Bipolar Neutrosophic Planar Graphs

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Abstract

Fuzzy graph theory is used for solving real-world problems in different fields, including theoretical computer science, engineering, physics, combinatorics and medical sciences. In this paper, we present conepts of bipolar neutrosophic multigraphs, bipolar neutrosophic planar graphs, bipolar neutrosophic dual graphs, and study some of their related properties. We also describe applications of bipolar neutrosophic graphs in road network and electrical connections.

Keywords: Bipolar neutrosophic planar graphs, bipolar neutrosophic dual graphs.

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1 Introduction

Fuzzy graph theory has a number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Kaufmann defined first fuzzy graph [12], then Rosenfeld [15] discussed several basic graph-theoretic concepts, including bridges, cut-nodes, connectedness, trees and cycles. Bhattacharya [9] gave some remarks on fuzzy graphs, and Sunitha and Vijayakumar [11] characterized fuzzy trees. Abdul-jabbar *et al.* [1] introduced the concept of a fuzzy dual graph and discussed some of its interesting properties. Samanta and Pal [16, 17] introduced and investigated the concept of fuzzy planar graphs and studied several properties. On other hand, Alshehri and Akram [7] introduced the concept of intuitionistic fuzzy planar graphs. Akram et al. [6] discussed the concept of bipolar fuzzy planar graphs. Dhavaseelan et al. [11] defined strong neutrosophic graphs. Akram and Shahzadi [2] introduced the notions of neutrosophic graphs and neutrosophic soft graphs. In this paper, we introduce the notions of bipolar single-valued neutrosophic multigraphs, bipolar single-valued neutrosophic planar graphs, bipolar single-valued neutrosophic dual graphs, and investigate some of their interesting properties. We also describe applications of bipolar neutrosophic graphs in road network and electrical connections.

2 Bipolar neutrosophic planar graphs

Smarandache [14] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets.

Definition 2.1. [14] A neutrosophic set C on a non-empty set X is characterized by a truth membership function $T_C : X \to [0,1]$, indeterminacy membership function $I_C : X \to [0,1]$ and a falsity membership function $F_C : X \to [0,1]$. There is no restriction on the sum of $T_C(x)$, $I_C(x)$ and $F_C(x)$ for all $x \in X$.

Deli et al. [10] defined bipolar neutrosophic (BN) sets a generalization of bipolar fuzzy sets.

Definition 2.2. [10] A BN set on a nonempty set X is an object of the form

$$C = \{(y, T_C^P(x), I_C^P(x), F_C^P(x), T_C^N(x), I_C^N(x), F_C^N(x)) : y \in X\}$$

where, $T_C^P, I_C^P, F_C^P : Y \to [0,1]$ and $T_C^N, I_C^N, F_C^N : Y \to [-1,0]$. The positive values $T_C^P(x), I_C^P(x), F_C^P(x)$ denote respectively the truth, indeterminacy and false membership degrees of an element $y \in Y$ whereas $T_C^N(x), I_C^N(x), F_C^N(x)$ denote the implicit counter property of the truth, indeterminacy and false membership degrees of the element $y \in X$ corresponding to the bipolar neutrosophic set C.

We define BN multisets based on the concept of Ye and Ye [18].

Definition 2.3. Let X be a nonempty set with generic elements in X denoted by x. A BN multiset C drawn from X is characterized by the three positive functions: count truth-membership of CT_C^P, count indeterminacy-membership of CI_C^P, and count falsity-membership of CF_C^P such that $CT_C^P(x) : X \to R^+$, $CI_C^P(x) : X \to R^+$, $CF_C^P(x) : X \to R^+$ for $x \in X$, where R^+ is the set of all real number multisets in the real unit interval $[0,1]$, and three negative functions: count truth-membership of CT_C^N, count indeterminacy-membership of CI_C^N, and count falsity-membership of CF_C^N such that $CT_C^N(x) : X \to R^-$, $CI_C^N(x) : X \to R^-$, $CF_C^N(x) : X \to R^-$ for $x \in X$, where R^- is the set of all real number multisets in the real unit interval $[-1,0]$,. Then, a bipolar single valued neutrosophic multiset A is denoted by
$A = \{\langle x, ((T^1)_C^P(x), (T^2)_C^P(x), \ldots, (T^q)_C^P(x)), ((I^1)_C^P(x), (I^2)_C^P(x), \ldots, (I^q)_C^P(x)), ((F^1)_C^P(x), (F^2)_C^P(x), \ldots, (F^q)_C^P(x)$
$(T^1)_C^N(x), (T^2)_C^N(x), \ldots, (T^q)_C^N(x)), ((I^1)_C^N(x), (I^2)_C^N(x), \ldots, (I^q)_C^N(x)), ((F^1)_C^N(x), (F^2)_C^N(x), \ldots, (F^q)_C^N(x))\rangle | x \in X\}$, where the positive truth, indeterminacy and falsity-membership sequences
$((T^1)_C^P(x), (T^2)_C^P(x), \ldots, (T^q)_C^P(x)), ((I^1)_C^P(x), (I^2)_C^P(x), \ldots, (I^q)_C^P(x)), ((F^1)_C^P(x), (F^2)_C^P(x), \ldots, (F^q)_C^P(x))$
may be in decreasing or increasing order, and sum of $(T_C^i)^P(x)$, $(I^i)_C^P(x)$, $(F^i)_C^P(x) \in [0,1]$ satisfies

the condition $0 \leq \sup(T^i)^P_C(x) + \sup(I^i)^P_C(x) + \sup(F^i)^P_C(x) \leq 3$ for $x \in X$ and $i = 1, 2, \ldots, q$, the negative truth, indeterminacy and falsity-membership sequences $((T^1)^P_C(x), (T^2)^P_C(x), \ldots, (T^q)^P_C(x)), ((I^1)^N_C(x), (I^2)^N_C(x), \ldots, (I^q)^N_C(x)), ((F^1)^N_C(x), (F^2)^N_C(x), \ldots, (F^q)^N_C(x))$ may be in decreasing or increasing order, and sum of $(T^i_C)^N(x)$, $(I^i)^N_C(x)$, $(F^i)^N_C(x) \in [-1, 0]$ satisfies the condition $-3 \leq \inf(T^i)^N_C(x) + \inf(I^i)^N_C(x) + \inf(F^i)^N_C(x) \leq 0$ for $x \in X$ and $i = 1, 2, \ldots, q$. For convenience, a BN multiset C can be denoted by the simplified form:

$$C = \{\langle x, (T)^P_C(x)_i, (I)^P_C(x)_i, (F)^P_C(x)_i, (T)^N_C(x)_i, (I)^N_C(x)_i, (F)^N_C(x)_i\rangle | x \in X, i = 1, 2, \ldots, q\}.$$

We now define the concept of bipolar neutrosophic graphs.

Definition 2.4. A *bipolar neutrosophic graph* on a nonempty set X is a pair $G = (C, D)$, where C is a bipolar neutrosophic set on X and D is a bipolar neutrosophic relation in X such that

(a) $T^P_D(yz) \leq \min(T^P_C(y), T^P_C(z))$,

(b) $I^P_D(yz) \leq \min(I^P_C(y), I^P_C(z))$,

(c) $F^P_D(yz) \leq \max(F^P_C(y), F^P_C(z))$,

(d) $T^N_D(yz) \geq \max(T^N_C(y), T^N_C(z))$,

(e) $I^N_D(yz) \geq \max(I^N_C(y), I^N_C(z))$,

(f) $F^N_D(yz) \geq \min(F^N_C(y), F^N_C(z))$

for all $y, z \in X$. Note that D is called a BN relation on C.

Example 2.5. Consider a bipolar neutrosophic graph $G = (C, D)$ on $X = \{x, y, z\}$ as shown in Fig. 2.1.

