k + (k+1) + (k+2) + ... + (k+n)$$

two different types of rearrangement are derived: The "compression", during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. The following theorem provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Theorem 3.1. ''1. The sequence $\mu(k_1,n_1), (k_1,n_1) \in \mathbb{N}^* \times A$ can be compressed

$$\mu(k_1,n_1) = \mu(k_1+\varphi, n_1-\omega) \quad (3.2)$$

if and only if there exist $\varphi,\omega \in \mathbb{N}^*, \omega \leq n_1 - 2$ which satisfies the equation

$$\begin{aligned} &\omega^2 - (2k_1 + 2n_1 + 1 + 2\varphi)\omega + 2(n_1+1)\varphi = 0 \\ &\varphi,\omega \in \mathbb{N}^* \\ &\omega \leq n_1 - 2 \end{aligned} \quad . \quad (3.3)$$

2. The sequence $\mu(k_2,n_2), (k_2,n_2) \in \mathbb{N}^* \times A$ can be decompressed

$$\mu(k_2,n_2) = \mu(k_2-\varphi, n_2+\omega) \quad (3.4)$$

if and only if there exist $\varphi,\omega \in \mathbb{N}^*, \varphi \leq k_2 - 1$ which satisfies the equation

$$\begin{aligned} &\omega^2 + (2k_2 + 2n_2 + 1 - 2\varphi)\omega - 2(n_2+1)\varphi = 0 \\ &\varphi,\omega \in \mathbb{N}^* \\ &\varphi \leq k_2 - 1 \end{aligned} \quad . \quad (3.5)$$

3. The odd number $\Pi \neq 1$ is prime if and only if the sequence

$$\begin{aligned} &\mu(k,n) = \Pi \cdot 2^l \\ &l,k \in \mathbb{N}^*, n \in A \end{aligned} \tag{3.6}$$

cannot be rearranged.

4. The odd Π is prime if and only if the sequence

$$\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2 \tag{3.7}$$

cannot be rearranged.''

Proof. 1, 2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k_1, n_1)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega).$$

In this equation the natural number $n_1 - \omega$ belongs to the set $A = \{2, 3, 4, \ldots\}$ and thus $n_1 - \omega \geq 2 \Leftrightarrow \omega \leq n_1 - 2$. Next, from equations (2.1) we obtain

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega)$$

$$\frac{(n_1+1)(2k_1+n_1)}{2} = \frac{(n_1-\omega+1)\left[2(k_1+\varphi)+n_1-\omega\right]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^l$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k,n)$ in equation (3.6) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2} = \Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ, ψ are unique $\chi = \Pi \wedge \psi = \Pi$, and from equation (2.5) we get $k = \dfrac{\Pi+1}{2} \wedge n = \Pi - 1$. Thus, the sequence

$\mu(k,n)=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)$ cannot be rearranged. Conversely, if the sequence $\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2=\Pi\cdot\Pi$ cannot be rearranged the odd number Π cannot be composite and thus Π is prime.□

We now prove the following corollary:

Corollary 3.1. ''1. The odd number Φ,

$$\begin{aligned}&\Phi=\Pi^2=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)\\&\Pi=odd\\&\Pi\neq1\end{aligned}\tag{3.8}$$

is decompressed and compressed if and only if the odd number Π is composite.

2. The even number α_1,

$$\begin{aligned}&\alpha_1=2^l\Pi=\mu\left(2^l-\frac{\Pi-1}{2},\Pi-1\right)\\&\Pi=odd\\&3\leq\Pi\leq2^l-1\\&l\in\mathbb{N},l\geq2\end{aligned}\tag{3.9}$$

cannot be decompressed, while it compresses if and only if the odd number Π is composite.

3. The even number α_2,

$$\begin{aligned}&\alpha_2=2^l\Pi=\mu\left(\frac{\Pi+1}{2}-2^l,2^{l+1}-1\right)\\&\Pi=odd\\&\Pi\geq2^{l+1}+1\\&l\in\mathbb{N}^*\end{aligned}\tag{3.10}$$

cannot be compressed, while it decompresses if and only if the odd number Π is composite.

4. Every even number that is not a power of can be written either in the form of equation (3.9) or in the form of equation (3.10).''

Proof. 1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2.5) since every composite odd Π can be written in the form of $\Pi=\chi\psi$, $\chi,\psi\in\mathbb{N}$, χ,ψ odds.

2,3. Let the even number α ,

$$\alpha = 2^l \Pi$$
$$\Pi = odd \,. \qquad (3.11)$$
$$l \in \mathbb{N}^*$$

From equation (2.4) we obtain

$$k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \qquad (3.12)$$
$$n = \Pi - 1$$

and since $k, n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1$$
$$\Pi - 1 \geq 2$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$

In the second of equations (3.12) the natural number n obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number k takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha_1 = \mu\left(2^l - \frac{\Pi - 1}{2}, \Pi - 1\right)$$

cannot decompress. If the odd number Π is composite then it can be written in the form of $\Pi = \chi\psi$, $\chi, \psi \in \mathbb{N}^*$, χ, ψ odds, $\chi, \psi < \Pi$, $\alpha_1 = 2^l \chi\psi$. Therefore, the natural number $\alpha_1 = 2^l \chi\psi$ decompresses since from equations (3.11) it can be written in the form of $\alpha_1 = \mu(k, n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10). □

By substituting $\Pi = P = prime$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite *impossible* diophantine equations.

Example 3.1. The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega, \varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0 \,.$$

We now prove the following corollary:

Corollary 3.2 ''The square of every prime number can be uniquely written as the sum of consecutive natural numbers.''

Proof. For $\Pi = P = prime$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right). \tag{3.13}$$

According with 4 of Theorem 3.1 the odd P^2 cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13).□

Example 3.2. The odd $P=17$ is prime. From equation (3.13) for $P=17$ we obtain

$$289 = \mu(9,16)$$

and from equation (2.1) we get

$$289 = 9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25$$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

4 NATURAL NUMBERS AS LINEAR COMBINATION OF CONSECUTIVE POWERS OF 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

Theorem 4.1. 'Every natural number, with the exception of 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.''

Proof. Let the odd number Π as given from equation

$$\begin{gathered} \Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ \beta_i = \pm 1, i = 0,1,2,........,\nu-1 \\ \nu \in \mathbb{N} \end{gathered} \quad . \tag{4.1}$$

From equation (4.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2+1 = 3.$$

We now examine the case where $\nu \in \mathbb{N}^*$. The lowest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\min}=\Pi(\nu)=2^{\nu+1}+2^{\nu}-2^{\nu-1}-2^{\nu-1}-........2^{1}-1$$

$$\Pi_{\min}=\Pi(\nu)=2^{\nu+1}+1. \qquad (4.2)$$

The largest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\max}=\Pi(\nu)=2^{\nu+1}+2^{\nu}+2^{\nu-1}+........2^{1}+1$$

$$\Pi_{\max}=\Pi(\nu)=2^{\nu+2}-1. \qquad (4.3)$$

Thus, for the odd numbers $\Pi=\Pi(\nu,\beta_i)$ of equation (4.1) the following inequality holds

$$\Pi_{\min}=2^{\nu+1}+1\leq\Pi(\nu,\beta_i)\leq 2^{\nu+2}-1=\Pi_{\max}\,. \qquad (4.4)$$

The number $N(\Pi(\nu,\beta_i))$ of odd numbers in the closed interval $\left[2^{\nu+1}+1,2^{\nu+2}-1\right]$ is

$$N(\Pi(\nu,\beta_i))=\frac{\Pi_{\max}-\Pi_{\min}}{2}+1=\frac{(2^{\nu+2}-1)-(2^{\nu+1}+1)}{2}+1$$

$$N(\Pi(\nu,\beta_i))=2^{\nu}\,. \qquad (4.5)$$

The integers $\beta_i, i=0,1,2,........,\nu-1$ in equation (4.1) can take only two values, $\beta_i=-1\vee\beta_i=+1$, thus equation (4.1) gives exactly $2^{\nu}=N(\Pi(\nu,\beta_i))$ odd numbers. Therefore, for every $\nu\in\mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $\left[2^{\nu+1}+1,2^{\nu+2}-1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha=2^{\nu},\nu\in\mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, according to corollary 3.1 the even number α is written in the form of

$$\alpha=2^{l}\Pi,\Pi=\text{odd},\Pi\neq 1,l\in\mathbb{N}^*. \qquad (4.6)$$

We now prove that the even number α can be uniquely written in the form of equation (4.6). If we assume that the even number α can be written in the form of

$$\begin{aligned}&\alpha=2^{l}\Pi=2^{l'}\Pi'\\&l\neq l'(l>l')\\&\Pi\neq\Pi'\\&l,l'\in\mathbb{N}^*\\&\Pi,\Pi'=odd\end{aligned} \qquad (4.7)$$

the we obtain

$$2^{l}\Pi = 2^{l^{'}}\Pi'$$
$$2^{l-l^{'}}\Pi = \Pi^{'}$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l^{'}$ and we take that $\Pi = \Pi^{'}$ from equation (4.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number Π of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{aligned} &\alpha = \alpha\left(l, \nu, \beta_i\right) = 2^{l}\left(2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{i}\right) \\ &l \in \mathbb{N}^{*}, \nu \in \mathbb{N} \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \end{aligned} \tag{4.8}$$

and equivalently

$$\begin{aligned} &\alpha = \alpha\left(l, \nu, \beta_i\right) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{l+i} \\ &l \in \mathbb{N}^{*}, \nu \in \mathbb{N} \qquad . \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \end{aligned} \tag{4.9}$$

For 1 we take

$$1 = 2^{0}$$
$$1 = 2^{1} - 2^{0}$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). □

In order to write an odd number $\Pi \neq 1,3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^{*}$ from inequality (4.4). Then, we calculate the sum

$$2^{\nu+1} + 2^{\nu}.$$

If it holds that $2^{\nu+1} + 2^{\nu} < \Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (4.1). The number of ν steps needed in order to write the odd number Π in the form of equation (4.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (4.4).

