Relativistic Quantum Theory of Atoms and Gravitation from 3 Postulates

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Abstract

We use three postulates P1, P2a/b and P3 :

P1 : $E = H = \gamma m_0 c^2 - k/r$, defines the Hamiltonian for central potential problems (which can be adapted to other potentials)
P2a : $k = q^2/4\pi\epsilon_0$ / **P2b** : $k = GM\gamma m_0$ define the electromagnetic/gravitationnal potentials
P3 : $\Psi = e^{iS/\hbar}$ defines the wavefunction, with S relativistic action, deduced from **P1**

Combining **P1** and **P2a** with "Sommerfeld's quantum rules" correspond to the original quantum theory of Hydrogen, which produces the correct relativistic energy levels for atoms (Sommerfeld's and Dirac's theories of matter produces the same energy levels, and Schrodinger's theory produces the approximation of those energy levels). **P3** can be found in Schrodinger's famous paper introducing his equation, **P3** being his first assumption (a second assumption, suppressed here, is required to deduce his equation). **P3** implies that Ψ is solution of both Schrodinger's and Klein-Gordon's equations in the non interacting case ($k = 0$ in **P1**) while, in the interacting case ($k \neq 0$), it immediatly implies "Sommerfeld's quantum rules" : **P1**, **P2a**, and **P3** then produce the correct relativistic energy levels of atoms, and we check that the required degeneracy is justified by pure deduction, without any other assumption (Schrodinger's theory only justifies one half of the degeneracy).
We observe that the introduction of an interaction in **P1** ($k = 0 \to k \neq 0$) is equivalent to a modification of the metric inside Ψ in **P3**, such that the equation of motion of a system can be deduced with two different methods, with or without the metric. Replacing the electromagnetic potential **P2a** by the suggested gravitationnal potential **P2b**, the equation of motion (deduced in two ways) is equivalent to the equation of motion of General Relativity in the low field approximation (with accuracy 10^{-6} at the surface of the Sun). We have no coordinate singularity inside the metric. Other motions can be obtained by modifying **P2b**, the theory is adaptable. First of all, we discuss classical Kepler problems (Newtonian motion of the Earth around the Sun), explain the link between Kelpler law of periods (1619) and Plank's law (1900) and observe the links between all historical models of atoms (Bohr, Sommerfeld, Pauli, Schrodinger, Dirac, Fock). This being done, we introduce **P1**, **P2a/b**, and **P3** to then describe electromagnetism and gravitation in the same formalism.

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I – New results in classical physics

We start with classical physics ($m = \gamma m_0 = (1 - v^2/c^2)^{-1/2} m_0 \approx m_0$) and the potential k/r can refer to the electromagnetic ($k = q^2/(4\pi\epsilon_0) = \alpha\hbar c$) or gravitationnal ($k = GMm_0$) potential. Equations (1) to (4) are well known results of classical physics. In bound systems, the classical energy $\epsilon \leq 0$ is given by :

$$\epsilon = \frac{p^2}{2m_0} - \frac{k}{r} = \frac{(\vec{p}.\vec{r})^2}{2m_0 r^2} + \frac{(\vec{p} \wedge \vec{r})^2}{2m_0 r^2} - \frac{k}{r} = \frac{p_r^2}{2m_0} + \frac{L^2}{2m_0 r^2} - \frac{k}{r} = \frac{m_0 \dot{r}^2}{2} + \frac{L^2}{2m_0 r^2} - \frac{k}{r} \tag{1}$$

With $L = m_0 r^2 \frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{m_0 r^2 d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$\epsilon = \frac{L^2}{2m_0}(\frac{du}{d\phi})^2 + \frac{L^2}{2m_0}u^2 - ku \Leftrightarrow 0 = \frac{du}{d\phi}(\frac{L^2}{m_0}\frac{d^2u}{d\phi^2} + \frac{L^2}{m_0}u - k) \Leftrightarrow 0 = \frac{d^2u}{d\phi^2} + u = \frac{m_0 k}{L^2} \tag{2}$$

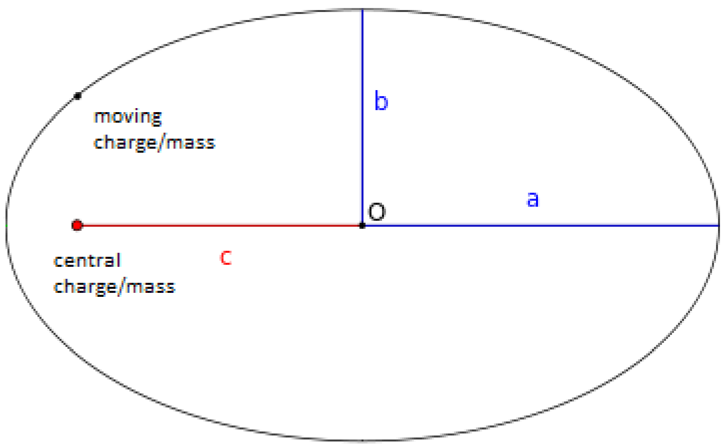
The solution, with u proportionnal to the potential k/r, takes the form

$$u = 1/r = \frac{m_0 k}{L^2}(1 + e cos(\phi - \phi_0)) \; with \; e = \sqrt{1 + \frac{2\epsilon L^2}{k^2 m_0}} \tag{3}$$

e (the eccentricity) and ϕ_0 are constants of integration, and r can be rewritten

$$r = \frac{L^2}{m_0 k + \sqrt{m_0^2 k^2 + 2m_0 \epsilon L^2} cos(\phi - \phi_0)} = \frac{L^2/(m_0 k)}{1 + e cos(\phi - \phi_0)} = \frac{l}{1 + e cos(\phi - \phi_0)} \tag{4}$$

l is called "semi latus rectum". It is well known that the solution is an ellipse, described by 3 parameters a, b, c where $a^2 = b^2 + c^2$, $e = c/a$ and $l = b^2/a$).



$$a = \frac{m_0 k}{2m_0|\epsilon|} = \frac{J}{\sqrt{2m_0|\epsilon|}}; \; b = \frac{L}{\sqrt{2m_0|\epsilon|}}; \; c = \frac{\sqrt{m_0^2 k^2/(2m_0|\epsilon|) - L^2}}{\sqrt{2m_0|\epsilon|}} = \frac{K}{\sqrt{2m_0|\epsilon|}} \tag{5}$$