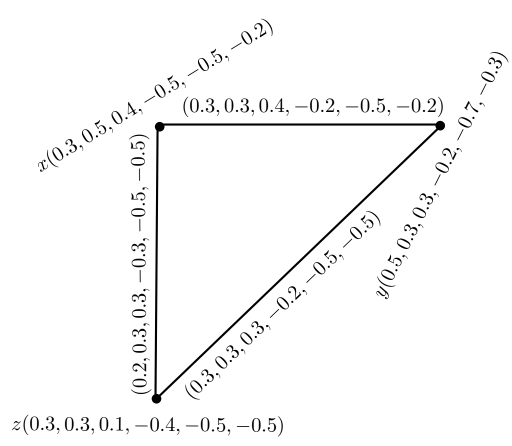

Figure 2.1: Bipolar neutrosophic graph

Definition 2.6. Let $C = (T_C^P, I_C^P, F_C^P, T_C^N, I_C^N, F_C^N)$ be a BN set on V and let $D = \{(xy, T_D^P(xy)_i, I_D^P(xy)_i, F_D^P(xy)_i, T_D^N(xy)_i, I_D^N(xy)_i, F_D^N(xy)_i), i = 1, 2, \ldots, m | xy \in V \times V\}$ be a BN multiset of $V \times V$ such that

(g) $T_D^P(xy)_i \leq \min\{T_C^P(x), T_C^P(y)\}$,

(h) $T_D^N(xy)_i \geq \max\{T_C^N(x), T_C^N(y)\}$,

(i) $I_D^P(xy)_i \leq \min\{I_C^P(x), I_C^P(y)\}$,

(j) $I_D^N(xy)_i \geq \max\{I_C^N(x), I_C^N(y)\}$,

(k) $F_D^P(xy)_i \leq \max\{F_C^P(x), F_C^P(y)\}$,

(l) $F_D^N(xy)_i \geq \min\{F_C^N(x), F_C^N(y)\}$

for all $i = 1, 2, \ldots, m$. Then $G = (C, D)$ is called *a BN multigraph.*

There may be more than one edge between the vertices x and y. The positive values $T_D^P(xy)_i$, $I_D^P(xy)_i$, $F_D^P(xy)_i$ represent truth, indeterminacy and falsity of the edge xy in G, whereas the negative values $T_D^N(xy)_i$, $I_D^N(xy)_i$, $F_D^N(xy)_i$ represent the implicit counter property of the truth, indeterminacy and false membership degrees of the edge xy in G. m denotes the number of edges between the vertices. In BN multigraph G, D is said to be BN multiedge set.

Example 2.7. Let $G^* = (V, E)$, where $V = \{a, b, c, d\}$, $E = \{ab, ab, ab, bc, bd\}$. Let $C = (T_C^P, I_C^P, F_C^P, T_C^N, I_C^N, F_C^N)$ be a BN set on V and $D = (T_D^P, I_D^P, F_D^P, T_D^N, I_D^N, F_D^N)$ be a BN multiedge set on $E \subseteq V \times V$ defined in Table 1 and Table 2.

Table 1: Single-valued neutrosophic set C

C	a	b	c	d
T_C^P	0.5	0.4	0.5	0.4
I_C^P	0.3	0.2	0.4	0.3
F_C^P	0.3	0.4	0.3	0.4
T_C^N	-0.5	-0.4	-0.5	-0.4
I_C^N	-0.3	-0.2	-0.4	-0.3
F_C^N	-0.3	-0.4	-0.3	-0.4

Table 2: BN multiedge set D

D	ab	ab	ab	bc	bd
T_D^P	0.2	0.1	0.2	0.3	0.1
I_D^P	0.2	0.1	0.2	0.1	0.2
F_D^P	0.2	0	0.2	0.3	0.2
T_D^N	-0.2	-0.1	-0.2	-0.3	-0.1
I_D^N	-0.2	-0.1	-0.2	-0.1	-0.2
F_D^N	-0.2	-0	-0.2	-0.3	-0.2

By direct calculations, we see from Fig. 2.2 that it is a BN multigraph.

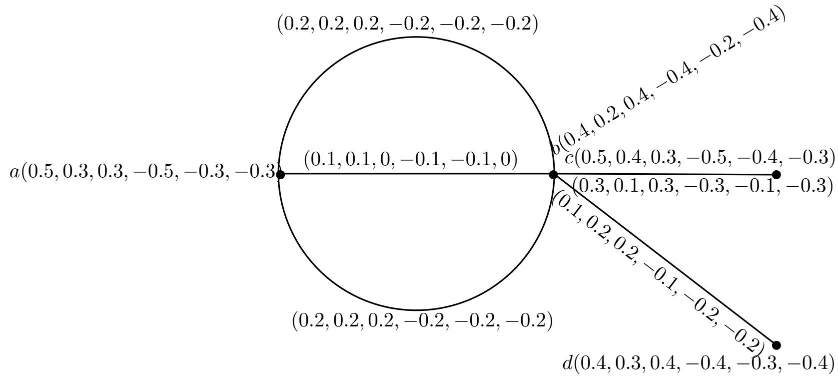

Figure 2.2: Neutrosophic multigraph

Definition 2.8. Let $D = \{(xy, T_D^P(xy)_i, I_D^P(xy)_i, F_D^P(xy)_i, T_D^N(xy)_i, I_D^N(xy)_i, F_D^N(xy)_i), i = 1, 2, \ldots, m | xy \in V \times V\}$ be a BN multiedge set in BN multigraph G. The degree of a vertex $x \in V$, denoted by $\deg(x)$, is defined by

$$\deg(x) = (\sum_{i=1}^{m} T_D^P(xy)_i, \sum_{i=1}^{m} I_D^P(xy)_i, \sum_{i=1}^{m} F_D^P(xy)_i, \sum_{i=1}^{m} T_D^N(xy)_i, \sum_{i=1}^{m} I_D^N(xy)_i, \sum_{i=1}^{m} F_D^N(xy)_i).$$

Example 2.9. In Example 2.7, the degree of vertices a, b, c, d are $deg(a) = (0.5, 0.5, 0.4, -0.5, -0.5, -0.4)$, $deg(b) = (0.9, 0.8, 0.9, -0.9, -0.8, -0.9)$, $deg(c) = (0.3, 0.1, 0.3, -0.3, -0.1, -0.3)$ and $deg(d) = (0.1, 0.2, 0.2, -0.1, -0.2, -0.2)$.

Definition 2.10. Let $D = \{(xy, T_D^P(xy)_i, I_D^P(xy)_i, F_D^P(xy)_i, T_D^N(xy)_i, I_D^N(xy)_i, F_D^N(xy)_i), i = 1, 2, \ldots, m | xy \in V \times V\}$ be a BN multiedge set in BN multigraph G. A multiedge xy of G is *strong* if the following conditions are satisfied:

(m) $\frac{1}{2}\min\{T_C^P(x), T_C^P(y)\} \leq T_D^P(xy)_i$,

(n) $\frac{1}{2}\max\{T_C^N(x), T_C^N(y)\} \geq T_D^N(xy)_i$,

(o) $\frac{1}{2}\min\{I_C^P(x), I_C^P(y)\} \leq I_D^P(xy)_i$,

(p) $\frac{1}{2}\max\{I_C^N(x), I_C^N(y)\} \geq I_D^N(xy)_i$,

(q) $\frac{1}{2}\max\{F_C^P(x), F_C^P(y)\} \geq F_D^P(xy)_i$,

(r) $\frac{1}{2}\min\{F_C^N(x), F_C^N(y)\} \leq F_D^N(xy)_i$

for all $i = 1, 2, \ldots, m$.