Example 4.1. For the odd number $\Pi = 23$ we obtain from inequality (4.4)

$$2^{\nu+1}+1<23<2^{\nu+2}-1$$
$$2^{\nu+1}+2<24<2^{\nu+2}$$
$$2^{\nu}<12<2^{\nu+1}$$

thus $\nu=3$. Then, we have

$2^{\nu+1}+2^{\nu}=2^4+2^3=24>23$ (thus 2^2 is subtracted)

$2^4+2^3-2^2=20<23$ (thus 2^1 is added)

$2^4+2^3-2^2+2^1=22<23$ (thus $2^0=1$ is added)

$2^4+2^3-2^2+2^1+1=23.$

Fermat numbers F_s can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\min}$,

$$\begin{gathered}F_s=2^{2^s}+1=\Pi_{\min}\left(2^s-1\right)=2^{2^s}+2^{2^s-1}-2^{2^s-2}-2^{2^s-3}-........-2^1-1\\ s\in\mathbb{N}^*\end{gathered}. \qquad (4.10)$$

Mersenne numbers M_p can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\max}$,

$$\begin{gathered}M_p=2^p-1=\Pi_{\max}\left(p-2\right)=2^{p-1}+2^{p-2}+2^{p-3}+........+2^1+1\\ p=prime\end{gathered}. \qquad (4.11)$$

In order to write an even number α that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number Π in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha=368$ by 2 we obtain $\alpha=368=2^4\cdot 23$. Then, we write the odd number $\Pi=23$ in the form of equation (4.1), $23=2^4+2^3-2^2+2^1+1$, and we get

$$\begin{gathered}368=2^4\left(2^4+2^3-2^2+2^1+1\right)\\ 368=2^8+2^7-2^6+2^5+2^4\end{gathered}.$$

This equation gives the unique way in which the even number $\alpha=368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

$$2^{\nu+1}+1 \leq \Pi \leq 2^{\nu+2}-1$$
$$2^{\nu+1} < 2^{\nu+1}+1 \leq \Pi \leq 2^{\nu+2}-1 < 2^{\nu+2}$$
$$2^{\nu+1} < \Pi < 2^{\nu+2}$$
$$(\nu+1)\log 2 < \log \Pi < (\nu+2)\log 2$$

from which we get

$$\frac{\log \Pi}{\log 2}-1 < \nu+1 < \frac{\log \Pi}{\log 2}$$

and finally

$$\nu+1=\left[\frac{\log \Pi}{\log 2}\right] \tag{4.12}$$

'where $\left[\frac{\log \Pi}{\log 2}\right]$ the integer part of $\frac{\log \Pi}{\log 2} \in \mathbb{R}$.

We now give the following definition:

Definition 4.1. (**The fundamental** Π^* **symmetry**) We define as the conjugate of the odd

$$\begin{aligned} &\Pi=\Pi(\nu,\beta_i)=2^{\nu+1}+2^{\nu}+\sum_{i=0}^{i=\nu-1}\beta_i 2^i \\ &\beta_i=\pm 1, i=0,1,2,........,\nu-1 \\ &\nu \in \mathbb{N}^* \end{aligned} \tag{4.13}$$

the odd Π^*,

$$\begin{aligned} &\Pi^*=\Pi^*(\nu,\gamma_j)=2^{\nu+1}+2^{\nu}+\sum_{j=0}^{j=\nu-1}\gamma_j 2^j \\ &\gamma_i=\pm 1, j=0,1,2,........,\nu-1 \\ &\nu \in \mathbb{N}^* \end{aligned} \tag{4.14}$$

for which it holds

$$\gamma_k=-\beta_k \forall k=0,1,2,........,\nu-1 . \tag{4.15}$$

For conjugate odds, the following corollary holds:

Corollary 4.1. '' For the conjugate odds $\Pi=\Pi(\nu,\beta_i)$ and $\Pi^*=\Pi^*(\nu,\gamma_i)$ the following hold:

1. $(\Pi^*)^*=\Pi$. (4.16)

2. $\Pi+\Pi^*=3\cdot 2^{\nu+1}$. (4.17)

3. Π is divisible by 3 if and only if Π^* is divisible by 3.''

Proof. 1.The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

$$\Pi+\Pi^*=\left(2^{\nu+1}+2^{\nu}\right)+\left(2^{\nu+1}+2^{\nu}\right)$$

and, equivalently

$$\Pi+\Pi^*=3\cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi=3x, x=odd$ and from equation (4.17) we get $3x+\Pi^*=3\cdot 2^{\nu+1}$ and equivalently $\Pi^*=3\left(2^{\nu+1}-x\right)$. Similarly we can prove the inverse. □

5 THE HARMONIC ODD NUMBERS AND A METHOD FOR DEFINING LARGE PRIME NUMBERS

The harmonic symmetry: We define as harmonic the odd numbers of equation (4.1) for which the signs of $\beta_i=\pm 1, i=0,1,2,3........,\nu-1$ alternate:

$$\Pi_1=2^{\nu+1}+2^{\nu}-2^{\nu-1}+2^{\nu-2}-........-2^1+1=\frac{2^{\nu+3}+1}{3}$$

$$\Pi_2=2^{\nu+1}+2^{\nu}+2^{\nu-1}-2^{\nu-2}+........+2^1-1=\frac{5\times 2^{\nu+1}-1}{3}. \quad (5.1)$$

$$\nu=2\lambda, \lambda\in\mathbb{N}^*$$

$$\Pi_1=2^{\nu+1}+2^{\nu}-2^{\nu-1}+2^{\nu-2}-........+2^1-1=\frac{7\times 2^{\nu}+1}{3}$$

$$\Pi_2=2^{\nu+1}+2^{\nu}+2^{\nu-1}-2^{\nu-2}+........-2^1+1=\frac{11\times 2^{\nu}-1}{3}. \quad (5.2)$$

$$\nu=2\lambda+1, \lambda\in\mathbb{N}^*$$

From equations (5.1), (5.2) and definition 4.1 we obtain

$$\Pi_2=\Pi_1^*=3\times 2^{\nu+1}-\Pi_1 \quad (5.3)$$

for the pair of harmonic odd numbers.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [1-5].When we consider the prime factorization of the odd integers

$$\Phi_1=2+\Pi_1=2+\frac{2^{\nu+3}+1}{3} \quad (5.4)$$

$$\nu=2\lambda, \lambda\in\mathbb{N}^*$$

$$\Phi_2 = -2 + \Pi_2 = -2 + \frac{5 \times 2^{\nu+1} - 1}{3} \quad (5.5)$$

$\nu = 2\lambda, \lambda \in \mathbb{N}^*$

$$\Phi_2 = \Phi_1^* = 3 \times 2^{\nu+1} - \Phi_1 \quad (5.6)$$

we have the following statement:

The factors of either Φ_1 or $\Phi_2 = \Phi_1^*$ consist of a set of small prime factors and one large factor. Hence from the factorization of Φ_1 and $\Phi_2 = \Phi_1^*$ of equations (5.4), (5.5) we get a large prime number. Following are 11 examples where we have chosen arbitrary even ν , $600 \le \nu \le 1000$, in equations (5.4), (5.5).

1. $\nu = 604$

$\Phi_1 =$ 3 × 5 × 1423 × 2677 × 1 039667 × 1 465469 × 2 033624 136455 907062 140355 606581 460617 960329 378244 909713 340374 546035 722007 834481 807880 893223 943637 129816 307143 853799 454110 280390 103176 523414 883666 509589 711687 765791.

2. $\nu = 626$

$\Phi_1 =$ 13 × 186653 × 306 032599 340492 581270 323029 570138 222136 733600 420877 092183 139417 574185 782109 955578 496315 765962 131603 014089 519221 871827 181120 845674 859725 387186 219442 305406 755275 821605 426602 403741 599957.

3. $\nu = 644$

$\Phi_1 =$ 5 × 79 × 12 671297 × 38892 671359 559494 324882 180204 888273 078001 950134 412751 881230 225550 378061 442396 379471 711953 850899 474720 409489 565536 036909 109253 945965 590266 361910 559333 142120 493266 182138 997818 136400 630503.

4. $\nu = 688$

$\Phi_1 =$ 3^3 × 5 × 137 × 2357 × 84239 × 14 276659 × 111598 463167 × 164995 567141 × 3547 493034 864246 374604 223939 439254 117526 195183 644765 258504 745395 441461 348003 624541 265182 053620 319595 210678 493117 621150 188802 864705 030169 622562 000148 389984 593085 080457.

5. $\nu = 732$

$\Phi_1 =$ 5^5 × 19 × 4357 × 10093 × 2 901193 × 373 058471 × 21 318693 003272 810610 223875 009176 985967 454565 131359 547239 807330 702392 207730 665072 351378 572215 387223 133953 567092 456647 869354 874941 347502 746701 928543 247909 407783 975122 056127 018272 539991 430427 637981.

6. $\nu = 818$

$\Phi_2 =$ 5 826599 918309 521729 414628 892756 111346 582385 085483 938095 996388 692690 239258 901551 139189 714409 550909 093308 382061 608683 211278 156913 402724 889465 422572 029940 710372 011896 628413 739441 379150 647273 555252 328499 350633 086191 299627 798178 454098 229036 211569 513811 is prime.

7. $\nu = 838$?

8. $\nu = 842$

$\Phi_1 =$ 13 × 811 × 7789 × 15271 × 66809 × 933 419184 297225 688884 848133 741618 091582 561157 362135 750330 558036 085494 747230 138415 970602 017694 350758 458917 589235 971861 548843 635060 827053 633582 882443 092203 262135 552296 661334 709021 156021 405492 515100 671199 284761 072521 866782 927154 434480 887521.

9. $\nu = 914$?

10. $\nu = 986$?

11. $\nu = 998$

$\Phi_2 =$ 23 × 277 × 4211 × 1 385899 × 240154 091459 652243 015812 929515 159070 212159 918817 425875 611004 712759 052716 135663 441910 181493 025014 669780 274245 881010 561780 858639 784499 969926 885693 756207 174479 909272 942309 784548 553831 369221 141895 942976 579419 394048 307219 568666 715750 728448 387606 183250 921312 430705 694057 415487 884739 523892 723969.