Here K, which is proportionnal to the eccentricity e, is the norm of the Laplace-Runge-Lenz-Pauli vector, the second converved angular momentum of Kepler Problems :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(\vec{p} \wedge \vec{L} - m_0 k \frac{\vec{r}}{r}) \quad , \quad \epsilon \; classical \; energy < 0 \tag{6}$$

since (with $\dot{\vec{L}} = 0$, and $\vec{v}.\vec{r} = \dot{x}x + \dot{y}y + \dot{z}z = \dot{r}r$ which can be easily checked from the right to the left) :

$$\frac{d(\vec{p} \wedge \vec{L})}{dt} = \dot{\vec{p}} \wedge \vec{L} = \frac{k\vec{r}}{r^3} \wedge (m_0 \vec{v} \wedge \vec{r}) = \frac{m_0 k}{r^3}(r^2 \vec{v} - (\vec{v}.\vec{r})\vec{r}) = \frac{m_0 k}{r^3}(r^2 \vec{v} - (\dot{r}r)\vec{r}) = \frac{d(m_0 k\vec{r}/r)}{dt} \tag{7}$$

Equation (7) shows that $\vec{K}$ is conserved, which is well known. We can now give a first new result (equations (11) to (13)), the conserved Runge-Lenz vector can be rewritten :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2\vec{v}\wedge\vec{r}\wedge\vec{v} - m_0 k\frac{\vec{r}}{r}) \tag{8}$$

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2 v^2 - m_0\frac{k}{r})\vec{r} - m_0^2(\vec{v}.\vec{r})\vec{v} \tag{9}$$

$$\vec{K} = (\frac{m_0^2 v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})\vec{r} - (\frac{\vec{p}.\vec{r}}{\sqrt{-2m_0\epsilon}})\vec{p} \tag{10}$$

$$\vec{K} = (p_w)\vec{r} - (w)\vec{p} \tag{11}$$

where we can check that

$$\dot{w} = \frac{\dot{\vec{p}}.\vec{r} + \vec{p}.\dot{\vec{r}}}{\sqrt{-2m_0\epsilon}} = \frac{m_0 v^2 - \frac{k}{r}}{\sqrt{-2m_0\epsilon}} = \frac{p_w}{m_0} \tag{12}$$

We can then define 6 rotations and a total angular momentum J such that

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} \qquad \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \vec{r}p_w - w\vec{p} \tag{13}$$

$$J^2 = K^2 + L^2 \ (\vec{K}.\vec{L} = 0 \ \textit{can be easily checked}) \tag{14}$$

The 6 independant previous rotations defines the SO(4) symmetry. The two angular momenta have opposite parity. In 1926, Pauli [Pauli, 1926] used this definition for J to deduce the non relativistic energy levels of Hydrogen, without solving for the wave function. In 1936 he described Dirac's Hydrogen [Dirac, 1928] as follow : "If, instead of Dirac's equation, one assumes as a basis the old scalar Klein-Gordon relativistic equation, it possesses the following properties: the charge density may be either positive or negative and the energy density is always ≥ 0, it can never be negative. This is exactly the opposite situation as in Dirac's theory and exactly what one wants to have. [...] Still am I happy to beat against my old enemy, the Dirac theory of the spinning electron".
In 1935, Fock [Fock, 1935], studying Schrodinger's Hydrogen [Schrodinger, 1926] in momentum space, observed that, with an $1/r$ potential (SO(3) symmetry) he could describe an intrinsic SO(4) in the model. Concerning this point, we now give a new result involving the symmetries of $\vec{J}$:

$$r^2P^2 = r^2[p_x^2 + p_y^2 + p_z^2 + p_w^2] = r^2[p^2 + (\frac{m_0^2 v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] \tag{15}$$

$$r^2P^2 = r^2[p^2 + (\frac{m_0^2 v^2 - 2m_0\frac{k}{r} + m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] = r^2[p^2 + (-\sqrt{-2m_0\epsilon} + \frac{m_0 k/r}{\sqrt{-2m_0\epsilon}})^2] \tag{16}$$

$$r^2P^2 = r^2[p^2 - 2m_0\epsilon - 2m_0 k/r + J^2/r^2) = J^2 \tag{17}$$

This shows that J combines both SO(3) and SO(4) symmetries. From this (and the definition of angular momentum) we easily deduce

$$J^2 = r^2(p^2 + p_w^2) \ ; \ L^2 = r^2(p^2 - p_r^2) \ ; \ K^2 = r^2(p_w^2 + p_r^2) \tag{18}$$

We now recall Kepler's third law of periods for Planetary motion, and observe that it can be rewritten in a new form (using equation 5):

$$T^2 = \frac{4\pi^2}{GM}a^3 \ (usual form) \tag{19}$$

$$T^2 = \frac{4\pi^2}{GM}(a)(a^2) = \frac{4\pi^2}{GM}(\frac{m_0^2GM}{2m_0|\epsilon|})(\frac{J^2}{2m_0|\epsilon|}) \Leftrightarrow |\epsilon|T = \pi J \ (newform) \tag{20}$$

Evidently, with h has Planck constant (and $\hbar = h/(2\pi)$), fixing J equal to one unit of angular momentum, $J = \hbar$, and $\nu = 1/T$ gives $|\epsilon| = h\nu/2$. In particular, when the motion around the central object is a circle (corresponding to Bohr's model of Hydrogen [Bohr, 1913]), the electromagnetic interaction energy $V(r) = E_i$ is constant, and $|E_i| = 2|\epsilon| = h\nu$: we recognize here Planck's law [Planck, 1900] for the electromagnetic field, hidden in Kepler's third law [Kepler, 1619].

II – Sommerfeld′s model for Atoms :

Starting with **P1** : $E = H = \gamma m_0c^2 - k/r$ which, in polar coordinates, becomes

$$(E + \frac{k}{r})^2 = p^2c^2 + m_0^2c^4 \Leftrightarrow E^2 - m_0^2c^4 = p^2c^2 - \frac{2Ek}{r} - \frac{k^2}{r^2} = p_r^2c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \tag{21}$$

