Some characterizations of Smarandache special definite groups

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Abstract

In this paper, we study Smarandache special definite groups. We give necessary and sufficient conditions for a group to be Smarandache special definite group(S-special definite group). Moreover we study Smarandache special definite *s*ubgroups, Smarandache special definite maximal ideals, Smarandache special definite minmal ideals.

Keywords: Smarandache special definite group, Smarandache special definite *s*ubgroup.

Introduction

Smarandache algebraic structures introduced by Raul Padilla and Florentine Smarandache [1],[2]. Smarandache special definite structures such as Smarandache special definite group, Smarandache special definite ring, Smarandache special definite field defined by Vasantha Kandasamy in 2009 as those strong algebraic structures which have in them a proper subset which is a weak algebraic structure[3]. In this paper we study Smarandache special definite groups. In section one we give a necessary and sufficient condition for a group to be Smarandache special definite group. We prove that the direct product of two groups G_1 ,G_2 is a Smarandache special definite group if and only if at least one of G_1 or G_2 is a Smarandache special definite group . In section two, we study many S-special definite substructures such as Smarandache special definite subgroups, Smarandache special definite ideals, we give a characterization of a Smarandache special definite group using its Smarandache special definite substructures. Moreover we study Smarandache special definite simple groups, it is shown that a commutative Smarandache special definite group can not be a S-special definite simple group and that a commutative Smarandache special definite group has no Smarandache special definite minimal ideal. Conditions are given under which a Smarandache special definite maximal ideal of a Smarandache special definite group is Smarandache special definite prime ideal.

1. *Smarandache special definite groups (S-special definite group)*

In this section we study Smarandache special definite groups (S- special definite group) we give a characterization of an S-special definite group. Condition under which every non trivial subgroups is S-special definite subgroup is given. We show that no finite group is Smarandache special definite group.

Definition 1.1, [3] A group (G, *) is said to be Smarandache special definite group if there is a non empty subset S in G such that S is just a semigroup (By just semigroup S of a group G we mean a subset S of G which is a semigroup under the induced operation of G but not a group).

***Definition* 1.2, [4]** A group G is said to be a torsion group if every element of G is of finite order and is said to be torsion free group if every element of G except the identity is of an infinite order. An element of finite order called a torsion element, otherwise called torsion free element.

***Lemma* 1.3** If G is a group and S is just a semigroup of G, then S is an infinite set containing a torsion free element.

***Proof*:** Let G be a group and S be a semigroup of G which is not a group. If S is a finite set, then S is a finite semigroup which satisfies cancelation laws. Then S is a group [5.p.50], which is a contradiction. So S is an infinite set. Now suppose that the order of any element of S is finite .Since S is just a semigroup, then there exists an element a in S such that a has no inverse in S, but a is of finite order, hence there exists $n \in Z^+$ such that $a^n = e_G$, so $a\,a^{n-1}= e_G$, thus $a^{-1}= a^{n-1} \in S$, which is a contradiction. This means that S contains a torsion free element. ♦

Now we give necessary and sufficient condition under which a group is S-special definite group.

***Theorem* 1.4** Let G be a group. Then G is a S-special definite group if and only if G contains a torsion free element.

***Proof*:** Suppose that G contains a torsion free element a. Then $S= \{ a^n; n \in Z^+\}$ is just a semigroup, so G is a S-special definite group. Conversely suppose that G is a S-special definite group. Then there exists $S \subset G$ which is just a semigroup. By Lemma 1.3, S contains a torsion free element, which is an element of G. ♦

Since every infinite cyclic group contain elements of infinite order we get the following result.

***Corollary* 1.5** Every infinite cyclic group is a S-special definite group.

Recall that every element of a torsion group is of finite order, we get the following result.

***Corollary* 1.6** If G is torsion group, then G can not be S-special definite group.

From Corollary 1.6, we deduce that a finite group can not be S-special definite group.

***Examples* 1.7**

1. (Z,+) is a S-special definite group because it contains $(Z^+,+)$ or follows from corollary 1.5,
2. (P(X),Δ) where X is an infinite set is an infinite group but can not be S-special definite group because p(X) is a torsion group.
3. $(Z_{p^\infty}, +)$ is an infinite torsion group [6]. This means that $(Z_{p^\infty}, +)$ is not a S-special definite group for every prime number p.

In the following proposition a necessary and sufficient condition is given under which the direct product of two group is a S-special definite group.

***Proposition* 1.8** Let G_1, G_2 be two groups. Then $G_1\times G_2$ is a S-special definite group if and only if at least one of G_1 or G_2 is a S-special definite group .