Equations (5.4), (5.5) and (5.6) are a special case of equations

$$\Phi_1(\nu,\xi) = \Phi_1\left(\nu, 2^{2\xi+1}\right) = 2^{2\xi+1} + \Pi_1(\nu) = 2^{2\xi+1} + \frac{2^{\nu+3}+1}{3}$$
$$\nu = 2\lambda, \lambda \in \mathbb{N}^* \tag{5.7}$$
$$\xi = 0,1,2,........,\frac{\nu-2}{2} = \lambda - 1$$

$$\Phi_2(\nu,\xi) = \Phi_2\left(\nu, 2^{2\xi+1}\right) = -2^{2\xi+1} + \Pi_2(\nu) = -2^{2\xi+1} + \frac{5\times 2^{\nu+1}-1}{3}$$
$$\nu = 2\lambda, \lambda \in \mathbb{N}^* \tag{5.8}$$
$$\xi = 0,1,2,........,\frac{\nu-2}{2} = \lambda - 1$$

$$\Phi_2(\nu,\xi) = 3\times 2^{\nu+1} - \Phi_1(\nu,\xi) = \Phi_1^*(\nu,\xi) \tag{5.9}$$

for $\xi = 0$. The general equations (5.7) and (5.8) or (5.9) give all possible variations of the method. For example, for $\nu = 838$, a value of ν that did not give a large prime number in the previous examples, from equation (5.7) for $\xi = 1$, $2^{2\xi+1} = 2^3$ we get

$\Phi_1\left(838, 2^3\right) =$ 3 × 251 × 124 958179 125661 642577 × 51 945201 394308 356447 274374 943957 749268 889249 128703 205379 933327 597692 534177 000888 147927 160249 734500 867000 722765 431922 957290 626876 299700 840201 468643 187688 745195 339241 792572 155819 582073 320776 475981 870379 650986 830637 696975 455178 897139.

For $\nu = 914$ and $\xi = 446$, $2^{2\xi+1} = 2^{893}$ from equation (5.7) we get

$\Phi_2\left(914, 2^{893}\right) = \Phi_1^*\left(914, 2^{893}\right) =$ 664331 × 2 846040 859139 × 244156 938515 639118 957128 090989 142139 910652 427045 074721 151803 096387 973957 589979 330000 475973 653713 544853 032484 133776 843427 696009 975250 479123 845904 732248 525911 944098 218552 642872 716830 980122 426431 711592 329448 776714 147775 168920 208010 693295 505312 315955 821688 402869.

For $\nu = 986$ and $\xi = 2$, $2^{2\xi+1} = 2^5$ from equation (5.7) we get

$\Phi_1\left(986, 2^5\right) =$ 29 × 4253 × 416 339548 376347 × 33 962793 894011 369474 190517 253893 737579 763016 839246 389349 317772 361943 015051 440254 338769 687497 378741 306027 784171 795586 328198 808732 364434 864222 375712 875623 070283 735838 206731 569184 817760 627665 419282 986706 989291 001175 496233 325374 140887 038097 481963 826327 954330 255038 364958 385620 947977.

For $\nu = 66$ and $\xi = 1, 2, 3, \ldots\ldots, \dfrac{\nu - 2}{2} = 32$ from equation (5.7) we get

$\Pi_1 = \Pi_1\left(66\right) =$ 196765270119568550571

$\Pi_2 = \Pi_2\left(66\right) =$ 245956587649460688213=3^2 × 27 328509 738828 965357

$\Phi_2\left(66, 2^1\right) = \Phi_1^*\left(66, 2^1\right) =$ 73 × 3 369268 323965 214907

$\Phi_1\left(66, 2^3\right) =$ 1 645337 × 119 589646 449067

$\Phi_2\left(66, 2^5\right) =$ 4140 643009 × 59400 577909

$\Phi_2\left(66, 2^7\right) = \Phi_1^*\left(66, 2^7\right) =$ 5 × 13907 × 3537 162402 379531

$\Phi_2\left(66, 2^9\right) = \Phi_1^*\left(66, 2^9\right) =$ 13 × 601 × 25184 342777 367023

$\Phi_2\left(66, 2^{11}\right) = \Phi_1^*\left(66, 2^{11}\right) =$ 5 × 49 191317 529892 137233

$\Phi_1\left(66, 2^{13}\right) =$ 196 765270 119568 558763 is prime

$\Phi_2\left(66, 2^{15}\right) = \Phi_1^*\left(66, 2^{15}\right) =$ 5 × 3259 × 36269 × 416167 836559

$\Phi_2\left(66,2^{17}\right)=\Phi_1^*\left(66,2^{17}\right)=$ 157 × 12 248491 × 127901 671243

$\Phi_1\left(66,2^{19}\right)=$ 13 × 19 × 4643 × 32083 × 5347 833013

$\Phi_1\left(66,2^{21}\right)=$ 271 × 5903 × 123 000357 013771

$\Phi_1\left(66,2^{23}\right)=$ 7 × 257879 × 2 350441 × 46 375123

$\Phi_1\left(66,2^{25}\right)=$ 634853 × 309 938316 617551

$\Phi_2\left(66,2^{27}\right)=\Phi_1^*\left(66,2^{27}\right)=$ 5 × 49 191317 529865 294097

$\Phi_1\left(66,2^{29}\right)=$ 7^2 × 107 × 37529 137921 057681

$\Phi_1\left(66,2^{31}\right)=$ 13 × 17 749367 × 852750 974689

$\Phi_1\left(66,2^{33}\right)=$ 233 × 1231 × 5227 × 131244 759703

$\Phi_2\left(66,2^{35}\right)=\Phi_1^*\left(66,2^{35}\right)=$ 5 × 14683 × 3350 222537 834243

$\Phi_2\left(66,2^{37}\right)=\Phi_1^*\left(66,2^{37}\right)=$ 73 × 3 369268 322082 489517

$\Phi_2\left(66,2^{39}\right)=\Phi_1^*\left(66,2^{39}\right)=$ 3^2 × 27 328509 738828 965357

$\Phi_1\left(66,2^{41}\right)=$ 7 × 37 × 759711 476133 559097

$\Phi_2\left(66,2^{43}\right)=\Phi_1^*\left(66,2^{43}\right)=$ 5 × 41 × 93 472763 × 12835 698347

$\Phi_1\left(66,2^{45}\right)=$ 541 × 13963 × 26 047888 286741

$\Phi_1\left(66,2^{47}\right)=$ 7 × 10061 × 2793 891701 436337

$\Phi_2\left(66,2^{49}\right)=\Phi_1^*\left(66,2^{49}\right)=$ 245 956024 699507 266901 is prime

$\Phi_2\left(66,2^{51}\right)=\Phi_1^*\left(66,2^{51}\right)=$ 5 × 229 × 8423 × 25 502467 080379

$\Phi_1\left(66,2^{53}\right)=$ 7 × 28 110611 045546 184509

$\Phi_2\left(66,2^{53}\right)=\Phi_1^*\left(66,2^{53}\right)=$ 245 947580 450205 947221 is prime

$\Phi_1\left(66,2^{55}\right)=$ 13 × 19 × 796766 392374 848237

$\Phi_1\left(66,2^{57}\right)=$ 601 × 327636 248432 020643

$\Phi_1\left(66,2^{59}\right)=$ 7 × 1447 × 19482 844394 498171

$\Phi_2\left(66,2^{61}\right)=\Phi_1^*\left(66,2^{61}\right)=$ 16183 × 75731 × 198808 531457

$\Phi_2\left(66,2^{63}\right)=\Phi_1^*\left(66,2^{63}\right)=$ 5 × 23 × 41 × 50208 529292 175167

$\Phi_1\left(66,2^{65}\right)=$ 7 × 46507 × 717 737600 997047.

Above, we saw the variation of the digits of the largest prime number for $\nu=66$, as ξ takes values from $\xi=0$ to $\xi=\frac{66-2}{2}=32$. A similar variation arises for every $\nu\in\mathbb{N}^*$ and $\xi=0,1,2,........,\frac{\nu-2}{2}$ in equations (5.7) and (5.9). From this distribution we come to an optimization of the method:

For a fixed computer network we choose a random time interval T,

$$T=T\left(\nu\right) \qquad (5.10)$$

and put for execution the following algorithmic procedure: The factorization of

$\Phi_1\left(\nu,2^{2\xi+1}\right)$ and $\Phi_2\left(\nu,2^{2\xi+1}\right)=\Phi_1^*\left(\nu,2^{2\xi+1}\right)=3\times2^{\nu+1}-\Phi_1\left(\nu,2^{2\xi+1}\right)$

is interrupted when the time interval $T=T\left(\nu\right)$ is exceeded. By this condition, the method gives only the largest primes for every $\nu\in\mathbb{N}^*$. Additionally, the required run time for the method is minimized for a fixed computer network and a fixed $\nu\in\mathbb{N}^*$. Next, we can see an example for a low computational power computer:

Example 5.1. For $\nu=998$ and $T=\mathbf{3s}$ from equations (5.7) and (5.8) we get

$\xi=0,1,2,3,........,\frac{998-2}{2}=498$.

1. $\xi=0$, $\Phi_2\left(998,2^1\right)=\Phi_1^*\left(998,2^1\right)=$ 23 × 277 × 4211 × 1 385899 × 240154 091459 652243 015812 929515 159070 212159 918817 425875 611004 712759 052716 135663 441910 181493 025014 669780 274245 881010 561780 858639 784499 969926 885693 756207 174479 909272 942309 784548 553831 369221 141895 942976 579419 394048 307219 568666 715750 728448 387606 183250 921312 430705 694057 415487 884739 523892 723969 (288 digits)

T=0.6s.

2. $\xi=1$, $\Phi_1\left(998,2^3\right)=$ 97 × 137 × 9923 × 1 081681 × 50080 644682 811942 296699 821816 279773 732717 457591 276109 457062 242788 762858 485087 453320 300320 950501 243057 574463 353237 805619 330431 715070 029378 288884 517452 798815 573077 642329 490846 332389 896689 408072

542886 142722 619257 370697 531361 739108 642652 205080 024148 915392 295551 861665 088028 145461 038179 893337 (287 digits)

T=0.6s.