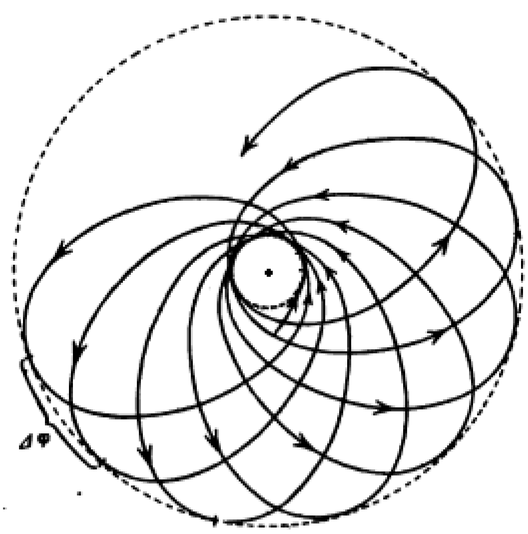
$$E^2 - m_0^2c^4 = (\gamma m_0\dot{r})^2c^2 + \frac{L'^2c^2}{r^2} - \frac{2Ek}{r} \tag{22}$$

$L'^2 = L^2 - (k/c)^2 = L^2 - (\alpha\hbar)^2$ will be important. With $L = \vec{p} \wedge \vec{r} = \gamma m_0 r^2 \frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{\gamma m_0 r^2 d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$E^2 - m_0^2c^4 = L^2c^2(\frac{du}{d\phi})^2 + L'^2c^2u^2 - 2Eku \tag{23}$$

$$0 = \frac{du}{d\phi}(2L^2c^2\frac{d^2u}{d\phi^2} + 2L'^2c^2u - 2Ek) \Leftrightarrow \frac{d^2u}{d\phi^2} + u(1 - (\frac{k}{Lc})^2) = \frac{Ek}{L^2c^2} \tag{24}$$

The solution takes the form $u = 1/r = \frac{Ek}{L^2c^2}(1 + ecos(\Gamma\phi - \phi_0))$ with $e = \sqrt{1 + \frac{(E^2 - m_0^2c^4)L'^2c^2}{E^2k^2}}$ and $\Gamma = 1 - (\frac{k}{Lc})^2$. The Γ factor produces a shift of the perihelion, as illustrated by Sommerfeld :



The 3 parameters of the ellipse are now given by (with $E < m_0c^2$) :

$$a = \frac{Ek}{m_0c^2 - E^2} = \frac{J}{\sqrt{m_0c^2 - E^2}} \ ; \ b = \frac{\sqrt{L^2 - (k/c)^2}}{\sqrt{m_0c^2 - E^2}} = \frac{L'}{\sqrt{m_0c^2 - E^2}} \tag{25}$$

$$c = \frac{\sqrt{(Ek/c)^2/(m_0c^2 - E^2) - (L^2 - (k/c)^2)}}{\sqrt{m_0c^2 - E^2}} = \frac{K}{\sqrt{m_0c^2 - E^2}} \ ; \ J^2 = L^2 + K^2 - (k/c)^2 \quad (26)$$

In classical physics $r(t)$ and $\phi(t)$ are cyclic functions of time with period T. In the relativistic domain, there is a precession of the perihelion, such that $r(t)$ and $\phi(t)$ have now different periods, T_r and T_ϕ. Sommerfeld's (postulated) quantum rules [Sommerfeld, 1916], with n_ϕ and n_r integers, are

$$\int_t^{t+T_r} (\gamma m_0 \dot{r}^2) \mathrm{d}t \int_{r(t)}^{r(t+T_r)} p_r \mathrm{d}r = \oint p_r \mathrm{d}r = n_r h \quad (27)$$

$$\int_t^{t+T_\phi} L\dot{\phi}\mathrm{d}t = \int_0^{\pm 2\pi} L\mathrm{d}\phi = \oint L\mathrm{d}\phi = \pm 2\pi L = n_\phi h \ \ (|L'| = \hbar\sqrt{n_\theta^2 - \alpha^2}) \quad (28)$$

Sommerfeld gave two methods to compute equation (27). We reproduce them is Annex. The final result is (with $E < m_0c^2$ for $V < 0$):

$$\oint p_r \mathrm{d}r = 2\pi(J - |L'|) = n_r h \Leftrightarrow J = \frac{Ek/c}{\sqrt{m_0^2c^4 - E^2}} = \frac{E\alpha\hbar}{\sqrt{m_0^2c^4 - E^2}} = (n_r + \sqrt{n_\phi^2 - \alpha^2})\hbar \quad (29)$$

This can be rewritten

$$E^2\alpha^2 = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2(m_0^2c^4 - E^2) \Leftrightarrow E^2(\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2) = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2 m_0^2c^4 \quad (30)$$

$$E^2 = \frac{m_0^2c^4}{\frac{\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} = \frac{m_0^2c^4}{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} \quad (31)$$

$$E = \frac{m_0c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}}} \quad (32)$$

In Dirac's theory, n_ϕ is replaced by $j + 1/2$ with $j = n_\theta \pm 1/2$. The energy levels are then the same : it was the great triumph of Dirac's theory that it reproduced Sommerfeld's energy levels.

III – Relativistic Hamilton – Jacobi equation

To show that our third postulate implies equations (27) and (28) and then energy levels (32), we first extend the well known Hamilton-Jacobi equation, to the relativistic domain, which is a new result :

$$H = \gamma m_0c^2 - \frac{k}{r} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - \frac{k}{r} = \frac{m_0c^2 - v^2 + v^2}{\sqrt{1 - v^2/c^2}} - \frac{k}{r} \quad (33)$$

$$H = \frac{m_0v^2}{\sqrt{1 - v^2/c^2}} + m_0c^2\sqrt{1 - v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} + m_0c^2\sqrt{1 - v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} - \mathcal{L} \quad (34)$$

Here $\mathcal{L} = -m_0c^2\sqrt{1 - v^2/c^2} + \frac{k}{r}$ is the usual Lagrangian. Its properties are well known, and the action S is then given by :

$$S = \int \mathcal{L}\mathrm{d}t = \int \vec{p}.\vec{v} - H\mathrm{d}t = -Ht + \int \vec{p}.\vec{v}\mathrm{d}t \quad (35)$$

We now write, in polar coordinates,

$$H = \gamma(r, \dot{r}, \dot{\phi}) m_0 c^2 - \frac{k}{r} \ ; \ S = -Ht + \int \gamma m_0 (\dot{r}^2 + r^2 \dot{\phi}^2) \mathrm{d}t \tag{36}$$

$$S = -Ht + \int (\gamma m_0 \dot{r}^2 + L\dot{\phi}) \ \mathrm{d}t = -Ht + \int \gamma m_0 \dot{r} \mathrm{d}r + \int L \mathrm{d}\phi = -Ht + \int p_r(r) \mathrm{d}r + L\phi \tag{37}$$