Definition 2.11. Let $D = \{(xy, T_D^P(xy)_i, I_D^P(xy)_i, F_D^P(xy)_i, T_D^N(xy)_i, I_D^N(xy)_i, F_D^N(xy)_i), i = 1, 2, \ldots, m | xy \in V \times V\}$ be a BN multiedge set in BN multigraph G. A BN multigraph G is *complete* if the following conditions are satisfied:

(s) $\min\{T_C^P(x), T_C^P(y)\} = T_D^P(xy)_i$,

(t) $\max\{T_C^N(x), T_C^N(y)\} = T_D^N(xy)_i$,

(u) $\min\{I_C^P(x), I_C^P(y)\} = I_D^P(xy)_i$,

(v) $\max\{I_C^N(x), I_C^N(y)\} = I_D^N(xy)_i$,

(w) $\max\{F_C^P(x), F_C^P(y)\} = F_D^P(xy)_i$

(x) $\min\{F_C^N(x), F_C^N(y)\} = F_D^N(xy)_i$

for all $i = 1, 2, \ldots, m$ and for all $x, y \in V$.

Example 2.12. Consider a BN multigraph G as shown in Fig. 2.3. By routine calculations, it is easy to see that Fig. 2.3 is a BN complete multigraph.

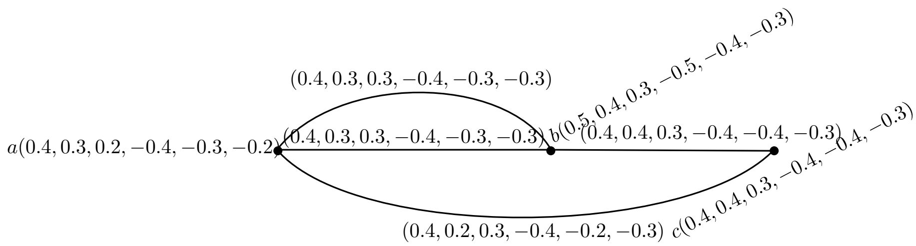

Figure 2.3: Bipolar neutrosophic complete multigraph.

Suppose that geometric insight for BN graphs has only one crossing between single valued neutrosophic edges $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i)$ and $(cd, T_D^P(cd)_i, I_D^P(cd)_i, F_D^P(cd)_i, T_D^N(cd)_i, I_D^N(cd)_i, F_D^N(cd)_i)$. We note that:

- If $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i) = (1, 1, 1, -1, -1, -1)$ and $(cd, T_D^P(cd)_i, I_D^P(cd)_i, F_D^P(cd)_i, T_D^N(cd)_i, I_D^N(cd)_i, F_D^N(cd)_i) = (0, 0, 0, 0, 0, 0)$ or $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i) = (0, 0, 0, 0, 0, 0)$, $(cd, T_D^P(cd)_i, I_D^P(cd)_i, F_D^P(cd)_i, T_D^N(cd)_i, I_D^N(cd)_i, F_D^N(cd)_i) = (1, 1, 1, -1, -1, -1)$, then BN graph has no crossing,

- If $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i) = (1, 1, 1, -1, -1, -1)$ and $(cd, T_D^P(cd)_i, I_D^P(cd)_i, F_D^P(cd)_i, T_D^N(cd)_i, I_D^N(cd)_i, F_D^N(cd)_i) = (1, 1, 1, -1, -1, -1)$, then there exists a crossing for the representation of the graph.

Definition 2.13. The *strength of the BN edge ab* can be measured by the value
$S_{ab} = ((S_{T^P})_{ab}, (S_{I^P})_{ab}, (S_{F^P})_{ab}, (S_{T^N})_{ab}, (S_{I^N})_{ab}, (S_{F^N})_{ab})$
$= (\frac{T_D^P(ab)_i}{\min(T_C^P(a), T_C^P(b))}, \frac{I_D^P(ab)_i}{\min(I_C^P(a), I_C^P(b))}, \frac{F_D^P(ab)_i}{\max(F_C^P(a), F_C^P(b))}, \frac{T_D^N(ab)_i}{\max(T_C^N(a), T_C^N(b))}, \frac{I_D^N(ab)_i}{\max(I_C^N(a), I_C^N(b))}, \frac{F_D^N(ab)_i}{\min(F_C^N(a), F_C^N(b))})$.

Definition 2.14. Let G be a BN multigraph. An edge ab is said to be a *BN strong* if $(S_{T^P})_{ab} \geq 0.5$, $(S_{I^P})_{ab} \geq 0.5$, $(S_{F^P})_{ab} \geq 0.5$, $(S_{T^N})_{ab} \leq -0.5$, $(S_{I^N})_{ab} \leq -0.5$, $(S_{F^N})_{ab} \leq -0.5$ otherwise, we call weak edge.

Definition 2.15. Let $G = (C, D)$ be a BN multigraph such that D contains two edges $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i)$ and $(cd, T_D^P(cd)_j, I_D^P(cd)_j, F_D^P(cd)_j, T_D^N(cd)_j, I_D^N(cd)_j, F_D^N$ intersected at a point P, where i and j are fixed integers. We define the intersecting value at the point Q by
$\mathcal{S}_Q = ((\mathcal{S}_{\mathcal{T}^\mathcal{P}})_Q, (\mathcal{S}_{\mathcal{I}^\mathcal{P}})_Q, (\mathcal{S}_{\mathcal{F}^\mathcal{P}})_Q, (\mathcal{S}_{\mathcal{T}^\mathcal{N}})_Q, (\mathcal{S}_{\mathcal{I}^\mathcal{N}})_Q, (\mathcal{S}_{\mathcal{F}^\mathcal{N}})_Q)$
$= (\frac{(S_{T^P})_{ab}+(S_{T^P})_{cd}}{2}, \frac{(S_{I^P})_{ab}+(S_{I^P})_{cd}}{2}, \frac{(S_{F^P})_{ab}+(S_{F^P})_{cd}}{2}, \frac{(S_{T^N})_{ab}+(S_{T^N})_{cd}}{2}, \frac{(S_{I^N})_{ab}+(S_{I^N})_{cd}}{2}, \frac{(S_{F^N})_{ab}+(S_{F^N})_{cd}}{2})$.

If the number of point of intersections in a BN multigraph increases, planarity decreases. Thus for BN multigraph, $\mathcal{S}_Q$ is inversely proportional to the planarity. We now introduce the concept of a BN planar graph.

Definition 2.16. Let G be a BN multigraph and $Q_1, Q_2, \ldots, Q_z$ be the points of intersection between the edges for a certain geometrical representation, G is said to be a BN planar graph with BN planarity value $f = (f_{T^P}, f_{I^P}, f_{F^P}, f_{T^N}, f_{I^N}, f_{F^N})$, where

$$\begin{aligned} f &= (f_{T^P}, f_{I^P}, f_{F^P}, f_{T^N}, f_{I^N}, f_{F^N}) \\ &= (\frac{1}{1+\{(S_{T^P})_{Q_1} + (S_{T^P})_{Q_2} + \ldots + (S_{T^P})_{Q_z}\}}, \frac{1}{1+\{(S_{I^P})_{Q_1} + (S_{I^P})_{Q_2} + \ldots + (S_{I^P})_{Q_z}\}} \\ &, \frac{1}{1+\{(S_{F^P})_{Q_1} + (S_{F^P})_{Q_2} + \ldots + (S_{F^P})_{Q_z}\}}, \frac{1}{-1-\{(S_{T^N})_{Q_1} + (S_{T^N})_{Q_2} + \ldots + (S_{T^N})_{Q_z}\}} \\ &, \frac{1}{-1-\{(S_{I^N})_{Q_1} + (S_{I^N})_{Q_2} + \ldots + (S_{I^N})_{Q_z}\}}, \frac{1}{-1-\{(S_{F^N})_{Q_1} + (S_{F^N})_{Q_2} + \ldots + (S_{F^N})_{Q_z}\}}). \end{aligned}$$

Clearly, $f = (f_{T^P}, f_{I^P}, f_{F^P}, f_{T^N}, f_{I^N}, f_{F^N})$ is bounded and $0 < f_{T^P} \leq 1$, $0 < f_{I^P} \leq 1$, $0 < f_{F^P} \leq 1$, $-1 < f_{T^N} \leq 0$, $-1 < f_{I^N} \leq 0$, $-1 < f_{F^N} \leq 0$.