3. $\xi = 9$, $\Phi_1\left(998, 2^{19}\right) =$ 4 536179 × 356 426027 × 4418 194305 492551 055438 288717 852823 131268 899490 581575 671658 140386 858327 881061 918039 334820 806178 879155 888881 005738 442642 836356 895420 852799 269710 636084 534157 398165 083264 381690 315538 696593 057882 325500 671380 059142 783057 514685 959697 790231 700820 951045 409420 029331 949677 361996 881725 184302 765883 (286 digits)

T=0.6s.

4. $\xi = 13$, $\Phi_2\left(998, 2^{27}\right) = \Phi_1^*\left(998, 2^{27}\right) =$ 5 × 1 785847 678643 778868 247375 081766 669684 269008 019509 222679 072917 313950 585085 208226 870821 997298 026159 763545 991121 529255 244708 645242 142820 523405 997429 595783 095800 655761 295804 038497 570179 100843 728523 646325 697025 507745 830596 990211 233127 926527 590657 679510 485761 866079 614423 694610 071638 608770 731139 534251 168017 (301 digits)

T=0.2s.

5. $\xi = 25$, $\Phi_2\left(998, 2^{51}\right) = \Phi_1^*\left(998, 2^{51}\right) =$ 5 × 11661 167857 × 16521 920251 × 4 880496 240331 × 1899 231315 908657 097875 196652 086767 417005 191295 227630 772878 186939 449441 507048 325820 190068 477005 532913 880320 995340 694858 276546 924458 586740 967203 764636 699403 607786 693467 423245 127615 083671 437704 878429 984084 139551 880694 959912 111807 982429 926306 542433 872186 453701 258022 963289 (268 digits)

T=2.6s.

6. $\xi = 29$, $\Phi_2\left(998, 2^{59}\right) = \Phi_1^*\left(998, 2^{59}\right) =$ 5^2 × 11 × 31 × 41 × 9 612847 × 8207 166203 × 323 810490 953628 022064 233800 849387 739635 421729 426241 826230 216969 635899 103869 379857 533713 745016 382039 410786 464059 201318 076449 245745 596630 400732 573528 033619 426479 931615 580202 341503 811059 519577 410940 997169 667449 031997 541911 776463 049074 801526 844609 127839 268529 094893 668906 920738 922813 (279 digits)

T=1.1s.

7. $\xi = 38$, $\Phi_2\left(998, 2^{77}\right) = \Phi_1^*\left(998, 2^{77}\right) =$ 12689 × 12791 × 362 367211 × 63438 358051 × 73076 239109 × 32749 562773 009140 773210 602629 062784 951656 363861 516124 158877 033055 872447 866810 557438 212636 674718 860993 747247 501118 202226 318125 517894 724914 745242 508958 914820 486945 219310 864886 943633 570729 820520 875419 898759 256524 768188 352102 694842 651897 783215 819590 508484 531342 501991 (263 digits)

T=2.6s.

8. $\xi = 40$, $\Phi_2\left(998, 2^{81}\right) = \Phi_1^*\left(998, 2^{81}\right) =$ 2213 × 46544 703463 × 86688 749106 696899 807354 911283 542897 445289 843103 778339 244209 182382 057748 453154 185684 585569 053616 848194 373798 960426 348482 124482 392398 508300 413367 195172 964872 513704 529280 109527 456726 374749 787017 361927 178522 932931 830671 329496 439241 623018 611617 086650 436534 193897 918305 285320 911790 616877 929319 (287 digits)

T=1.7s.

9. $\xi = 53$, $\Phi_1\left(998, 2^{107}\right) =$ 232109 × 30 776017 795842 106393 933455 088198 556441 482372 842228 826612 891655 454128 622073 391843 846158 439319 908487 194309 417067 485625 196931 532032 671210 912217 922262 312673 714516 124084 732673 674826 399305 513249 870080 803858 480722 552694 304779 051415 207991 530317 060651 323481 394721 722632 287825 024460 688001 173139 722930 671031 (296 digits)

T=1.6s.

10. $\xi = 57$, $\Phi_2\left(998, 2^{115}\right) = \Phi_1^*\left(998, 2^{115}\right) =$ 5 × 1 785847 678643 778868 247375 081766 669684 269008 019509 222679 072917 313950 585085 208226 870821 997298 026159 763545 991121 529255 244708 645242 142820 523405 997429 595783 095800 655761 295804 038497 570179 100843 728523 646325 697025 507745 830596 990211 233127 926527 590657 679510 485761 866079 614423 686302 396664 953046 525490 740151 259409 (301 digits)

T=0.2s.

11. $\xi = 58$, $\Phi_1\left(998, 2^{117}\right) =$ 43691 × 163 497990 766407 623377 572047 494144 760638 942392 667526 280384 785636 762773 576728 225663 941955 761875 549633 887619 063104 919113 291858 299617 112954 467144 027794 812029 553057 221056 584105 513498 906328 611725 849593 401450 827448 009505 902541 961615 264281 113052 176208 327617 656896 484824 281773 010317 594643 051590 431302 090753 (297 digits)

T=0.1s.

12. $\xi = 68$, $\Phi_2\left(998, 2^{137}\right) = \Phi_1^*\left(998, 2^{137}\right) =$ 105417 695783 × 84 703410 816335 186156 806261 492001 372948 895961 712695 154018 727889 782535 917962 496814 120959 304160 650293 671672 252582 708267 822424 427945 951346 549078 717762 234124 652165 456699 024462 611819 549559 013895 512611 534924 096905 108400 219574 841626 337387 088721 482649 840062 562405 309768 293988 524532 287345 514667 771427 (290 digits)

T=1.1s

13. $\xi = 73$, $\Phi_1\left(998, 2^{147}\right) =$ 97 × 379 × 5749 × 12511 × 2 701525 852731 749067 825970 181538 841081 500468 570988 102390 004713 750684 212272 485899 120548 747955 954166 771149 039178 447733 176943 626139 060614 247893 193527 941427 464561 994282 082884 395184 241965 420437 569582 191909 943209 546029 373459 071393 266722 914761 826479 840486 134806 578495 819150 003168 800130 472275 972947 (289 digits)

T=0.1s

14. $\xi = 77$, $\Phi_2\left(998, 2^{155}\right) = \Phi_1^*\left(998, 2^{155}\right) =$ 5 × 23 × 593 × 920 421803 063369 × 142257 439151 650269 759512 229987 535797 603859 632871 650964 499028 781520 394883 025379 626882 238970 651138 172615 583681 318636 124615 473290 390325 908383 034899 430684 465196 844839 590606 618826 905902 899124 401196 506691 143719 061249 533242 261455 674589 486954 446740 453073 599526 221890 802680 087230 458993 840559 (282 digits)

T=0.9s

15. $\xi = 83$, $\Phi_1\left(998, 2^{167}\right) =$ 114773 × 155251 × 400 894654 169098 163595 483363 914852 531928 781693 381969 626300 885755 924123 202706 408560 360730 024076 936163 945689 446112 988849 825030 667217 976200 944962 868154 794364 056209 768391 777045 901886 394962 168952 671859 065166 558803 794909 960252 142707 475904 498255 382992 174430 898197 050208 510399 326290 198098 814515 285573 (291 digits)

T=0.4s

16. $\xi = 90$, $\Phi_1\left(998, 2^{181}\right) =$ 13 × 239 × 2029 × 345814 471567 × 3 276709 505948 663007 436071 497798 167728 716237 262266 046137 399918 770194 945601 014426 083832 974746 796394 366422 444638 956063 948505 660872 339745 731543 151887 495733 528762 939605 565934 906885 716395 630970 274169 097750 084818 105248 219940 313650 115083 027840 487493 893745 793535 513039 225289 854656 163199 872403 (283 digits)

T=2.7s

17. $\xi = 95$, $\Phi_2\left(998, 2^{191}\right) = \Phi_1^*\left(998, 2^{191}\right) =$ 5 × 17 × 157 × 2543 × 263117 345854 197111 775236 642637 849621 102132 569635 056743 616085 430844 342072 473092 169620 260599 446899 578806 313516 402000 281513 699879 810654 350772 703862 923881 305362 033902 791182 966354 140802 049019 690624 171161 337402 154320 115975 545121 656939 078207 412594 447644 799532 466810 505324 177709 194418 308158 570392 926619 (294 digits)

T=0.1s

18. $\xi = 96$, $\Phi_1\left(998, 2^{193}\right) =$ 13 × 1129 × 335029 × 8 269211 × 175 679317 658327 620139 490814 384990 040100 858556 390114 264495 835752 393980 708984 505460 993306 839388 894630 597492 726644 756282 306765 232024 031693 331107 485972 274891 069941 204175 847675 178674 462934 901401 328940 460822 002358 705536 882813 222690 476700 154781 597384 933080 860102 029870 056922 793612 063768 927961 (285 digits)

T=0.3s

19/20. $\xi = 101$, $\Phi_1\left(998, 2^{203}\right) =$ 773 730593 × 9232 400501 158825 799446 578593 625131 140544 202415 991183 014126 558203 447359 798054 291855 159964 176451 581841 722477 278492 490704 493289 912566 869504 181528 102486 613687 773132 999515 783853 709246 722784 522123 832558

993551 521661 213830 778384 182273 779765 239763 039295 444140 618834 911176 106897 416666 287373 629641 164363 (292 digits)

T=1.5s

$\Phi_2\left(998, 2^{203}\right) = \Phi_1^*\left(998, 2^{203}\right) =$ 5 × 1 785847 678643 778868 247375 081766 669684 269008 019509 222679 072917 313950 585085 208226 870821 997298 026159 763545 991121 529255 244708 645242 142820 523405 997429 595783 095800 655761 295804 038497 570179 100843 728523 646325 697025 507745 830596 990211 233127 926527 590655 108409 614947 481638 747284 346864 211474 573245 941087 065741 529361 (301 digits)