$p_r(r)$ is given by equations (21-22). The Lagrangian is now

$$\mathcal{L} = -m_0 c\sqrt{c^2 - v^2} + \frac{k}{r} = -m_0 c\sqrt{c^2 - \dot{r}^2 + r^2\dot{\theta}^2} + \frac{k}{r} \tag{38}$$

We recall the usual relations :

$$p_r = \frac{\partial H}{\partial \dot{r}} = \frac{\partial S}{\partial r} \ ; L = \frac{\partial H}{\partial \dot{\phi}} = \frac{\partial S}{\partial \phi} \tag{39}$$

From equation (35) we can now write $H + \frac{\partial S}{\partial t} = 0$ which is the classical Hamilton-Jacobi equation, naturally extended to the relativistic domain.
We now use our third postulate : **P3** : $\Psi = e^{iS/\hbar}$ and observe that, according to (37) Ψ is a seperable function $\Psi(r, \phi, t) = \Re(r)\Phi(\phi)e^{-iHt/\hbar}$. Since $r(t)$ and $\phi(t)$ are cyclic variables $\Re(r)$ and $\Phi(\phi)$ are cyclic functions :

$$\Re(r(t)) = \Re(r(t + T_r)) \Leftrightarrow e^{i(\int_{r(t_0)}^{r(t)} p_r(r)\mathrm{d}r)/\hbar} = e^{i(\int_{r(t_0)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \tag{40}$$

or

$$1 = e^{i2\pi n_r} = e^{i(\int_{r(t)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \Leftrightarrow \oint p_r \mathrm{d}r = n_r h \tag{41}$$

and similarily,

$$\Phi(\phi(t)) = \Phi(\phi(t + T_\phi)) \Leftrightarrow \oint L\mathrm{d}\phi = n_\phi h \tag{42}$$

We recognize here "Sommerfeld's quantum rules" producing the required energy levels. If we are now interested by the degeneracy, the previous analysis must be reproduced in three dimensions. Three quantum numbers will then appear, such that : $L_z = n_\phi \hbar$, $L = n_L \hbar$, $\oint p_r \mathrm{d}r = n_r h$. We observe that if $x, y, z \neq 0$, which we require to define Ψ,

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} = \begin{bmatrix} yz(p_z/z - p_y/y) \\ zx(p_x/x - p_z/z) \\ xy(p_y/y - p_x/x) \end{bmatrix} \tag{43}$$

$L_z \neq \ 0 \Rightarrow p_y/y \neq p_x/x \Rightarrow L^2 > L_z^2$ and $|n_L| > |n_z|$. If the rotation is inverted $\dot{p_i} \to -\dot{p_i}, \vec{L} \to -\vec{L}, \dot{\phi} \to -\dot{\phi}$. n_L an n_z can be either positive or negative integers, depending on the rotation. On the contrary, n_r does not depend on the rotation, it is positive. Then, in this model, for each value of $|n_L|$ there exist two possible values of n_L. The three quantum numbers of this model are exactly the same as the ones obtained by Schrodinger, except that we have two possibilities for n_L (In Schrodinger theory the solution is non singular only if $n_L > 0$): our degeneracy is then twice the degeneracy of Schrodinger, as required. The fact that observed degeneracy was twice the degeneracy of Schrodinger's theory was called "duplexity phenomena" by Dirac.

IV – Free wavefunction and interaction

Considering a non interacting system, in cartesian coordinates, the action S, according to equation (35) will take the form

$$S = -Ht + \int p_x v_x + p_y v_y + p_z v_z \; dt = -Ht + p_x(x - x_0) + p_y(y - y_0) + p_z(z - z_0) \qquad (44)$$

Then Ψ will take the form

$$\Psi = e^{iS/\hbar} = e^{i(-Ht + p_x(x-x_0)+p_y(y-y_0)+p_z(z-z_0))/\hbar} = e^{i(-wt + k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0))} \qquad (45)$$

which is the fundamental solution of both Klein-Gordon equation (with $H^2 = p^2c^2 + m_0^2c^4$) and Schrodinger equation (with $H = \frac{p^2}{2m}$) :

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = -\hbar^2 c^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}\right) + m_0^2 c^4 \Psi \quad (Klein - Gordon) \qquad (46)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m_0}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}\right) \quad (Schrodinger) \qquad (47)$$

The general solution of the previous equations take the forms (fixing $x_0 = y_0 = z_0 = 0$ in (45)) :

$$\Psi' = \int\int\int\int \psi(E, p_x, p_y, p_z) e^{i(-Ht + p_x x + p_y y) + p_z z)/\hbar} dE \; dp_x \; dp_y \; dp_z \; ; \qquad (48)$$

with

$$\int\int\int\int |\psi|^2 dE \; dp_x \; dp_y \; dp_z \; < \infty \qquad (49)$$

where we recognize the Fourier transform, in the previous equations.
Then, in this theory, a free particle obeys a wave-equation : free particles exhibit a wave-like behavior like in the famous double slit experiment. The previous equations being linear, if Ψ_1 and Ψ_2 are solutions, their sum is a solution : in the free case, superposed states are allowed.
With equation (39), equation (21) can be written :

$$E^2 - m_0^2 c^4 = p_r^2 c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} = \left(\frac{\partial S}{\partial r}\right)^2 c^2 + \frac{(\frac{\partial S}{\partial \phi})^2 c^2 - k^2}{r^2} - \frac{2Ek}{r} \qquad (50)$$

With $\Psi = e^{iS/\hbar}$ we deduce ($\Leftrightarrow S = -i\hbar ln(\Psi)$, $\frac{\partial S}{\partial r} = -i\hbar \frac{\partial \Psi}{\Psi \partial r}$, $\frac{\partial S}{\partial \phi} = -i\hbar \frac{\partial \Psi}{\Psi \partial \phi}$)

$$(E^2 - m_0^2 c^4)\Psi^2 = -\hbar^2 \left(\frac{\partial \Psi}{\partial r}\right)^2 c^2 + \frac{-\hbar^2 (\frac{\partial \Psi}{\partial \phi})^2 c^2 - k^2 \Psi^2}{r^2} - \frac{2Ek}{r}\Psi^2 \qquad (51)$$

This equation is not linear anymore, superposed states are not allowed for interacting matter (Hydrogen atom). The difference is induced by the integral in equation (37) (right hand side) while there is no integral in (44)(right hand side). Without the integral (=without interaction), the wave equation is obeyed since :

$$-\hbar^2 \left|\frac{\partial \Psi}{\partial x}\right|^2 = -\hbar^2 \left|\frac{\partial^2 \Psi}{\partial x^2}\right| \Psi = p_x^2 \Psi^2 \qquad (52)$$