If there is no point of intersection for a certain geometrical representation of a BN planar graph, then its BN planarity value is $(1, 1, 1, -1, -1, -1)$. We conclude that every BN graph is a BN planar graph with certain BN planarity value.

Example 2.17. Consider a multigraph $G^* = (V, E)$ such that $V = \{a, b, c, d, e\}$,

$$E = \{ab, ac, ad, ad, bc, bd, cd, ce, ae, de, be\}.$$

Let $C = (T_C^P, I_C^P, F_C^P, T_C^N, I_C^N, F_C^N)$ be a BN set of V and let $D = (T_D^P, I_D^P, F_D^P, T_D^N, I_D^N, F_D^N)$ be a BN multiedge set of $V \times V$ defined in Table 3 and Table 4.

Table 3: BN set C

A	a	b	c	d	e
T_C^P	0.5	0.4	0.3	0.6	0.6
I_C^P	0.5	0.4	0.3	0.6	0.6
F_C^P	0.2	0.1	0.1	0.2	0.1
T_C^N	-0.5	-0.4	-0.3	-0.6	-0.6
I_C^N	-0.5	-0.4	-0.3	-0.6	-0.6
F_C^N	-0.2	-0.1	-0.1	-0.2	-0.1

Table 4: BN multiedge set D

B	ab	ac	ad	ad	bc	bd	cd	ae	ce	de	be
T_D^P	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
I_D^P	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
F_D^P	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
T_D^N	-0.2	-0.2	-0.2	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2
I_D^N	-0.2	-0.2	-0.2	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2
F_D^N	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1

The BN multigraph as shown in Fig. 2.4 has two point of intersections P_1 and P_2. P_1 is a point between the edges $(ad, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$ and $(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$ and P_2 is between $(ad, 0.3, 0.3, 0.1, -0.3, -0.3, -0.1)$ and $(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$.
For the edge $(ad, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$, $S_{ad} = (0.4, 0.4, 0.5, -0.4, -0.4, -0.5)$,
For the edge $(ad, 0.3, 0.3, 0.1, -0.3, -0.3, -0.1)$, $S_{ad} = (0.6, 0.6, 0.5, -0.6, -0.6, -0.5)$ and
for the edge $(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$, $S_{bc} = (0.6667, 0.6667, 1, -0.6667, -0.6667, -1)$.

For the first point of intersection P_1, intersecting value $\mathcal{S}_{P_1}$ is $(0.5334, 0.5334, , 0.75, -0.5334, -0.5334, -0.75)$ and that for the second point of intersection $P_2, \mathcal{S}_{P_2} = (0.63335, 0.63335, , 0.75, -0.63335, -0.63335, -0.75)$.
Therefore, the BN planarity value for the BN multigraph shown in Fig. 2.4 is $(0.461, 0.461, 0.4, -0.461, -0.461, -0.4$

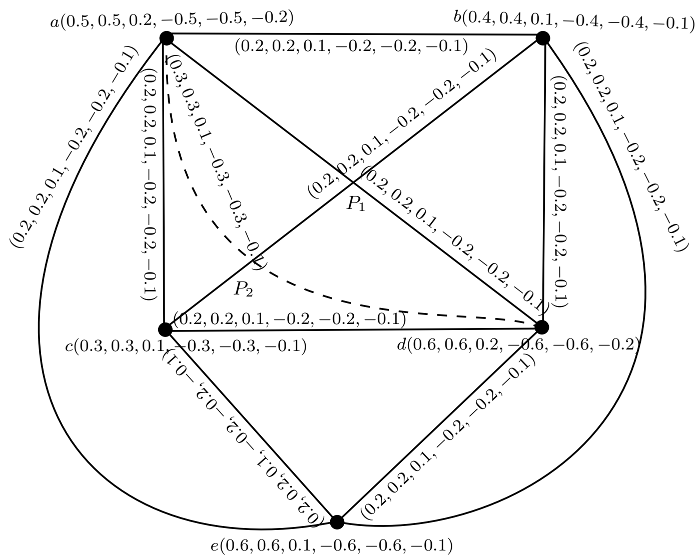

Figure 2.4: Neutrosophic planar graph

Theorem 2.18. *Let G be a BN complete multigraph. The planarity value, $f = (f_{T^P}, f_{I^P}, f_{F^P}, f_{T^N}, f_{I^N}, f_{F^N})$ of G is given by $f_{T^P} = \frac{1}{1+n_Q}$, $f_{I^P} = \frac{1}{1+n_Q}$ and $f_{F^P} = \frac{1}{1+n_Q}$ such that $f_{T^P} + f_{I^P} + f_{F^P} \leq 3$, $f_{T^N} = \frac{1}{-1-n_Q}$, $f_{I^N} = \frac{1}{-1-n_Q}$ and $f_{F^N} = \frac{1}{-1-n_Q}$ such that $-3 \leq f_{T^N} + f_{I^N} + f_{F^N} \leq 0$ where n_Q is the number of point of intersections between the edges in G.*

Definition 2.19. A BN planar graph G is called *strong BN planar graph* if the BN planarity value $f = (f_{T^P}, f_{I^P}, f_{F^P}, f_{T^N}, f_{I^N}, f_{F^N})$ of the graph is $f_{T^P} \geq 0.5$, $f_{I^P} \geq 0.5$, $f_{F^P} \leq 0.5$, $f_{T^N} \leq -0.5$, $f_{I^N} \leq -0.5$, $f_{F^P} \geq -0.5$.

Theorem 2.20. *Let G be a strong BN planar graph. The number of point of intersections between strong edges in G is at most one.*

Proof. Let G be a strong BN planar graph. Assume that G has at least two point of intersections P_1 and P_2 between two strong edges in G. For any strong edge $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i)$,

$$T_D^P(ab)_i \geq \frac{1}{2}\min\{T_C^P(a), T_C^P(b)\}, I_D^P(ab)_i \geq \frac{1}{2}\min\{I_C^P(a), I_C^P(b)\}, F_D^P(ab)_i \leq \frac{1}{2}\max\{F_C^P(a), F_C^P(b)\},$$

$$T_D^N(ab)_i \leq \frac{1}{2}\max\{T_C^N(a), T_C^N(b)\}, I_D^N(ab)_i \leq \frac{1}{2}\max\{I_C^N(a), I_C^N(b)\}, F_D^N(ab)_i \geq \frac{1}{2}\min\{F_C^N(a), F_C^N(b)\}.$$

This shows that $(S_{T^P})_{ab} \geq 0.5$, $(S_{I^P})_{ab} \geq 0.5$, $(S_{F^P})_{ab} \leq 0.5$, $(S_{T^N})_{ab} \leq -0.5$, $(S_{I^N})_{ab} \leq -0.5$, $(S_{F^N})_{ab} \geq -0.5$.