T=0.2s

21/22. $\xi = 105$, $\Phi_1\left(998, 2^{211}\right) =$ 587 × 720 547769 × 16 888984 061692 358355 153236 651114 719240 611806 510989 941762 465501 533611 645216 696095 964090 405285 020017 953218 387778 165387 576927 041108 914263 284391 156734 424355 035281 950141 293201 852464 222562 316879 263965 425696 648530 916626 519821 543011 793530 618310 055052 637183 984261 934271 230666 461625 182128 549715 877033 (290 digits)

T=0.5s

$\Phi_2\left(998, 2^{211}\right) = \Phi_1^*\left(998, 2^{211}\right) =$ 5 × 37 × 30491 × 1 582963 939420 120308 648786 112132 928621 621628 730063 211101 789821 288825 665956 554505 557086 847335 568368 657783 813142 495087 380422 087547 448933 113099 388148 736652 548603 757919 967348 839752 953400 605445 584318 320182 824905 805386 818260 940278 551958 997672 853398 014378 684431 869765 404191 641547 634034 247079 091754 668359 (295 digits)

T=0.2

23. $\xi =$ 123, $\Phi_1\left(998, 2^{247}\right) =$ 1 371641 × 16951 549039 × 128216 263801 × 2396 135860 767614 483702 567537 826479 613108 014773 336354 737978 026710 481589 070522 440576 983324 884283 105189 114843 934743 233073 871810 765202 096962 463805 602250 649971 484702 784738 167787 009328 401038 173512 902558 630958 832427 970019 825426 400796 258991 194077 880602 439158 915163 161887 574665 465621 (274 digits)

T=2.0s

24. $\xi = 126$, $\Phi_2\left(998, 2^{253}\right) = \Phi_1^*\left(998, 2^{253}\right) =$ 107 × 229 × 557 × 654244 322790 130219 004207 626709 347972 072231 517142 195345 835320 100382 162959 860418 978785 508072 116827 875158 506997 578377 075107 223247 035379 511073 680652 739397 497217 999306 022691 259692 441638 920278 669033 252267 170826 591982 849056 108034 999793 424443 706688 901363 124456 987270 278959 702729 922794 969298 190775 649151 (294 digits)

T=0.5s

25. $\xi = 141$, $\Phi_1\left(998, 2^{283}\right) =$ 1879 × 9151 × 34511 × 622103 × 19 350368 061089 137024 255081 876408 454341 134176 649502 367875 724581 146137 062600 954709 914300 522080 798917 742040 679924 014966 680273 004127 182045 662741 209853 383601 157577 602353 579690 670975 286235 843453 744059 341402 517087 478888 286731 306664 690147 302321 158859 165390 399832 045177 744372 761706 710364 119787 (284 digits)

T=0.6s

26. $\xi = 145$, $\Phi_2\left(998, 2^{291}\right) = \Phi_1^*\left(998, 2^{291}\right) =$ 5 × 53 × 1187 × 28879 × 42223 × 24 611183 × 3672 479219 × 257569 687091 602152 079137 833651 131746 881393 271929 762300 825308 494232 053410 197389 850727 531290 943856 262690 752668 357924 658836 012815 730385 298925 272892 377202 756649 576381 765451 465790 588151 386237 951574 352732 533550 835834 637309 905649 834713 883058 580340 426909 925190 838256 544266 400827 (270 digits)

T=1.0s

27/28. $\xi = 149$, $\Phi_1\left(998, 2^{299}\right) =$ 7 143390 714575 115472 989500 327066 678737 076032 078036 890716 291669 255802 340340 832907 483287 989192 104639 054183 964486 117020 978834 580968 571282 093623 989718 383132 383202 623045 183216 153990 280716 403374 914094 585302 788102 030984 340905 949012 175554 840333 206835 407122 468781 661151 425820 096510 511231 625732 806226 490203 744939 (301 digits) is prime

T=0.3s

$\Phi_2\left(998, 2^{299}\right) = \Phi_1^*\left(998, 2^{299}\right) =$ 5^3 × 11 × 31 × 41^2 × 101 × 251 × 601 × 1801 × 3529 × 4051 × 8101 × 8431 × 268501 × 8598 989029 × 2 014531 706153 918551 568531 289400 771225 903728 446888 818738 054449 663123 056799 858197 716102 446457 294008 497198 921956 466935 821473 840872 382211 084459 286634 758646 635538 904641 233614 993957 107131 230161 611503 584857 862662 299225 150156 806591 845847 074598 298104 560413 016403 (253 digits)

T=0.8s

29. $\xi = 176$, $\Phi_2\left(998, 2^{353}\right) = \Phi_1^*\left(998, 2^{353}\right) =$ 23 × 388227 756226 908449 618994 582992 754279 188914 786849 831017 189764 633467 518496 784397 145830 868977 831773 861640 432852 506359 835806 227226 552787 070305 651615 129518 064304 490382 890392 182282 080473 717574 723592 096229 586791 287745 959134 712343 206475 830707 192755 437013 192996 026942 738118 933232 281727 325378 488296 376778 157427 (300 digits)

T=0.5s

30. $\xi = 185$, $\Phi_1\left(998, 2^{371}\right) =$ 2099 × 12 937471 × 680 377837 × 386627 212955 742182 618863 921271 402758 475803 518040 436431 392999 790429 241969 001800 633833 709644 148165 858361 909113 085591 159491 017541 821555 845870 762554 472234 194100 606449 540385 960430 250521 051832 567287 314534 455016 244516 023635 797172 266397 392463 416148 563961 924352 900518 142198 481657 007282 974163 (282 digits)

T=1.7s

31. $\xi = 196$, $\Phi_2\left(998, 2^{393}\right) = \Phi_1^*\left(998, 2^{393}\right) =$ 5 402543 × 1 652784 326421 630395 396552 217878 385867 793192 964414 371786 650210 200965 161300 158302 183640 183241 508822 570728 258082 841039 159437 181011 000579 285908 504041 148569 382789 415800 388635 535614 959639 466829 753219 118784 455475 999024 511133 864710 625928 040615 180536 068168 167287 105977 962653 037457 737160 438527 849330 239547 (295 digits)

T=0.5s

32. $\xi = 197$, $\Phi_1\left(998, 2^{395}\right) =$ 11 × 1999 × 27091 × 282 372962 966749 × 42 466921 406365 948875 948770 737359 680169 193512 590162 300543 819779 157936 035393 558127 029345 608326 557832 279148 245930 623123 641140 114550 223537 988714 596718 353365 008099 378198 523221 591670 666786 220417 262224 405322 567840 139735 569537 878617 569393 087598 107572 123285 078427 876051 267706 022614 262169 (278 digits)

T=2.6s

33/34. $\xi = 199$, $\Phi_1\left(998, 2^{399}\right) =$ 201575 740129 × 2 480055 715723 × 14 289094 363187 213691 208680 611171 705945 303353 242452 569850 169220 526954 050535 261843 940156 925970 575589 033037 189974 594162 587028 204034 474059 758365 011979 903011 741983 421318 377292 028446 684764 288036 524278 562883 228210 081177 068286 210999 538629 937470 406625 038023 246173 066205 804579 013580 057217 (278 digits)

T=2.5s

$\Phi_2\left(998, 2^{399}\right) = \Phi_1^*\left(998, 2^{399}\right) =$ 5^3 × 11 × 17 × 31 × 41 × 101 × 157 × 251 × 401 × 601 × 1801 × 4051 × 8101 × 15671 × 61681 × 268501 × 340801 × 2 787601 × 18 004897 × 3173 389601 × 793803 149263 × 474063 717704 894849 562835 374581 495827 311871 974257 278381 183325 961420 859899 717498 041765 514624 204329 617053 587500 499410 353583 821054 381178 850330 054652 795756 561865 556583 500734 550291 163982 011165 238195 274533 641100 352256 127741 (216 digits)

T=1.6s

35. $\xi = 201$, $\Phi_1\left(998, 2^{403}\right) =$ 262957 × 3 887239 × 28 552837 × 244753 639896 519737 750246 725613 155331 950850 159334 731536 596181 252980 547298 934418 475724 098648 820062 993092 677736 512139 629575 073122 323492 039685 203991 554440 481720 353819 310553 897215 152630 195141 276654 005494 101148 387952 195777 353069 857283 492453 953255 273438 038066 180257 642229 578928 507868 082909 (282 digits)

T=0.9s

36. $\xi = 205$, $\Phi_2\left(998, 2^{411}\right) = \Phi_1^*\left(998, 2^{411}\right) =$ 5 × 1 785847 678643 778868 247375 081766 669684 269008 019509 222679 072917 313950 585085 208226 870821 997298 026159 763545 991121 529255 244708 645242 142820 523405 997429 595783 095800 655761 295804 038497 569121 411293 664125

888002 632532 655312 166871 542718 490277 975672 152585 240461 219783 767926 294120 957906 527182 852624 791899 496904 724753 (301 digits)

T=0.2s

37. $\xi = 214$, $\Phi_2\left(998, 2^{429}\right) = \Phi_1^*\left(998, 2^{429}\right) =$ 11 × 31 × 3539 × 45953 × 161 014729 636910 389111 812999 030059 341331 702777 776625 204197 132458 005105 750940 625571 667284 320311 378697 556043 726460 762240 869632 050174 893072 751159 659708 275344 562220 222729 353141 825480 373379 280680 480875 360857 256748 360641 295229 595555 109930 503027 083453 002486 844005 086833 270345 095169 720212 907905 003483 (291 digits)

T=0.2s

38. $\xi = 225$, $\Phi_2\left(998, 2^{451}\right) = \Phi_1^*\left(998, 2^{451}\right) =$ 5 × 83 × 157 × 593 × 1 547469 837457 × 149344 688969 630053 142970 351119 285083 892632 041523 756578 296865 620098 073334 610600 116811 906551 196474 836660 748584 447955 749792 450847 018523 472038 962609 946451 521773 976632 269231 508876 581693 377964 213322 116002 873927 538556 522954 025942 807762 980646 429619 608682 337730 938114 538150 182192 249431 760423 (282 digits)