IV – Metric and wavefunction

We now investigate the connection between the (quantum principle of minimal) coupling and the wave function, and observe the modification "*non interacting system* $\rightarrow$ *interacting system*" with $V = -k/r$:

$$E = \gamma m_0 c^2 \rightarrow E = \gamma m_0 c^2 + V \Leftrightarrow E - V = \gamma m_0 c^2 \qquad (53)$$

$$E\Psi = \hbar\frac{\partial}{\partial t}\Psi \rightarrow (E-V)\Psi = E(1-\frac{V}{E})\Psi = \hbar\frac{\partial}{\partial t'}\Psi = \hbar\frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\Psi \tag{54}$$

then

$$\frac{\partial t}{\partial t'} = (1-\frac{V}{E}) \Leftrightarrow t' = \frac{t}{1-\frac{V}{E}} = t(\frac{E}{E-V}) = t(1+\frac{V}{E-V}) = t(1+\frac{V}{\gamma m_0 c^2}) \tag{55}$$

We deduce that the metric corresponding to the minimal coupling is then :

$$c^2dt^2(1-\frac{k}{\gamma m_0 c^2 r})^2 = dx^2+dy^2+dz^2+ds^2 \tag{56}$$

Or, in polar corrdinates,

$$c^2dt^2(1-\frac{k}{\gamma m_0 c^2 r})^2 = dr^2+r^2d\phi^2+ds^2 \tag{57}$$

Precisely, from this metric, we will now deduce the equation of motion (22), with :

$$E = \gamma m_0 c^2 - \frac{k}{r} \Leftrightarrow \gamma = (E+\frac{k}{r})/(m_0c^2) \quad (\gamma^{-2} = 1-\frac{dr^2}{c^2dt^2}-\frac{r^2d\phi^2}{c^2dt^2}) \tag{58}$$

using the previous equation and the definition of L :

$$L = \gamma m_0 r^2\frac{d\phi}{dt} \Leftrightarrow d\phi^2 = (\frac{L}{\gamma m_0 r^2})^2dt^2 = (\frac{L}{(E+\frac{k}{r})r^2})^2c^2dt^2 \tag{59}$$

$$\frac{d\phi^2}{dt^2} = (\frac{L}{\gamma m_0 r^2})^2 = (\frac{Lc^2}{(E+\frac{k}{r})r^2})^2 \tag{60}$$

$$\frac{ds^2}{c^2dt^2} = (1-\frac{k}{\gamma m_0 c^2 r})^2 - 1 + \frac{1}{\gamma^2} \tag{61}$$

using the metric (57) and the previous equation,

$$dr^2 = [(1-\frac{k}{\gamma m_0 c^2 r})^2 - \frac{r^2d\phi^2}{c^2dt^2} - \frac{ds^2}{c^2dt^2}]c^2dt^2 = [-\frac{r^2d\phi^2}{c^2dt^2}+1-\frac{1}{\gamma^2}]c^2dt^2 \tag{62}$$

using equations (60) and (58)

$$dr^2 = [-(\frac{(Lc)^2}{(E+\frac{k}{r})r})^2 + 1 - (\frac{m_0c^2}{E+\frac{k}{r}})^2]c^2dt^2 = [\frac{-(Lc)^2+(E+\frac{k}{r})^2r^2-m_0^2c^4r^2}{(E+\frac{k}{r})^2r^2}]c^2dt^2 \tag{63}$$

using the previous equation and (59)

$$\frac{dr^2}{d\phi^2} = \frac{-L^2c^2+(E+\frac{k}{r})^2r^4-m_0^2c^4r^4}{L^2c^2} \tag{64}$$

$$L^2c^2(\frac{1}{r^2}\frac{dr^2}{d\phi^2}) = -\frac{L^2c^2}{r^2}+E^2+2E\frac{k}{r}+\frac{k^2}{r^2}-m_0c^4 \tag{65}$$

With $u=1/r, (\frac{du}{d\phi})^2 = (-\frac{dr}{rd\phi})^2 = (\frac{1}{r^2}\frac{dr^2}{d\phi^2})$ and $L'^2 = L^2-(k/c)^2$ this equation becomes

$$E^2-m_0c^4 = L^2c^2(\frac{du^2}{d\phi^2}) + L'^2c^2u^2 - 2Eku \tag{66}$$

We recognize equation (22) deduce from the metric (57).

V – Gravitation :

We now simply repeat the analysis of the previous part, for gravitation. Fixing, for gravitation, inertial mass $\leftrightarrow$ gravitationnal mass, (or energy $\gamma m_0 c^2$ (mass+kinetic) $\leftrightarrow$ gravitationnal charge) we obtain the postulate **P2b** :

$$V = -\frac{GM\gamma m_0}{r} = -\frac{GM\gamma m_0 c^2}{c^2 r} \tag{67}$$

produces the metric :

$$c^2 dt^2 (1 - \frac{GM}{c^2 r})^2 = dx^2 + dy^2 + dz^2 + ds^2 \tag{68}$$

we write, as prelimaries :

$$E - V = \gamma m_0 c^2 \Leftrightarrow E + \frac{GM\gamma m_0 c^2}{c^2} = \gamma m_0 c^2 \tag{69}$$

$$E = \gamma m_0 c^2 (1 - \frac{GM}{c^2 r}) \Leftrightarrow \gamma = \frac{E}{m_0 c^2 (1 - \frac{GM}{c^2 r})} \tag{70}$$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = 1 - (\frac{dr}{dct})^2 - r^2 (\frac{d\phi}{dct})^2 \tag{71}$$

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{p} \wedge \vec{r})}{dt} = \frac{d\vec{p}}{dt} \wedge \vec{r} + \frac{d\vec{r}}{dt} \wedge \vec{p} = \vec{F} \wedge \vec{r} = \gamma m_0 \vec{v} \wedge \vec{v} = 0 \tag{72}$$

using (70) and the definition of $(\vec{p} \wedge \vec{r})$

$$L = \gamma m_0 r^2 \frac{d\phi}{dt} = \frac{E}{c^2 (1 - \frac{GM}{c^2 r})} r^2 \frac{d\phi}{dt} \tag{73}$$

$$d\phi = \frac{Lc^2}{E} (1 - \frac{GM}{c^2 r}) \frac{1}{r^2} dt \Leftrightarrow r^2 \frac{d\phi}{dt} = \frac{Lc^2}{E} (1 - \frac{GM}{c^2 r}) \tag{74}$$