Thus for two intersecting strong edges $(ab, T_D^P(ab)_i, I_D^P(ab)_i, F_D^P(ab)_i, T_D^N(ab)_i, I_D^N(ab)_i, F_D^N(ab)_i)$ and $(cd, T_D^P(cd)_j, I_D^P(cd)_j, F_D^P(cd)_j, T_D^N(cd)_j, I_D^N(cd)_j, F_D^N(cd)_j)$,

$$\frac{(S_{T^P})_{ab}+(S_{T^P})_{cd}}{2} \geq 0.5, \frac{(S_{I^P})_{ab}+(S_{I^P})_{cd}}{2} \geq 0.5, \frac{(S_{F^P})_{ab}+(S_{F^P})_{cd}}{2} \leq 0.5,$$
$$\frac{(S_{T^N})_{ab}+(S_{T^N})_{cd}}{2} \leq -0.5, \frac{(S_{I^N})_{ab}+(S_{I^N})_{cd}}{2} \leq -0.5, \frac{(S_{F^N})_{ab}+(S_{F^N})_{cd}}{2} \geq -0.5.$$

That is,

$$(S_{T^P})_{Q_1} \geq 0.5, (S_{I^P})_{Q_1} \geq 0.5, (S_{F^P})_{Q_1} \leq 0.5, (S_{T^N})_{Q_1} \leq -0.5, (S_{I^N})_{Q_1} \leq -0.5, (S_{F^N})_{Q_1} \geq -0.5.$$

Similarly,

$$(S_{T^P})_{Q_2} \geq 0.5, (S_{I^P})_{Q_2} \geq 0.5, (S_{F^P})_{Q_2} \leq 0.5, (S_{T^N})_{Q_2} \leq -0.5, (S_{I^N})_{Q_2} \leq -0.5, (S_{F^N})_{Q_2} \geq -0.5.$$

This implies that $1+(S_{T^P})_{Q_1}+(S_{T^P})_{Q_2} \geq 2$, $1+(S_{I^P})_{Q_1}+(S_{I^P})_{Q_2} \geq 2$, $1+(S_{F^P})_{Q_1}+(S_{F^P})_{Q_2} \leq 2$, $-1+(S_{T^N})_{Q_1}+(S_{T^N})_{Q_2} \leq -2$, $-1+(S_{I^N})_{Q_1}+(S_{I^N})_{Q_2} \leq -2$, $-1+(S_{F^N})_{Q_1}+(S_{F^N})_{Q_2} \geq -2$. Therefore,

$$f_{T^P} = \frac{1}{1+(S_{T^P})_{Q_1}+(S_{T^P})_{Q_2}} \leq 0.5, f_{I^P} = \frac{1}{1+(S_{I^P})_{Q_1}+(S_{I^P})_{Q_2}} \leq 0.5, f_{F^P} = \frac{1}{1+(S_{F^P})_{Q_1}+(S_{F^P})_{Q_2}} \geq 0.5.$$

$$f_{T^N} = \frac{1}{-1+(S_{T^N})_{Q_1}+(S_{T^N})_{Q_2}} \geq -0.5, f_{I^N} = \frac{1}{-1+(S_{I^N})_{Q_1}+(S_{I^N})_{Q_2}} \geq -0.5, f_{F^N} = \frac{1}{-1+(S_{F^N})_{Q_1}+(S_{F^N})_{Q_2}} \leq -0.5$$

It contradicts the fact that the BN graph is a strong BN planar graph. Thus number of point of intersections between strong edges can not be two. Obviously, if the number of point of intersections of strong BN edges increases, the BN planarity value decreases. Similarly, if the number of point of intersection of strong edges is one, then the BN planarity value $f_{T^P} > 0.5$, $f_{I^P} > 0.5$, $f_{I^P} > 0.5$, $f_{T^N} < -0.5$, $f_{I^N} < -0.5$, $f_{I^N} < -0.5$. Any BN planar graph without any crossing between edges is a strong BN planar graph. Thus, we conclude that the maximum number of point of intersections between the strong edges in G is one. □

Face of a BN planar graph is an important parameter. Face of a BN graph is a region bounded by BN edges. Every BN face is characterized by BN edges in its boundary. If all the edges in the boundary of a BN face have T^P, I^P, F^P, T^N, I^N and F^N values $(1,1,1,-1,-1,-1)$ and $(0,0,0,0,0,0)$, respectively, it becomes crisp face. If one of such edges is removed or has T^P, I^P, F^P, T^N, I^N and F^N values $(0,0,0,0,0,0)$ and $(1,1,1,-1,-1,-1)$, respectively, the BN face does not exist. So the existence of a BN face depends on the minimum value of strength of BN edges in its boundary. A BN face and its T^P, I^P, F^P, T^N, I^N, and F^N values of a BN graph are defined below.

Definition 2.21. Let G be a BN planar graph and
$D = \{(xy, T_D^P(xy)_i, I_D^P(xy)_i, F_D^P(xy)_i, T_D^N(xy)_i, I_D^N(xy)_i, F_D^N(xy)_i), i = 1,2,\ldots,m | xy \in V \times V\}$. A BN face of G is a region, bounded by the set of BN edges $E' \subset E$, of a geometric representation of G. The truth, indeterminacy and falsity values of the BN face are:

1. $\min\left\{\frac{T^P_D(xy)_i}{\min\{T^P_C(x),T^P_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$,

2. $\max\left\{\frac{T^N_D(xy)_i}{\max\{T^N_C(x),T^N_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$,

3. $\min\left\{\frac{I^P_D(xy)_i}{\min\{I^P_C(x),I^P_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$,

4. $\max\left\{\frac{I^N_D(xy)_i}{\max\{I^N_C(x),I^N_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$,

5. $\max\left\{\frac{F^P_D(xy)_i}{\max\{F^P_C(x),F^P_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$,

6. $\min\left\{\frac{F^N_D(xy)_i}{\min\{F^N_C(x),F^N_C(y)\}}, i=1,2,\ldots,m|\, xy\in E'\right\}$.

Definition 2.22. A BN face is called *strong BN face* if its positive true and indeterminacy value is greater than 0.5 but false value is lesser than 0.5, and negative true and indeterminacy value is less than -0.5 but false value is greater than -0.5. Otherwise, face is weak. Every BN planar graph has an infinite region which is called *outer BN face*. Other faces are called *inner BN faces*.

Example 2.23. Consider a BN planar graph as shown in Fig. 2.5. The BN planar graph has the following faces:

- BN face F_1 is bounded by the edges
  $(v_1v_2, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1), (v_2v_3, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1), (v_1v_3, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$.
- outer BN face F_2 surrounded by edges
  $(v_1v_3, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1), (v_1v_4, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1), (v_2v_4, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$,
  $(v_2v_3, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$,
- BN face F_3 is bounded by the edges
  $(v_1v_2, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1), (v_2v_4, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1), (v_1v_4, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$.

Clearly, the positive truth, indeterminacy and falsity values of a BN face F_1 are 0.833, 0.833 and 0.333, respectively, and the negative truth, indeterminacy and falsity values of a BN face F_1 are -0.833, -0.833 and -0.333, respectively. The positive truth, indeterminacy and falsity values of a BN face F_3 are 0.833, 0.833 and 0.333, respectively, and the negative truth, indeterminacy and falsity values of a BN face F_3 are -0.833, -0.833 and -0.333, respectively. Thus F_1 and F_3 are strong BN faces.