T=1.6s

39. $\xi = 235$, $\Phi_2\left(998, 2^{471}\right) = \Phi_1^*\left(998, 2^{471}\right) =$ 5 × 17 × 105049 863449 634051 073375 004809 804099 074647 530559 366039 945465 724350 034416 776954 521813 058664 589774 103737 999477 737015 014394 626190 714283 560200 352789 976222 535047 097325 991928 033606 653223 274377 990621 650478 979463 295837 733811 331295 522321 593591 242336 018824 465544 275854 610595 076449 457186 855308 021508 224252 444929 (300 digits)

T=0.2s

40. $\xi = 239$, $\Phi_1\left(998, 2^{479}\right) =$ 137 × 2 434997 × 827 931791 × 25 863711 562732 751591 526226 429290 028370 217018 792199 675768 141948 292179 277815 856585 213443 474425 000554 419832 365597 844897 342275 207952 315785 919467 304239 926018 076282 134454 047167 786222 235401 711152 690088 464424 367353 699351 674265 583706 525493 638382 489694 557908 274402 227935 227883 791745 829585 571561 (284 digits)

T=1.2s

41. $\xi = 240$, $\Phi_1\left(998, 2^{481}\right) =$ 13 × 451313 × 32 388773 × 37591 418449 933545 132030 943408 610794 177762 760160 064590 467852 865002 096742 764103 807119 613757 199281 228573 494102 705882 841103 511342 128120 867188 414940 162429 183971 386811 945315 764966 480741 277831 641063 626381 603864 827290 129144 053231 083021 001899 477302 962558 981548 628055 404202 683736 887772 673100 322619 (287 digits)

T=0.5s

42/43. $\xi = 244$, $\Phi_1\left(998, 2^{489}\right) =$ 109 × 15737 × 1 810423 × 63 897091 × 35999 363077 864234 579333 215835 234616 573581 578631 320360 833997 586420 240784 265629 453544 798943 077771 079413 782788 321913 294236 465957 355591 614068 051739 603078 286428 192972 852372 442468 665132 012334 653651 970015 916188 872946 706635 088401 901774 944026 904859 962958 461711 445028 659621 722389 359879 084227 (281 digits)

T=0.4s

$\Phi_2\left(998, 2^{489}\right) = \Phi_1^*\left(998, 2^{489}\right) =$ 11 × 29 × 31 × 6131 × 15859 071337 × 46487 196589 × 199 765099 267706 307384 558500 862817 361682 347934 156991 993085 611675 011340 665792 527850 931325 277078 135380 228309 779364 438194 606702 523877 725780 747131 984481 430096 817822 024230 157706 713138 235841 861068 101965 998665 309826 441512 161687 509188 054991 629800 107672 152543 620661 007412 372772 259323 (273 digits)

T=1.7s

44. $\xi = 250$, $\Phi_1\left(998, 2^{501}\right) =$ 701 × 88993 × 114506 605306 822613 786971 308969 205959 876278 590409 848400 237087 189130 125532 815439 145413 046065 959549 000002 035206 091992 408687 172561 258064 328892 687270 749983 315938 949995 433402 489002 599979 639067 275489 260978 578588 631521 276723 222191 236183 570319 390598 743975 491594 455620 068289 121723 044383 566285 939440 760871 (294 digits)

T=0.2s

45. $\xi = 278$, $\Phi_2\left(998, 2^{557}\right) = \Phi_1^*\left(998, 2^{557}\right) =$ 229 × 727 × 53 634535 617563 921488 902022 481775 006585 327271 238181 156006 106248 504369 367599 341280 215457 352943 728782 024170 369392 948686 794106 462586 054444 869954 794455 207116 935716 897118 296816 058532 693088 478493 873803 596149 649557 250353 168036 287281 695021 634012 300003 716337 092657 262753 514640 377523 759265 284891 362434 694327 (296 digits)

T=0.2s

46. $\xi = 279$, $\Phi_2\left(998, 2^{559}\right) = \Phi_1^*\left(998, 2^{559}\right) =$ 5^2 × 11 × 17 × 31 × 41 × 827 × 61681 × 308 397511 × 95 525533 582727 010358 730178 212996 182198 046190 995203 859912 007761 218911 979801 056527 425688 304386 103373 878566 313258 491033 358792 611575 525127 777226 451065 150174 051526 998643 029803 956912 334773 819612 574956 958053 168970 683768 780685 731223 766665 224274 355223 675088 259411 620017 146224 397018 314357 (278 digits)

T=1.5s

47. $\xi = 290$, $\Phi_2\left(998, 2^{581}\right) = \Phi_1^*\left(998, 2^{581}\right) =$ 4271 × 812 807357 × 47 351674 899017 × 54320 264421 708924 159885 550778 684086 375546 537121 076070 874827 461311 350281 259344 523914 762800 663168 792865 931037 276353 338365 029872 927679 239415 307317 576197 627916 475652 149701 055322 844380 011460 685315 101399 407826 653375 607407 270811 361373 662256 367298 430634 913311 553196 800451 968236 359839 (275 digits)

T=1.2s

48. $\xi = 291$, $\Phi_1\left(998, 2^{583}\right) =$ 349 × 2907 294413 × 23810 368063 × 73546 783547 × 4020 315822 611107 366990 471894 187419 248314 220588 895788 829779 811649 899365 757256 877628 752645 126017 473183 589056 566855 137954 948482 141605 536988 074000 928848 554254 420994 950268 969954 835038 601519 213871 482150 608991 854938 042192 490200 506235 111973 470762 341180 295165 477898 598875 810087 (268 digits)

T=2.9s

49. $\xi = 295$, $\Phi_2\left(998, 2^{591}\right) = \Phi_1^*\left(998, 2^{591}\right) =$ 5 × 17 × 1406 097989 × 74 710201 046759 374231 190230 946133 654611 639964 844132 950356 894169 734876 433833 627025 984643 993818 964279 692174 054974 069815 268490 576987 062960 378081 613146 189500 797838 615894 934933 459515 736005 792665 911397 038134 695613 281592 216260 281415 828570 054065 928088 832312 687295 573298 633464 464089 053383 199273 268621 (290 digits)

T=0.7s

50. $\xi = 310$, $\Phi_1\left(998, 2^{621}\right) =$ 36855 430043 × 193 821933 599493 272177 567582 943172 082662 436241 377515 669660 033690 026704 115775 738622 638000 110666 749358 113953 414387 433082 786982 344762 852751 929213 620383 019188 075853 104891 533278 633490 641964 432434 239056 108392 905102 005458 066852 263624 668794 880022 911325 282562 473719 604069 735818 320825 292180 309996 201521 (291 digits)

T=0.8s

51. $\xi = 322$, $\Phi_2\left(998, 2^{645}\right) = \Phi_1^*\left(998, 2^{645}\right) =$ 277 × 32235 517665 050160 076667 420248 495842 676335 884828 686329 947164 572453 981680 238415 647487 761684 079894 580569 422221 883206 413211 671405 105402 928644 839944 950886 654321 477837 646010 323536 862980 306233 004600 826793 019239 143486 688083 316231 173760 641133 887614 708958 874886 812632 805841 496062 168876 045362 639002 611983 749953 (299 digits)

T=0.1s

52. $\xi = 324$, $\Phi_1\left(998, 2^{649}\right) =$ 13 × 572821 × 959272 780552 485045 736465 174053 792712 138163 187511 643215 203845 966621 918317 191168 120755 133089 918764 937601 525785 016559 368378 137849 325617 522371 822700 381247 853164 132156 968593 265532 025688 149083 061002 134066 754678 057925 758911 347548 291107 226398 649014 798037 634476 921719 554770 271195 412905 882872 170718 999931 (294 digits)

T=0.5s

53. $\xi = 328$, $\Phi_2\left(998, 2^{657}\right) = \Phi_1^*\left(998, 2^{657}\right) =$ 29 × 661 × 465 816599 364541 412762 109416 705793 125428 819453 155934 759004 882183 199589 202673 125064 119671 682932 380344 187456 706110 367950 710601 934115 989371 821453 948705 306139 900675 983648 459184 031151 052206 260865

961407 827115 708474 752170 303864 828688 979588 262185 397208 543982 696577 825655 462101 679409 961161 593780 039205 203189 (297 digits)

T=0.1s

54. $\xi = 335,\ \Phi_2\left(998, 2^{671}\right) = \Phi_1^*\left(998, 2^{671}\right) =$ 5 × 17 × 1579 × 66 529362 539350 254004 670680 690186 256538 725478 504981 659243 486842 526937 566039 869868 152665 398726 899368 982097 299007 797871 695768 173511 983186 737589 118822 301408 609592 427860 122784 039449 766451 405543 530905 392162 706272 467897 814273 122233 108312 621456 492962 446789 429307 147288 453987 268908 221454 645762 134691 941251 (296 digits)

T=0.1s

55. $\xi = 336,\ \Phi_1\left(998, 2^{673}\right) =$ 13 × 643 × 947 × 4817 × 15 563039 × 6452 647781 × 1865 482111 694196 984405 721028 109267 512169 662287 520722 309037 678721 775415 465389 736977 685952 209537 805133 533760 037167 161890 202547 864033 575096 403601 034516 354194 440294 113199 035912 051501 044428 259660 643442 949856 321344 053411 279634 717439 526154 204509 705020 485323 674761 245712 042546 515597 (274 digits)

T=0.7s

56. $\xi = 337,\ \Phi_2\left(998, 2^{675}\right) = \Phi_1^*\left(998, 2^{675}\right) =$ 5 × 1 785847 678643 778868 247375 081766 669684 269008 019509 222679 072917 313950 585085 208226 870821 997298 026159 732193 137933 322263 648242 402673 172327 054463 714020 815840 943906 796237 351353 694071 648607 168608 152146 800183 721008 933505 037501 681553 156304 073250 246384 262689 411986 276880 407170 664997 779636 981338 120611 558293 836049 (301 digits)