And we now start, in polar coordinates (spherical coordinates with $\theta = \pi/2$) :

$$ds^2 = (1 - \frac{GM}{c^2 r})^2 c^2 dt^2 - dr^2 - r^2 d\phi^2 \Leftrightarrow \frac{ds^2}{c^2 dt^2} = (1 - \frac{GM}{c^2 r})^2 - 1 + 1 - (\frac{dr^2}{c^2 dt^2}) - r^2 (\frac{d\phi^2}{c^2 dt^2}) \tag{75}$$

and using (71) and (70),

$$\frac{ds^2}{c^2 dt^2} = (1 - \frac{GM}{c^2 r})^2 - 1 + (\frac{m_o c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2 \tag{76}$$

From (75), (left hand side),

$$dr^2 = [(1 - \frac{GM}{c^2 r})^2 - \frac{ds^2}{c^2 dt^2} - r^2 \frac{d\phi^2}{c^2 dt^2}] c^2 dt^2 \tag{77}$$

$$dr^2 = [(1 - \frac{GM}{c^2 r})^2 - \frac{ds^2}{c^2 dt^2} - \frac{1}{r^2 c^2} (r^2 \frac{d\phi}{dt})^2] c^2 dt^2 \tag{78}$$

using now (76) and (74)

$$dr^2 = [(1 - \frac{GM}{c^2 r})^2 + 1 - (1 - \frac{GM}{c^2 r})^2 - (\frac{m_o c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2 - \frac{1}{r^2 c^2} (\frac{Lc^2}{E})^2 (1 - \frac{GM}{c^2 r})^2] c^2 dt^2 \tag{79}$$

$$dr^2 = [1 - (\frac{m_o c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2 - \frac{1}{r^2 c^2} (\frac{Lc^2}{E})^2 (1 - \frac{GM}{c^2 r})^2] c^2 dt^2 \tag{80}$$

using (74),

$$\frac{dr^2}{d\phi^2} = \frac{[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2 - \frac{1}{r^2c^2}(\frac{Lc}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2}{[(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2\frac{1}{r^4c^2}]c^2dt^2} \tag{81}$$

$$\frac{dr^2}{d\phi^2} = r^4c^2(\frac{E}{Lc^2})^2\frac{1}{(1-\frac{GM}{c^2r})^2} - r^4c^2(\frac{m_0}{L})^2 - r^2 \tag{82}$$

$$\frac{dr^2}{r^4d\phi^2} = (\frac{E}{Lc})^2\frac{1}{(1-\frac{GM}{c^2r})^2} - c^2(\frac{m_0}{L})^2 - \frac{1}{r^2} \tag{83}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{r^2d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{GM}{c^2r})^2((\frac{m_0c}{L})^2 + \frac{1}{r^2}) \tag{84}$$

with

$$u = \frac{1}{r} \quad ; \quad (\frac{du}{d\phi})^2 = (-\frac{\frac{dr}{d\phi}}{r^2})^2 = (\frac{dr}{r^2d\phi})^2 \tag{85}$$

$$(1-\frac{GMu}{c^2})^2(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{GMu}{c^2})^2((\frac{m_0c}{L})^2 + u^2) \tag{86}$$

Taking the low field approximation : $(1-\frac{GMu}{c^2})^2 \to (1-\frac{2GMu}{c^2})$,

$$(1-\frac{2GMu}{c^2})(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{2GMu}{c^2})((\frac{m_0c}{L})^2 + u^2) \tag{87}$$

This equation can be compared with the equation of motion in general relativity (See Magnan reproducing Wheeler and Weinberg for an example)

$$(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{2GMu}{c^2})((\frac{m_0c}{L})^2 + u^2) \tag{88}$$

where we have the three same roots for $du/d\phi = 0$ and then the same aphelion and perihelion. If we examine Mercury :

$G = 6.67384\ 10^{-11}, r = 59.91\ 10^9, M = 1.989\ 10^{30}$ and test with Microsoft Excel :
$\frac{2GM}{c^2r} \approx 4.6\ 10^{-8} << 1$ This theory, which is aaptable, then reproduces the motion General Relativity, in the low field approximation.

Studying the electromagnetic potential, we observed that the equation of motion could be deduced with or without the metric. We now show that our definition of energy was sufficient to deduce the equation of motion (77), without the use of the metric.

$$E - V = \gamma m_0c^2 \Leftrightarrow E + \frac{GM\gamma m_0c^2}{c^2} = \gamma m_0c^2 \Leftrightarrow E = \gamma m_0c^2(1-\frac{GM}{c^2r}) \tag{89}$$

we deduce :

$$(\frac{E}{\gamma m_0c^2})^2 = (1-\frac{v^2}{c^2})(\frac{E}{m_0c^2})^2 = (\frac{E}{m_0c^2})^2[1-(\frac{dr^2}{c^2dt^2}) - r^2(\frac{d\phi^2}{c^2dt^2})] = (1-\frac{GM}{c^2r})^2 \tag{90}$$

from which we deduce

$$r^2(\frac{d\phi^2}{c^2dt^2}) = 1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2 - (\frac{dr^2}{c^2dt^2}) \tag{91}$$

Then with (91) and the definition of $|\vec{L}| = |\vec{p}\wedge\vec{r}| = |\gamma m_0r^2(\frac{d\phi}{dt})|$

$$r^2\frac{d\phi}{dt} = \frac{L}{\gamma m_0} = \frac{Lc^2}{E}(1-\frac{GM}{c^2r}) \Leftrightarrow r^2(\frac{d\phi^2}{c^2dt^2}) = \frac{1}{r^2c^2}(r^4\frac{d\phi^2}{dt^2}) = (\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2 \tag{92}$$

we rewrite (90)

$$[1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(\frac{dr^2}{c^2dt^2})](\frac{E}{m_0c^2})^2=(1-\frac{GM}{c^2r})^2 \tag{93}$$

$$[1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(\frac{dr^2}{c^2dt^2})]=(1-\frac{GM}{c^2r})^2(\frac{m_0c^2}{E})^2 \tag{94}$$

$$1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(1-\frac{GM}{c^2r})(\frac{m_0c^2}{E})^2=(\frac{dr^2}{c^2dt^2}) \tag{95}$$

$$[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]c^2dt^2=dr^2 \tag{96}$$

rewriting (82)

$$r^2d\phi^2=[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2-dr^2 \tag{97}$$