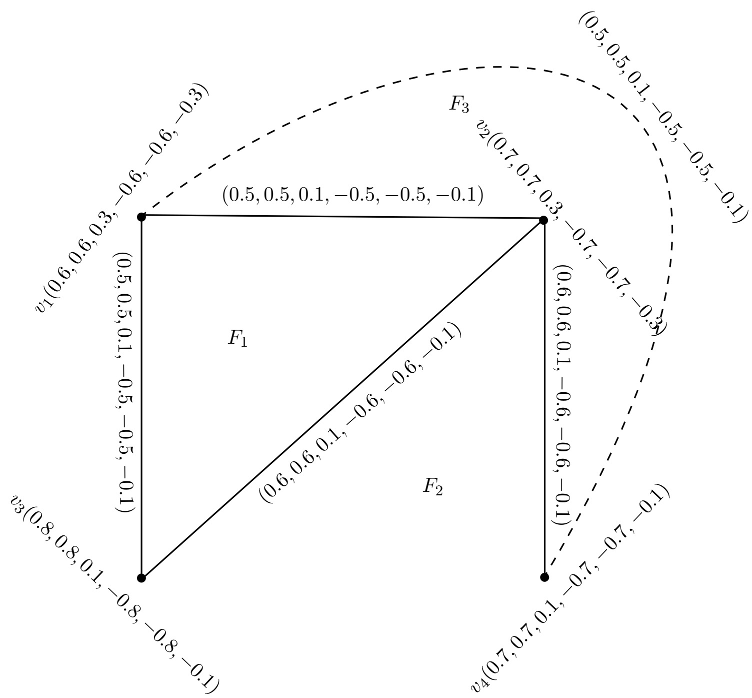

Figure 2.5: Faces in BN planar graph

We now introduce dual of BN planar graph. In BN dual graph, vertices are corresponding to the strong BN faces of the BN planar graph and each BN edge between two vertices is corresponding to each edge in the boundary between two faces of BN planar graph. The formal definition is given below.

Definition 2.24. Let G be a BN planar graph and let
$D = \{(xy, T^P_D(xy)_i, I^P_D(xy)_i, F^P_D(xy)_i, T^N_D(xy)_i, I^N_D(xy)_i, F^N_D(xy)_i), i = 1, 2, \ldots, m | xy \in V \times V\}$. Let $F_1, F_2, \ldots, F_k$ be the strong BN faces of G. The BN dual graph of G is a BN planar graph $G' = (V', C', D')$, where $V' = \{x_i, i = 1, 2, \ldots, k\}$, and the vertex x_i of G' is considered for the face F_i of G. The truth- membership, indeterminacy and false-truth- membership values of vertices are given by the mapping $C' = (T^P_{C'}, I^P_{C'}, F^P_{C'}, T^N_{C'}, I^N_{C'}, F^N_{C'}) : V' \to [0,1] \times [0,1] \times [0,1] \times [-1,0] \times [-1,0] \times [-1,0]$ such that
$T^P_{C'}(x_i) = \max\{T^P_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$,
$T^N_{C'}(x_i) = \min\{T^N_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$,
$I^P_{C'}(x_i) = \max\{I^P_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$,
$I^N_{C'}(x_i) = \min\{I^N_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$,
$F^P_{C'}(x_i) = \min\{F^P_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$,
$F^N_{C'}(x_i) = \max\{F^N_{D'}(uv)_i, i = 1, 2, \ldots, p | uv$ is an edge of the boundary of the strong BN face $F_i\}$.

There may exist more than one common edges between two faces F_i and F_j of G. Thus there may be more than one edges between two vertices x_i and x_j in BN dual graph G'. Let $(T^P)^l_D(x_ix_j)$, $(I^P)^l_D(x_ix_j)$ and $(F^P)^l_D(x_ix_j)$ denote the positive truth, indeterminacy and falsity membership values of the l-th edge between x_i and x_j, and let $(T^N)^l_D(x_ix_j)$, $(I^N)^l_D(x_ix_j)$ and $(F^N)^l_D(x_ix_j)$ denote the negative truth, indeterminacy and falsity membership values of the l-th edge between x_i and x_j. The positive and negative truth, indeterminacy and falsity values of the BN edges of the BN dual graph are given by $T^P_{D'}(x_ix_j)_l = (T^P)^l_D(uv)_j$, $I^P_{D'}(x_ix_j)_l = (I^P)^l_D(uv)_j$, $F^P_{D'}(x_ix_j)_l = (F^P)^l_D(uv)_j$, $T^N_{D'}(x_ix_j)_l = (T^N)^l_D(uv)_j$, $I^N_{D'}(x_ix_j)_l = (I^N)^l_D(uv)_j$, $F^N_{D'}(x_ix_j)_l = (F^N)^l_D(uv)_j$, where $(uv)_l$ is an edge in the boundary between two strong BN faces F_i and F_j and $l = 1, 2, \ldots, s$, where s is the number of common edges in the boundary between F_i and F_j or the number of edges between x_i and x_j.If there be any strong pendant edge in the BN planar graph, then there will be a self loop in G' corresponding to this pendant edge. The edge truth- membership, indeterminacy-membership and falsity-membership value of the self loop is equal to the truth- membership, indeterminacy-membership and falsity-membership value of the pendant edge. Single-valued neutrosophic dual graph of BN planar graph does not contain point of intersection of edges for a certain representation, so it is BN planar graph with planarity value $(1, 1, 1, -1, -1, -1)$. Thus the BN face of BN dual graph can be similarly described as in BN planar graphs.

Example 2.25. Consider a BN planar graph $G = (V, A, B)$ as shown in Fig. 2.6 such that $V = \{a, b, c, d\}$,
$C = (a, 0.6, 0.6, 0.2, -0.6, -0.6, -0.2), (b, 0.7, 0.7, 0.2, -0.7, -0.7, -0.2), (c, 0.8, 0.8, 0.2, -0.8, -0.8, -0.2),$
$(d, 0.9, 0.9, 0.1, -0.9, -0.9, -0.1)$, and $D = \{(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01), (ac, 0.4, 0.4, 0.01, -0.4, -0.4, -0.0$
$(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01), (bc, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01), (bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.$
$(cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01)\}$.

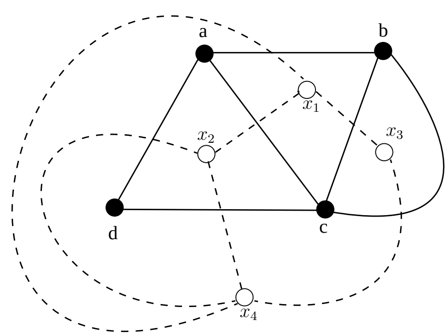

Figure 2.6: Neutrosophic dual graph

The BN planar graph has the following faces:

- BN face F_1 is bounded by
  $(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01), (ac, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01), (bc, 0.45, 0.45, 0.01, -0.45, -0.45, -0$

- BN face F_2 is bounded by
  $(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01), (cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01), (ac, 0.4, 0.4, 0.01, -0.4, -0.4, -$

- BN face F_3 is bounded by
  $(bc, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01), (bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01)$

- outer BN face F_4 is surrounded by
  $(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01), (bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01), (cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01)$
  $(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01)$.