T=0.3s

57. $\xi = 338,\ \Phi_2\left(998, 2^{677}\right) = \Phi_1^*\left(998, 2^{677}\right) =$ 107 × 83450 826104 849479 824643 695409 657461 881729 346706 038442 947332 584764 046032 019076 022001 027911 122717 740099 746652 743050 881254 377334 872002 179796 115130 582991 331225 477461 005514 143027 751583 708967 356215 713166 252007 439756 198982 044653 220833 295019 542689 907113 373395 304172 092776 241876 677739 817712 163038 674315 014463 (299 digits)

T=0.2s

58. $\xi = 352,\ \Phi_2\left(998, 2^{705}\right) = \Phi_1^*\left(998, 2^{705}\right) =$ 23 × 10651 × 13679 × 37 146433 × 71013 472483 × 877312 715381 × 1151 408834 272769 616264 028242 815038 516160 668354 436413 248269 807293 123140 307312 762024 536657 068208 059014 781694 377143 542542 796647 355483 353174 367781 256346 523370 543923 764970 333121 124603 680203 764935 092487 648250 393836 749893 110771 633048 079830 195769 053937 473038 020613 683577 (262 digits)

T=2.9s

59. $\xi = 359$, $\Phi_2\left(998, 2^{719}\right) = \Phi_1^*\left(998, 2^{719}\right) =$ 5^2 × 11 × 17 × 31 × 41 × 30497 × 61681 × 14 160555 016603 × 56 415501 183158 349160 686674 861979 191473 446248 684175 761835 579034 727284 183059 885039 649800 383964 528477 523288 742866 508024 938336 809504 120750 319466 108619 183801 412427 589818 862624 067559 950557 831114 436890 657855 493642 551670 132567 915170 686046 491491 868627 512790 813166 283642 607570 231603 (272 digits)

T=2.6s

60. $\xi = 361$, $\Phi_1\left(998, 2^{723}\right) =$ 97 × 149 × 494 249686 194915 621185 186489 107221 942646 926733 414300 886756 498253 359326 115051 055662 030866 561282 049052 277990 335001 485386 369573 505392 926243 163853 032142 313843 098458 228170 223858 991616 355435 276080 622915 380214 117152 667048 483900 878820 826383 015214 525959 955087 003729 585719 578697 235091 619395 197652 142456 697503 (297 digits)

T=0.1s

61/62. $\xi = 366$, $\Phi_1\left(998, 2^{733}\right) =$ 13 × 197 × 347 × 8 038321 119806 536613 815411 540055 699983 318872 061229 786541 293498 302291 342359 773579 449301 270927 620941 258502 572911 327940 378630 289589 009744 357063 355128 733522 983279 225000 677642 907764 056353 997324 939563 558957 447586 982342 515974 063254 044459 282552 480876 926346 614092 110408 702983 976784 026113 890618 854881 100529 (295 digits)

T=0.2s

$\Phi_2\left(998, 2^{733}\right) = \Phi_1^*\left(998, 2^{733}\right) =$ 28949 × 308 447213 831873 098940 788124 247239 919214 654741 011645 079117 226383 286224 927494 598580 570166 953981 115716 140254 037527 107340 838079 506065 455187 470916 309861 335942 285540 184738 161506 669113 646204 354305 663259 099907 946899 335904 477867 595817 341664 358017 561769 321520 790058 421222 113323 078307 182041 732080 360122 475329 (297 digits)

T=0.2s

63. $\xi = 398$, $\Phi_2\left(998, 2^{797}\right) = \Phi_1^*\left(998, 2^{797}\right) =$ 29 × 163 × 2 463029 × 766 936293 641854 859996 241608 214948 660659 056423 226601 016664 353466 779874 594217 421947 834134 116883 884398 435553 582671 602574 262856 904998 834876 720809 438464 792987 481224 305027 834865 905116 302509 209202 130027 079252 960983 420913 328570 423486 026674 436711 658835 296797 774424 285951 284461 891051 756446 211637 340327 (291 digits)

T=1.1S

64. $\xi = 399$, $\Phi_2\left(998, 2^{799}\right) = \Phi_1^*\left(998, 2^{799}\right) =$ 5^3 × 11 × 17 × 31 × 41 × 101 × 251 × 401 × 601 × 727 × 1381 × 1801 × 4051 × 8101 × 18341 × 61681 × 268501 × 340801 × 2 787601 × 3 096427 × 22 236047 × 385 568933 × 3173 389601 × 8409 382921 × 4 055093 275862 234632 967152 290766 093157 926308 080381 263801 953434 067536 396366 528848 607438 580599 723331 031593 797701 209811 504204

951986 391814 164791 656984 656394 323556 901357 576913 272343 943139 972622 741238 478271 (199 digits)

T=0.7s

65. $\xi = 400$, $\Phi_1\left(998, 2^{801}\right) =$ 10 125793 × 705464 817874 028777 103136 547139 239241 516791 038295 656519 572509 786821 753723 440865 880133 197531 750869 620157 217229 103275 345580 361409 387467 996717 702283 786320 155499 051237 867793 400561 986760 758488 567002 437780 775768 785283 778019 178464 352524 265035 566971 129062 738206 965345 676132 876866 458749 051176 325397 992971 (294 digits)

T=0.7s

66. $\xi = 401$, $\Phi_1\left(998, 2^{803}\right) =$ 61927 × 115 351796 705396 926590 816611 931252 583478 547839 844282 634655 186309 685885 048196 539475 868398 516998 514641 630476 900437 326273 659261 308974 585958 219848 561852 515693 568830 219268 210972 620282 014040 948021 816808 596342 386203 911540 406554 848540 517445 097906 684596 465781 569352 560895 680105 144058 973959 975176 195334 287517 (297 digits)

T=0.1s

67. $\xi = 408$, $\Phi_2\left(998, 2^{817}\right) = \Phi_1^*\left(998, 2^{817}\right) =$ 142699 × 1 846951 × 33 879587 548349 245200 557671 253217 043154 307156 362321 278579 421774 944084 759667 412391 141840 582765 349102 067532 624290 317216 351463 868244 562251 789887 979126 573411 373583 249089 114801 281543 585495 422391 457605 383196 910025 888366 843381 947903 287967 919488 663050 817933 454479 804737 156647 874763 758725 791514 916009 (290 digits)

T=1.0s

698. $\xi =$ 415, $\Phi_2\left(998, 2^{831}\right) = \Phi_1^*\left(998, 2^{831}\right) =$ 5 × 17 × 13 738939 × 324 012083 × 1214 035279 × 19437 919695 183749 611689 831947 804714 931428 224339 795663 392913 311810 739121 573995 786880 746093 626329 331972 413585 305942 189476 499679 941948 766165 414860 971838 232913 112919 122610 133519 492893 365897 765604 686820 606017 425865 907912 209695 993061 042084 194243 462261 483785 539732 509260 071971 982063 (275 digits)

T=2.6s

69. $\xi = 422$, $\Phi_2\left(998, 2^{845}\right) = \Phi_1^*\left(998, 2^{845}\right) =$ 30047 × 1 307821 × 227 229647 255682 705642 525444 617843 660928 229364 164101 761305 625487 466711 149585 594347 205782 604644 883381 310419 363594 017430 217640 792216 224451 435993 727300 680274 695078 127345 861091 023469 036113 840120 003285 534680 970159 441873 792882 422851 664415 544249 782082 010908 086736 203950 105875 888687 234292 184995 935063 (291 digits)

T=0.4s

70. $\xi = 459$, $\Phi_1\left(998, 2^{919}\right) =$ 71 × 100611 136825 001626 380133 869841 160689 830515 534524 580692 750171 927220 523525 437326 718001 073473 506106 358207 698232 001259 282480 038994 356902 384918 601202 973028 509572 215872 738695 569479 384835 982727 925613 832554 139247 839295 999079 475912 287843 371686 635199 635230 461199 158862 424572 210574 543796 146342 802140 426645 878909 (300 digits)

T=0.2s

71. $\xi = 467$, $\Phi_2\left(998, 2^{935}\right) = \Phi_1^*\left(998, 2^{935}\right) =$ 5 × 17 × 193 × 4951 × 2 994643 × 167360 569151 × 316897 488881 × 692194 682407 018939 975765 511703 243084 765560 211784 736983 754900 808956 805839 935450 928887 642673 596592 964383 170982 790765 412920 328379 441353 607935 129845 750509 440143 843391 470934 822922 000125 919848 831773 693595 718121 554122 562347 408091 408627 813895 611285 464692 861131 672925 217043 (264 digits)

T=2.2s

72. $\xi = 468$, $\Phi_1\left(998, 2^{937}\right) =$ 13 × 549491 593428 855036 473171 714370 380519 606693 723439 730809 914939 714488 364789 382325 326798 351371 577784 997770 471325 524395 625210 220093 964028 488755 358258 700478 026214 371894 736515 569214 744127 874805 730163 161771 272752 698160 788635 564042 299582 463334 676254 458952 750916 938297 354237 223522 572830 236240 988957 823368 523671 (300 digits)

T=0.2s

73. $\xi = 470$, $\Phi_2\left(998, 2^{941}\right) = \Phi_1^*\left(998, 2^{941}\right) =$ 773 × 457 035727 × 25 274628 064090 325719 460651 342540 761864 690763 486298 197189 937998 391605 967426 911349 278035 633411 559301 717752 687815 184121 224633 404134 985360 068405 148401 351842 766264 662482 564264 083436 142575 797378 518119 852203 977674 522509 813997 648730 863599 746227 144490 943614 288832 177284 925162 410943 526337 233834 536991 (290 digits)

T=0.9

74. $\xi = 475$, $\Phi_2\left(998, 2^{951}\right) = \Phi_1^*\left(998, 2^{951}\right) =$ 5 × 17 × 105049 863449 633827 146122 904626 001701 248765 456221 399337 826282 415641 221758 966603 942471 406762 787703 734583 241530 080529 278519 053667 014635 853778 956423 588283 793168 007616 098451 399793 715578 091490 967719 433477 442424 900079 883426 811023 032581 199593 582414 723910 870799 328163 944101 834339 682451 529432 582488 249488 965889 (300 digits)