(96) and (97) give

$$(\frac{rd\phi}{dr})^2=-1+\frac{[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]} \tag{98}$$

$$(\frac{rd\phi}{dr})^2=\frac{[-1+(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]+[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]} \tag{99}$$

$$(\frac{rd\phi}{dr})^2=\frac{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]} \tag{100}$$

$$(\frac{dr}{rd\phi})^2=\frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]}{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]} \tag{101}$$

$$(\frac{dr}{rd\phi})^2=\frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{m_0c^2}{E})^2]}{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]}-1 \tag{102}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2=\frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{m_0c^2}{E})^2]}{[(\frac{Lc}{rE})^2]}-(1-\frac{GM}{c^2r})^2 \tag{103}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2=[(\frac{rE}{Lc})^2-(1-\frac{GM}{c^2r})^2[(\frac{m_0cr}{L})^2]-(1-\frac{GM}{c^2r})^2 \tag{104}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2=[(\frac{rE}{Lc})^2-(1-\frac{GM}{c^2r})^2[(\frac{m_0cr}{L})^2+1] \tag{105}$$

We recognize (84).

Conclusion :

We first examined Kepler problems, producing new results involving connection between Kepler's third law and Planck's law and the symmetries of the three angular momenta existing in classical physics. We then extended the Hamilton-Jacobi equation to the relativistic domain, observing that our third postulate naturally implied the energy levels of atoms, when combined with the two others. The degeneracy is the required one. We finally gave the connexion between a definition of a metric and a definition of energy, for central problems. Our theory of gravitation reproduces General Relativity in the low field approximation, it has no coordinate singularity, the motion can be computed in two ways, the formalism is exactly the same for gravitationnal and electromagnetic potentials. Everything came from 3 postulates.

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Annex : Sommerfeld's integral

Sommerfeld uses ϵ for e, γ an Γ. $p_r = \gamma m_0 \dot{r} = \gamma m_0 \frac{dr}{d\phi}\dot{\phi} = \frac{L}{r^2}\frac{dr}{d\phi}$ and $dr = \frac{dr}{d\phi}d\phi$ such that $p_r dr = L(\frac{dr}{rd\phi})^2 d\phi$

$$\text{(6)} \qquad \frac{1}{r} = \frac{1}{a}\,\frac{1+\varepsilon\cos\gamma\varphi}{1-\varepsilon^2}.$$

Wir merken sogleich die Formel an

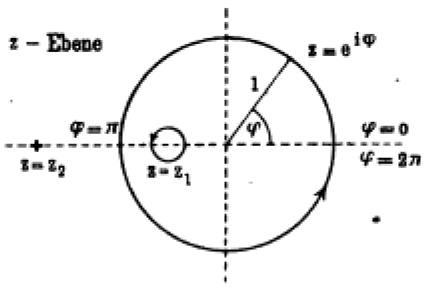
$$\text{(6a)} \qquad \frac{1}{r}\,\frac{dr}{d\varphi} = \frac{\varepsilon\gamma\sin\gamma\varphi}{1+\varepsilon\cos\gamma\varphi}.$$

Evidently, this quantity is independant of the constant factor $(a(1-e^2))$ of the first equation.

7. Ausführung einiger Integrale auf komplexem Wege.

Die Methode der komplexen Integration hat den wohlbekannten Vorteil, daß sie Integrale von geschlossenem Integrationswege ohne Kunstgriffe und fast ohne Rechnung auszuwerten gestattet. Sie hängt, wie wir in der folgenden Nummer sehen werden, mit den Aufgaben der Quantentheorie innerlich eng zusammen.

Fig. 100.



a) Es handle sich zunächst um ein reelles Integral, welches sich aber unmittelbar als geschlossenes komplexes Integral auffassen läßt:

$$\text{(1)} \qquad J_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{1+\varepsilon\cos\varphi}.$$

Wir führen als neue Variable die komplexe Größe

$$z = e^{i\varphi}$$

ein, welche den Einheitskreis im positiven Sinne umläuft, während φ von 0 bis 2π geht, vgl. Fig. 100. In dieser Variabeln schreiben wir

$$\text{(2)} \qquad J_1 = \frac{1}{2\pi i}\oint\frac{dz}{z\left[1+\frac{\varepsilon}{2}\left(z+\frac{1}{z}\right)\right]} = \frac{1}{\pi i\varepsilon}\oint\frac{dz}{z^2+2\eta z+1},$$

wo

$$\eta = \frac{1}{\varepsilon};$$

da wir wegen der Bedeutung von ε (numerischer Exzentrizität) und auch wegen der Konvergenz des Integrals (1) $\varepsilon < 1$ voraussetzen müssen, wird $\eta > 1$. Die Wurzeln des Nenners in (2) sind

$$\text{(3)} \qquad \begin{cases} z_1 = -\eta + \sqrt{\eta^2-1} \;\ldots\ldots\; -z_1 < 1 \\ z_2 = -\eta - \sqrt{\eta^2-1} \;\ldots\ldots\; -z_2 > 1 \end{cases}.$$

Statt (2) erhält man durch Partialbruchzerlegung

$$\text{(4)} \qquad \begin{cases} J_1 = \dfrac{1}{\pi i\varepsilon}\displaystyle\oint\frac{dz}{(z-z_1)(z-z_2)} \\ \quad\; = \dfrac{1}{\pi i\varepsilon\,(z_1-z_2)}\left(\displaystyle\oint\frac{dz}{z-z_1} - \oint\frac{dz}{z-z_2}\right). \end{cases}$$

Von den beiden letzten Integralen verschwindet das zweite, da seine singuläre Stelle außerhalb des Einheitskreises liegt; das erste läßt sich

7. Integrale auf komplexem Wege. 477

auf einen Umgang um den Punkt $z = z_1$ im Innern des Einheitskreises zusammenziehen und hat den Wert $2\pi i$. Somit

$$J_1 = \frac{2\pi i}{\pi i \varepsilon (z_1 - z_2)}. \tag{5}$$

Nach (3) ist aber

$$z_1 - z_2 = 2\sqrt{\eta^2 - 1} = \frac{2}{\varepsilon}\sqrt{1 - \varepsilon^2},$$

also geht (5) über in

$$J_1 = \frac{1}{\sqrt{1 - \varepsilon^2}}. \tag{6}$$

b) Auf das Integral a) läßt sich das folgende Integral zurückführen, das uns bei der Ellipsenbewegung in Gl. (13) von S. 266 entgegentrat:

$$J_2 = \frac{\varepsilon^2}{2\pi}\int_0^{2\pi} \frac{\sin^2\varphi}{(1 + \varepsilon\cos\varphi)^2}\, d\varphi.$$