Routine calculations show that all faces are strong BN faces. For each strong BN face, we consider a vertex for the BN dual graph. So the vertex set $V' = \{x_1, x_2, x_3, x_4\}$, where the vertex x_i is taken corresponding to the strong BN face F_i, $i = 1, 2, 3, 4$. Thus

$$T^P_{C'}(x_1) = \max\{0.5, 0.4, 0.45\} = 0.5, T^P_{C'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7,$$

$$T^N_{C'}(x_1) = \min\{-0.5, -0.4, -0.45\} = -0.5, T^N_{C'}(x_2) = \min\{-0.55, -0.7, -0.4\} = -0.7,$$

$$I^P_{C'}(x_1) = \max\{0.5, 0.4, 0.45\} = 0.5, I^P_{C'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7,$$

$$I^N_{C'}(x_1) = \min\{-0.5, -0.4, -0.45\} = -0.5, I^N_{C'}(x_2) = \min\{-0.55, -0.7, -0.4\} = -0.7,$$

$$F^P_{C'}(x_1) = \min\{0.01, 0.01, 0.01\} = 0.01, F^P_{C'}(x_2) = \min\{0.01, 0.01, 0.01\} = 0.01,$$

$$F^N_{C'}(x_1) = \max\{-0.01, -0.01, -0.01\} = -0.01, F^N_{C'}(x_2) = \max\{-0.01, -0.01, -0.01\} = -0.01,$$

$$T^P_{C'}(x_3) = \max\{0.45, 0.6\} = 0.6, T^P_{C'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7,$$

$$T^N_{C'}(x_3) = \min\{-0.45, -0.6\} = -0.6, T^N_{C'}(x_4) = \min\{-0.5, -0.6, -0.7, -0.55\} = -0.7,$$

$$I^P_{C'}(x_3) = \max\{0.45, 0.6\} = 0.6, I^P_{C'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7,$$

$$F^P_{C'}(x_3) = \min\{0.01, 0.01\} = 0.01, F^P_{C'}(x_4) = \min\{0.01, 0.01, 0.01, 0.01\} = 0.01.$$

$$F^N_{C'}(x_3) = \max\{-0.01, -0.01\} = -0.01, F^N_{C'}(x_4) = \max\{-0.01, -0.01, -0.01, -0.01\} = -0.01.$$

There are two common edges ad and cd between the faces F_2 and F_4 in G. Hence between the vertices x_2 and x_4, there exist two edges in the BN dual graph of G. Truth-membership, indeterminacy-membership and falsity-membership values of these edges are given by

$$T^P_{D'}(x_2x_4) = T^P_D(cd) = 0.7, T^P_{D'}(x_2x_4) = T^P_D(ad) = 0.55, I^P_{D'}(x_2x_4) = I^P_D(cd) = 0.7, I^P_{D'}(x_2x_4) = I^P_D(ad) = 0.55,$$

$$F^P_{D'}(x_2x_4) = F^P_D(cd) = 0.01, F^P_{D'}(x_2x_4) = F^P_D(ad) = 0.01.$$

$$T^N_{D'}(x_2x_4) = T^N_D(cd) = -0.7, T^N_{D'}(x_2x_4) = T^N_D(ad) = -0.55, I^N_{D'}(x_2x_4) = I^N_D(cd) = -0.7, I^N_{D'}(x_2x_4) = I^N_D(ad) = -0$$

$$F^N_{D'}(x_2x_4) = F^N_D(cd) = -0.01, F^N_{D'}(x_2x_4) = F^N_D(ad) = -0.01.$$

The truth- membership, indeterminacy-membership and falsity-membership values of other edges of the BN dual graph are calculated as

$T^P_{D'}(x_1x_3) = T^P_D(bc) = 0.45, T^P_{D'}(x_1x_2) = T^P_D(ac) = 0.4, T^P_{D'}(x_1x_4) = T^P_D(ab) = 0.5, T^P_{D'}(x_3x_4) = T^P_{D'}(bc) = 0.6,$

$T^N_{D'}(x_1x_3) = T^N_D(bc) = -0.45, T^N_{D'}(x_1x_2) = T^N_D(ac) = -0.4, T^N_{D'}(x_1x_4) = T^N_D(ab) = -0.5, T^N_{D'}(x_3x_4) = T^N_{D'}(bc) = -$

$I^P_{D'}(x_1x_3) = I^P_D(bc) = 0.45, I^P_{D'}(x_1x_2) = I^P_D(ac) = 0.4, I^P_{D'}(x_1x_4) = I^P_D(ab) = 0.5, I^P_{D'}(x_3x_4) = I^P_{D'}(bc) = 0.6,$

$I^N_{D'}(x_1x_3) = I^N_D(bc) = -0.45, I^N_{D'}(x_1x_2) = I^N_D(ac) = -0.4, I^N_{D'}(x_1x_4) = I^N_D(ab) = -0.5, I^N_{D'}(x_3x_4) = I^N_{D'}(bc) = -0.6,$

$F^P_{D'}(x_1x_3) = T^P_D(bc) = 0.01, F^P_{D'}(x_1x_2) = F^P_D(ac) = 0.01, F^P_{D'}(x_1x_4) = F^P_D(ab) = 0.01, F^P_{D'}(x_3x_4) = F^P_D(bc) = 0.01.$

$F^N_{D'}(x_1x_3) = T^N_D(bc) = 0.01, F^N_{D'}(x_1x_2) = F^N_D(ac) = 0.01, F^N_{D'}(x_1x_4) = F^N_D(ab) = 0.01, F^N_{D'}(x_3x_4) = F^N_D(bc) = 0.01$

Thus the edge set of BN dual graph is
$D' = \{(x_1x_3, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01), (x_1x_2, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01),$
$(x_1x_4, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01), (x_3x_4, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01), (x_2x_4, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01$
$(x_2x_4, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01)\}$. In Fig. 2.6, the BN dual graph $G' = (V', C', D')$ of G is drawn by dotted line.

Weak edges in planar graphs are not considered for any calculation in BN dual graphs. We state the following Theorem without its proof.

Theorem 2.26. *Let $G = (V, C, D)$ be a BN planar graph without weak edges and the BN dual graph of G be $G' = (V', C', D')$. The truth- membership indeterminacy-membership and falsity-membership values of BN edges of G' are equal to truth- membership, indeterminacy-membership and falsity-membership values of the BN edges of G.*

3 Applications

Graph is considered an important part of Mathematics for solving countless real World problems in information technology, psychology, engineering, combinatorics and medical sciences. Everything in this World is connected, for instance, cities and countries are connected by roads, railways are linked by railway lines, flight networks are connected by air, electrical devices are connected by wires, pages on internet by hyperlinks, components of electric circuits by various paths, and many more. Scientists, analysts and engineers are trying to optimize these networks to find a way to save millions of lives by reducing traffic accidents, plane crashes and circuit shots. Planar graphs are used to find such graphical representations of networks without any crossing or minimum number of crossings. But there is always an uncertainty and degree of indeterminacy in data which can be dealt using bipolar neutrosophic graphs. We now present applications of bipolar neutrosophic graphs in road networks.

1. Road network model to monitor traffic: Roads are a mean of frequent and unacceptable number of fatalities every year. Road accidents are increasing due to dense traffic, negligence of drivers and speed of vehicles. Traffic accidents can be minimized by modeling road networks to monitor the traffic, apply quick emergency services and to take action against the speedily going vehicles quickly. The practical approach of bipolar neutrosophic planar graphs can be applied to construct road networks, as these are the combination of vertices and edges along with the degree of truth, indeterminacy and falsity. The method for the construction of road network is given in Algorithm 1.

Algorithm 1

1. Input the n number of location $L_1, L_2, \ldots, L_n$.
2. Input the bipolar neutrosophic set of cities.
3. Input the adjacency matrix of $\xi = [\xi_{ij}]_{n\times n}$ of cities.
4. **do** i from $1 \to n$
5. **do** j from $1 \to n$
6. **if**$(i < j, \xi_{ij} \neq (0,0,1,0,0,-1))$**then**
7. Draw an edge between L_i and L_j.
8. $B(L_iL_j) = \xi_{ij}$
9. **end if**
10. **end do**
11. **end do**

Consider the problem of road networks between 6 locations $L_1, L_2, L_3, L_4, L_5, L_6$. The degree of memberships of cities and roads between cities is given in Table 5 and Table 6.