T=0.1s

75. $\xi = 477$, $\Phi_1\left(998, 2^{955}\right) =$ 11 × 137 × 1151 × 811 560341 × 545823 333697 × 9297 001807 780365 378157 339991 931775 374994 174254 862155 487110 020205 625810 301420 380556 294833 009868 905666 872651 884681 274096 795146 081058 186377 800015 494430 395149 833924 359413 745004 673172 691476 704437 295834 061266 627712 469047 819535 451890 375103 771868 768002 035400 864495 188032 572747 188171 (274 digits)

T=1.6s

76. $\xi = 480$, $\Phi_1\left(998, 2^{961}\right) =$ 13 × 705161 × 779242 745175 008707 887255 024600 207012 392236 279773 603602 582286 366120 216752 873275 462643 339829 137717 953152 185595 437449 100676 413317 276005 491813 350221 773819 196464 704314 714605 297071 348641 204952 830267 337342 699086 693740 590320 878996 501931 735825 212243 467971 217296 301264 717544 619220 781889 038242 175762 602271 (294 digits)

T=0.5s

77. $\xi = 483$, $\Phi_2\left(998, 2^{967}\right) = \Phi_1^*\left(998, 2^{967}\right) =$ 5 × 17 × 37 × 53 × 53 569537 702681 669902 976725 719459 552303 428718 803455 358056 168375 116620 172178 167663 087211 710405 946546 415676 625338 005442 566911 275166 851365 409528 398787 779065 282595 019268 928339 016361 739010 447221 894845 540400 264801 154160 168251 483698 796430 981716 466960 686431 808676 862342 923285 876071 696115 531569 104470 746521 (296 digits)

T=0.2s

By increasing the values of $\nu \in \mathbb{N}^*$ the method gives even bigger primes.

As ξ takes values from $\xi = 0$ to $\xi = \dfrac{\nu - 2}{2}$ we have neighborhoods of $\mathbb{N}$ where the method gives big primes. We have such a neighborhood for small values of $\xi \in \mathbb{N}$. Next, we have an example for ten consecutive even numbers.

Example 5.2.

1.v=900, $\xi_{\max} = 449$

$\xi = 0, \Phi_2\left(900, 2^1\right) = \Phi_1^*\left(900, 2^1\right) =$ 17 × 1303 × 14 612289 234017 × 87 048891 637396 895102 119311 765818 573790 576976 052955 671628 663425 575545 670256 411482 623556 091041 420326 465040 293560 953469 136495 015271 490122 045045 761776 713210 784123 751846 457476 167032 775232 117142 820499 562144 984787 267535 228165 750170 673557 613732 612894 777923 633653 (254 digits)

T=2.9s

2.v=902, $\xi_{\max} = 450$

$\xi = 13, \Phi_1\left(902, 2^{27}\right) =$ 79 × 229 × 1 460479 × 3412 454982 716461 315441 492312 197972 257894 940472 842179 546378 500415 486322 912154 774592 223763 649210 130907 126821 500568 874032 042527 791656 219277 508682 946942 316385 551856 434126 811018 680404 462255 372535 998280 156025 411962 803812 239151 596564 022946 652154 073044 186875 329433 668351 (262 digits)

T=0.3s

3.v=904, $\xi_{\max} = 451$

$\xi = 0, \Phi_1\left(904, 2^1\right) =$ $3^3 \times 5 \times 55349\ 376221 \times 269\ 949933\ 997411 \times 178794\ 883900\ 768710\ 779431$
681652 203639 002785 735652 238602 781429 372818 766589 763466 986895 723986 365238 027066
477543 364144 304665 741374 114296 962532 152246 702412 638211 290697 066742 179336 399463
717647 854627 879305 921306 176521 561550 312327 666557 219221 818976 363757 (246 digits)

T=3.8s

4.v=906, $\xi_{\max} = 452$

$\xi = 0, \Phi_1\left(906, 2^1\right) =$ 2137 × 7904 941489 × 85 396804 898645 518273 535449 436485 582682 070411
560076 424907 006632 243214 309176 544188 795999 633591 556075 663051 094702 367573 582291
715723 564664 172471 347636 830470 189043 981602 095079 931244 982740 899398 967082 280998
040450 772949 380551 511813 090613 292727 464205 777687 764915 232261 (260 digits)

T=0.4s

5.v=908, $\xi_{\max} = 453$

$\xi = 2, \Phi_1\left(908, 2^5\right) =$ 5 × 11 × 13 × 193 × 31337 × 5193 211631 × 78675 252767 × 639414 588139 ×
5107 722783 935179 304172 239044 061832 508951 237990 324440 802449 265317 534358 446647
711092 117210 040675 533286 426276 096350 873679 327060 545552 370413 520023 090391 873435
398320 355633 012294 950598 332790 746090 550911 652325 384164 764505 681874 426917 792187
(232 digits)

T=1.6s

6.v=910, $\xi_{\max} = 455$

$\xi = 2, \Phi_1\left(910, 2^5\right) =$ $3^2 \times 14341 \times 4\ 775591 \times 37446\ 886656\ 847465\ 991689\ 801239\ 315679$
624739 231358 928800 721195 922116 731834 107587 307131 483040 971907 188058 407841
315162 734199 983990 866115 713353 711711 628902 689770 160555 698001 725152 373447
996175 935434 554759 465933 006827 249223 042787 690762 164517 396758 215478 383294
266960 578897 (263 digits)

T=0.4s

7.v=912, $\xi_{\max} = 457$

$\xi = 2, \Phi_2\left(912, 2^5\right) = \Phi_1^*\left(912, 2^5\right) =$ 23 × 149 × 659 × 60719 × 11 041001 × 26876 151571 × 2 836188
753129 069940 492734 226989 316227 184045 585740 335285 079297 169982 072938 028004 816032
845909 190630 974616 004415 530717 300703 618346 044670 780911 714583 808861 379615 020293
776481 905988 243508 215935 481111 593654 563054 563999 425532 469293 817706 631274 780775
366553 (247 digits)

T=1.0s

8.v=914, $\xi_{\max} = 459$

$\xi = 6, \Phi_2\left(914, 2^{13}\right) = \Phi_1^*\left(914, 2^{13}\right) =$ 29 × 61 × 1949 × 25601 × 5 229957 750865 496264 357355 480088 625174 001536 852714 167066 864397 704695 999698 801925 859661 277596 675508 722238 218991 071382 073662 384074 750194 404457 201630 701080 016677 936353 199094 775345 829505 129480 596437 475996 233880 466205 440690 156352 658022 840786 124621 665088 846591 316666 865841 (265 digits)

T=0.2s

9.v=916, $\xi_{\max} = 461$

$\xi = 1, \Phi_2\left(916, 2^{3}\right) = \Phi_1^*\left(916, 2^{3}\right) =$ 3 × 5 × 643 × 1879 × 15413 × 5 074187 × 41 543569 × 171086 649479 × 183295 663130 862074 067638 363019 172742 346164 777991 554895 205082 555475 491571 720023 959818 603684 236280 364495 760895 496783 666104 655742 353071 597443 639020 055221 481430 590963 609534 893845 473115 725480 837024 043233 943621 581429 100015 236844 781051 117821 631559 (240 digits)

T=0.9s

10.v=918, $\xi_{\max} = 463$

$\xi = 17, \Phi_1\left(918, 2^{35}\right) =$ 7^2 × 11 × 83 × 132 080253 637656 840970 507882 020668 068565 970146 410211 302784 266214 466081 883374 047212 523145 565954 422459 787372 682931 670181 898434 610927 462717 777805 520241 382481 097327 948016 317217 695750 676011 922498 521129 532676 289355 824507 277149 661974 118151 713754 816883 037972 236909 202853 042215 034987 (273 digits)

T=0.1s

From the study above a method for calculating big prime numbers arises: There are $\nu, \xi \in \mathbb{N}$ for which either $\Phi_1(\nu, \xi)$ or $\Phi_2(\nu, \xi)$ is the product of a set of small primes with a big prime. $\Phi_1(\nu, \xi)$ and $\Phi_2(\nu, \xi)$ can be determined theoretically or statistically. In a first statistical study with small values of the even numbers ν , $\nu < 10^5$, the method gave a big set of primes. Indicatively we list eight of them:

Example 5.3.

1. $\Phi_2(\nu = 3500, \xi = 2) = \Phi_2\left(3500, 2^5\right) = 29^2 \times 257 \times P$ where the prime P has 1049 digits.

2. The number $\Phi_2(\nu = 6696, \xi = 980) = \Phi_2\left(6696, 2^{1961}\right)$ is a prime with 2017 digits.

3. $\Phi_1(\nu = 10032, \xi = 592) = \Phi_1\left(10032, 2^{1185}\right) = 5 \times P$ where the prime P has 3020 digits.

4. $\Phi_1(\nu = 16188, \xi = 0) = \Phi_1\left(16188, 2^{1}\right) = 5 \times P$ where the prime P has 4873 digits.

5. $\Phi_1(\nu = 17448, \xi = 2190) = \Phi_1(17448, 2^{4381}) = 5 \times P$ where the prime P has 5253 digits.

6. $\Phi_1(\nu = 25700, 0) = \Phi_1(25700, 2^1) = 5 \times P$ where the prime P has 7737 digits.

7. $\Phi_1(\nu = 99934, \xi = 18) = \Phi_1(99934, 2^{37}) = 3 \times P$ where the prime P has 30084 digits.

8. The number $\Phi_1(\nu = 99990, \xi = 16318) = \Phi_1(99990, 2^{32637})$ is a prime with 30101 digits.

Mathematical calculations have been done by a low computational power computer, so the method could not be applied for values of $\nu > 10^5$. We expect that the application of the method in computer network with high computational power will give extremely high primes. The number of digits of the big prime calculated by the method is of order $\nu \log 2$.

Theorem 4.1 highlights additional symmetries of the internal structure of the natural numbers. These symmetries determine a priori the signs of $\beta_i = \pm 1, i = 0,1,2,........,\nu - 1$ in equation (4.1). In this article we will not make any more mention in other symmetries of equation (4.1).

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