Es geht nämlich durch partielle Integration über in

$$J_2 = \frac{\varepsilon}{2\pi}\, \frac{\sin\varphi}{1 + \varepsilon\cos\varphi}\Big|_0^{2\pi} - \frac{\varepsilon}{2\pi}\int_0^{2\pi} \frac{\cos\varphi}{1 + \varepsilon\cos\varphi}\, d\varphi.$$

Der erste Term rechts verschwindet, der zweite läßt sich umformen in

$$J_2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1 + \varepsilon\cos\varphi} - 1\right) d\varphi = J_1 - 1.$$

Also wird

$$J_2 = \frac{1}{\sqrt{1 - \varepsilon^2}} - 1. \tag{7}$$

From this result we deduce, with $\varphi = \Gamma\phi$

$$\oint p_r dr = L\int_0^{2\pi} (\frac{e\Gamma sin(\Gamma\phi)}{1 + ecos(\Gamma\phi)})^2 d\phi = L\Gamma e^2 \int_0^{2\pi} (\frac{sin(\varphi)}{1 + ecos(\varphi})^2 d\varphi \tag{106}$$

with $e = K/J (= c/a)$

$$\oint p_r dr = 2\pi L'(\frac{1}{\sqrt{1 - e^2}} - 1) = 2\pi L'(\frac{J}{\sqrt{J^2 - K^2}} - 1) = 2\pi L'(\frac{J}{L'} - 1) = 2\pi(J - L') \tag{107}$$

The second method starts from the equation

$$E^2 - m_0^2 c^4 = p_r^2 c^2 + \frac{L^2 c^2 - k^2}{r^2} - \frac{2Ek}{r} \Leftrightarrow p_r = \sqrt{[(\frac{E}{c})^2 - m_0^2 c^2] + 2\frac{Ek}{r} - \frac{L2'}{r^2}} = \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} \tag{108}$$

From which we deduce,

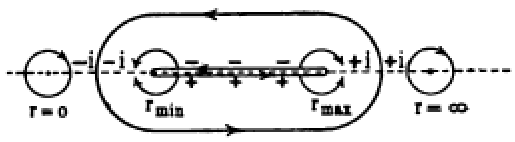
$$\oint p_r dr = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} dr \tag{109}$$

c) Die natürliche und allgemeine Grundlage für die Behandlung der radialen Quantenbedingung bildet aber nicht das vorhergehende Integral in φ, sondern das in r geschriebene Integral

$$J_3 = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}}\, dr. \tag{8}$$

Die Konstanten A, B und C haben im relativistischen und nichtrelativistischen Falle etwas verschiedene Bedeutung. Für die Zeichnung nehmen wir sie so an, daß die Verzweigungspunkte des Integranden — wir nennen sie r_{min} und r_{max}, „Perihel- und Apheldistanz" — reelle positive Werte haben. Der Integrationsweg läuft ursprünglich von r_{min} bis r_{max} und wieder zurück zu r_{min} und wird, wie Fig. 101 zeigt, in einen geschlossenen Umlauf in der komplexen r-Ebene auseinandergezogen. Die r-Ebene ist zwischen r_{min} und r_{max} aufgeschlitzt zu denken und stellt das obere Blatt einer zweiblätterigen Riemannschen Fläche dar. Wegen des positiven Charakters der Phasenintegrale ist bei positivem dr (unteres Ufer des Schlitzes) das Vorzeichen der Quadratwurzel positiv, bei negativem dr (oberes Ufer desselben) negativ zu nehmen, wie in der Figur angedeutet ist. Daraus folgt zugleich, daß die Quadratwurzel außerhalb des Schlitzes auf der reellen Achse der r-Ebene imaginär ist, und zwar positiv imaginär für $r > r_{max}$, negativ imaginär für $0 < r < r_{min}$, wie ebenfalls in der Figur angedeutet ist. Man erkennt dies, wenn man von dem positiven oder negativen Ufer des Verzweigungsschnittes aus je einen halben Umlauf um die Verzweigungspunkte $r = r_{max}$ oder $r = r_{min}$ macht.

Fig. 101.



Wir fahren mit der Erweiterung des Integrationsweges fort und ziehen diesen auf die Pole des Integranden zusammen. Es sind dies die Stellen

$$r = 0 \quad \text{und} \quad r = \infty.$$

An der Stelle $r = 0$ verhält sich J_3 wie

$$\sqrt{C} \int \frac{dr}{r} \left(1 + \frac{B}{C}\, r + \cdots\right).$$

Die Integration ist, wie die Figur zeigt, im Uhrzeigersinne zu nehmen und liefert daher vom ersten Reihengliede her den Wert $-2\pi i$; die folgenden Reihenglieder dagegen verschwinden bei der Integration. Somit kommt als Beitrag der Stelle $r = 0$ im ganzen

$$-2\pi i \sqrt{C}. \tag{9}$$

Die Unendlichkeitsstelle ist in der Figur im Endlichen angedeutet. Wir setzen

$$s = \frac{1}{r}, \quad dr = -\frac{ds}{s^2}$$

und erhalten aus (8)

$$\begin{aligned} J_3 &= -\int \sqrt{A + 2Bs + Cs^2}\, \frac{ds}{s^2} \\ &= -\sqrt{A} \int \left(1 + \frac{B}{A}\, s + \cdots\right) \frac{ds}{s^2}. \end{aligned}$$

Das Residuum dieses Integrals für die Stelle $s = 0$ wird allein durch das Glied mit s^{-1} bestimmt; dieses Glied hat den Koeffizienten

$$-\frac{B}{\sqrt{A}}.$$

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Der Beitrag der Unendlichkeitsstelle wird daher (vgl. den Umlaufssinn in der Figur)

$$+2\pi i \frac{B}{\sqrt{A}}. \tag{9a}$$

Aus der Summe von (9) und (9a) ergibt sich als Wert von J_3 die für das Folgende fundamentale Formel:

$$J_3 = -2\pi i\left(\sqrt{C} - \frac{B}{\sqrt{A}}\right). \tag{10}$$

Ergänzend fügen wir eine Bemerkung über das Vorzeichen von $\sqrt{C}$ an. $\sqrt{C}$ war in (8) definiert als Residuum des Ausdrucks

$$\sqrt{A + \frac{2B}{r} + \frac{C}{r^2}}$$

We deduce,

$$\oint p_r dr = -2\pi i(iL' - iJ) = 2\pi(J - L') \tag{110}$$