Table 5: Bipolar neutrosophic set of cities

A	L_1	L_2	L_3	L_4	L_5	L_6
T_A^p	0.7	0.5	0.8	0.6	0.5	0.4
I_A^p	0.4	0.4	0.2	0.1	0.4	0.5
F_A^p	0.2	0.3	0.2	0.1	0.4	0.5
T_A^n	-0.2	-0.3	-0.2	-0.1	-0.4	-0.5
I_A^n	-0.4	-0.4	-0.2	-0.1	-0.4	-0.5
F_A^n	-0.7	-0.5	-0.8	-0.6	-0.5	-0.4

Table 6: Bipolar neutrosophic set of roads

A	L_1L_3	L_1L_6	L_2L_3	L_2L_4	L_3L_5	L_5L_6	L_2L_5	L_3L_6	L_4L_6
T_B^p	0.4	0.4	0.5	0.5	0.5	0.4	0.5	0.4	0.4
I_B^p	0.2	0.4	0.2	0.1	0.2	0.4	0.4	0.2	0.1
F_B^p	0.2	0.5	0.3	0.1	0.4	0.4	0.3	0.5	0.5
T_B^n	-0.2	-0.2	-0.3	-0.1	-0.2	-0.4	-0.3	-0.2	-0.1
I_B^n	-0.4	-0.4	-0.2	-0.1	-0.2	-0.4	-0.4	-0.2	-0.1
F_B^n	-0.7	-0.4	-0.8	-0.6	-0.8	-0.4	-0.5	-0.8	-0.6

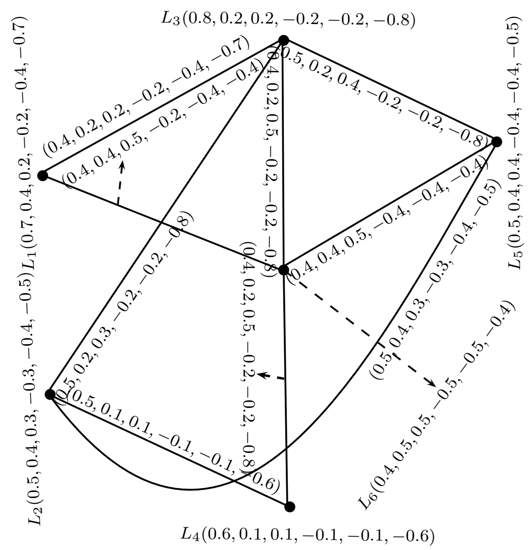

Figure 3.1: Bipolar neutrosophic road model

The positive degree of membership $T^p(x)$ of each vertex x represents the percentage that vehicles traveling to or from this city are dense, $I^p(x)$ and $F^p(x)$ represent the indeterminacy and falsity in this percentage. The negative degree of membership $T^n(x)$ represents the percentage that traffic is not dense, $I^n(x)$ and $F^n(x)$ represent the indeterminacy and falsity in this percentage. The positive degree of memberships of each edge xy indicate the percentage of truth, indeterminacy and falsity of road accidents through this road. The negative degree of memberships of xy show the percentage of truth, indeterminacy and falsity that the road is safer. The bipolar neutrosophic model of road connections between the cities is shown in Fig. 3.1. This bipolar neutrosophic model can be used to check and monitor the percentage of annual accidents. Also, by monitoring and taking special security actions, the total number of accidents can be minimized.

2. Electrical connections: Graph theory is extensively used in designing circuit connections and installation of wires in order to prevent crossing which can cause dangerous electrical hazards. The twisted and crossing wires are a serious safety risk to human life. There is a need to install electrical

wires to reduce crossing. Bipolar neutrosophic planar graphs can be used to model electrical connections and to study the degree of damage that can cause due to the connection.
Consider the problem of setting electrical wires between 5 electrical utilities and power plugs E_1, E_2, E_3, E_4, E_5 in a factory as shown in Fig. 3.2.

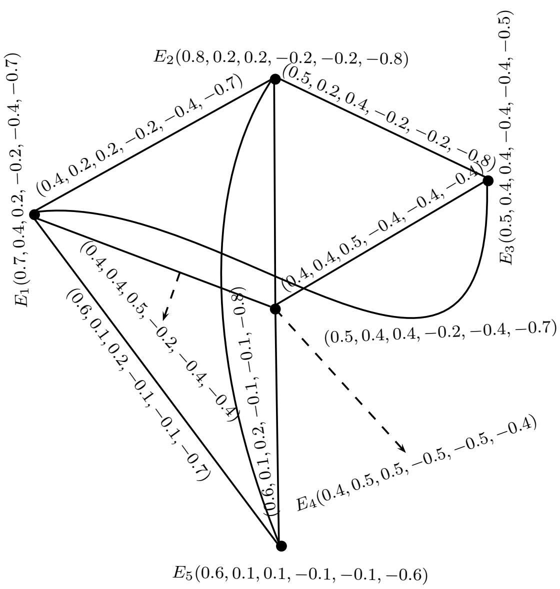

Figure 3.2: Electrical connections

The positive degree of membership $T^p(E_i)$ of each vertex E_i represents the percentage of faults and electrical sparks of utility or power plug E_i , $I^p(E_i)$ and $F^p(E_i)$ represent the indeterminacy and falsity in this percentage. The negative degree of membership $T^n(E_i)$ represents the percentage that E_i is update and safer, $I^n(x)$ and $F^n(x)$ represent the indeterminacy and falsity in this percentage. The positive degree of memberships of each edge E_iE_j indicate the percentage of truth, indeterminacy and falsity of electrical hazards through this connection. The negative degree of memberships of E_iE_j show the percentage of truth, indeterminacy and falsity that the connection is safer. The crossing of wires can be reduced if we change the geometrical representation of Fig. 3.2. The other representation is shown in Fig. 3.3 which has only one crossing, at point P_1, between the edges E_1E_4 and E_2E_5. The electrical damage at crossing point P_1 can be reduced by using better electrical wires between E_1 and E_4, E_2 and E_5.

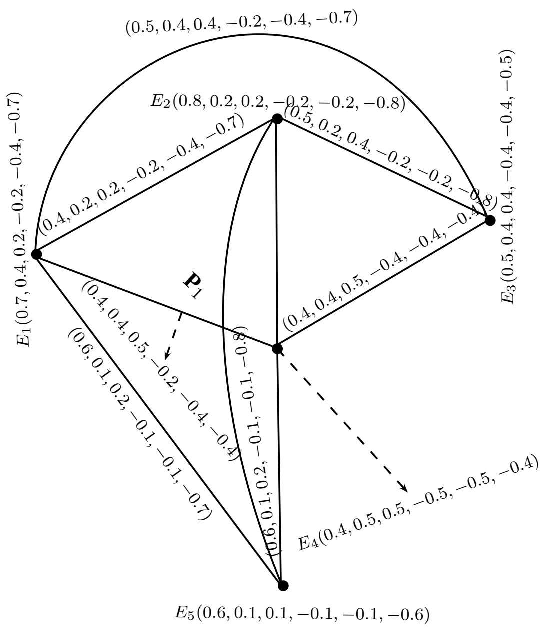

Figure 3.3: Bipolar neutrosophic planar graph

The method for the construction of bipolar neutrosophic planar graph in given in Algorithm 2.

Algorithm 2

1. Input the n number of utilities $E_1, E_2, \ldots, E_n$ and p number of connections $e_1, e_2, \ldots, e_p$.
2. Input the bipolar neutrosophic set of utilities.
3. Input the points of intersection $P_1, P_2, \ldots, P_r$.
4. **do** i from $1 \to r$
5. $\quad$ P_i is a point of intersection between e_j and e_k.
6. $\quad$ Change the graphical representation of one of the edges e_j and e_k.
7. $\quad$ **if** There is no new point of intersection in this representation **then**
8. $\quad\quad$ Keep this graphical representation.
9. $\quad$ **else**
10. $\quad\quad$ Keep the previous graphical representation.

11. **end if**

12. **end do**

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