***Proof*:** Suppose G_1 is a S-special definite group. Then there exists $S \subset G_1$ such that S is just a semigroup . Now , $S\times \{ e_{G2}\} \subset G_1\times G_2$ and $S\times\{ e_{G2}\}$ is just a semigroup of $G_1\times G_2$, then $G_1\times G_2$ is a S-special definite group .Similarly if G_2 is a S-special definite group, then $G_1 \times G_2$ is S-special definite group. Conversely suppose that $G_1\times G_2$ is a S-special definite group. Then by Theorem 1.4, there exists a torsion free element (a,b) of $G_1\times G_2$. Hence a is a torsion free element of G_1 or b is a torsion free element of G_2, because if a is a torsion element of G_1 and b is a torsion element of G_2, then there exist n, m $\in Z^+$ such that $a^n =e_{G1}$ and $b^m=e_{G2}$, thus $(a,b)^{nm}=(a^{nm},b^{nm}) =((a^n)^m,(b^m)^n)=(e_{G1},e_{G2})$ which is a contradiction with (a,b) is a torsion free element, so at least one of G_1 or G_2 is a S-special definite group. ♦

***Proposition* 1.9** Every group can be imbedded in a S-special definite group.

***Proof*:** Let G be a group. Since (Z, +) is a S-special definite group, then by Theorem 1.8, G×Z is a S-special definite group. G × {0} is subgroup of G×Z which is isomorphic to G. Then G imbedded in G×Z. ♦

***Proposition* 1.10** Let N be a normal subgroup of a S-special definite group G. Then at least one of N or G⁄N is a S-special definite group.

***Proof*:** If N is not a S-special definite group, then every element of N is of finite order. Since G is a S-special definite group, then by Theorem 1.4, there exists $x\in G$ such that x is a torsion free element. We claim that x+N is a torsion free element of G⁄N. If x+N is a torsion element of G⁄ N, then there exists $n\in Z^+$ such that $(x+N)^n = N$. So $x^n+N = N$, hence $x^n\in N$, consequently x^n is of finite order, thus $(x^n)^m=e$ for some $m\in Z^+$, which means x is a torsion element of G which is a contradiction. Then x+N is a torsion free element of G⁄N and G⁄N is a S-special definite group. Then at least one of N or G⁄N is a S-special definite group. ♦

***Proposition* 1.11** Let G be a S-special definite group and H a group isomorphic to G. Then H is S-special definite group.

Proof: Since $G \cong H$, then there exists a group isomorphism $\phi: G\to H$. G is S-special definite group, means that there exists $S \subset G$ such that S is just a semigroup. Then clearly $\varphi(S)$ is a semigroup of H. Since S is just a semigroup, then there exists $x \in S$ such that x has no inverse in S. If $\varphi(y) \in \varphi(S)$ such that $\varphi(x)\varphi(y) = e_H$, then $\varphi(xy) = \varphi(e_G)$, thus $xy = e_G$, so $x^{-1}=y\in S$ which is a contradiction with x has no inverse in S. Hence $\varphi(x)$ has no inverse in $\varphi(S)$, then $\varphi(S)$ is just a semigroup of H, consequently H is a S-special definite group. ♦

Now we give the following useful lemma.

Lemma 1.12 Let G be a group and S be a semigroup of G. Then S is a subgroup of G if and only if x S=S for all $x \in S$.

***Proof*:** Suppose that xS=S for all $x \in S$. Choose any $x \in S$, then xS=S. Hence there exists $x_l \in S$ such that $x x_l=x$ so $x x_l = x = x e$, then $e = x_l \in S$, thus $e\in xS$. So there exists $x_2 \in S$ such that $x x_2=e$ .Consequently $x x_2=e= x x^{-1}$. Then $x^{-1}= x_2\in S$, which means that every element in S has inverse in S, then S is a subgroup. Conversely suppose that S is a subgroup of G. Let $y\in S$, then $y=x(x^{-1}y) \in xS$, so $S \subseteq xS$. But $xS \subseteq S$, hence $xS=S$ for all $x\in S$.

***Corollary* 1.13** Let G be a group and S be a semigroup of G which is not a group, then there exists an element x in S such that $xS \subset S$.

***Definition* 1.14 ,[7]** A Smarandache semigroup is a semigroup S such that a proper subset A of S is a group with respect to the induced operation of S. A Smarandache semigroup called Smarandache weakly cyclic semigroup If there exists at least a proper subset M of S, which is a non trivial cyclic subgroup.

***Theorem* 1.15** Let G be a S-special definite group and S be a semigroup of G. Then S is a Smarandache weakly cyclic semigroup if and only if there exists $x\neq e$ in S such that $xS=S$.

Proof: Suppose that there exists $x\neq e\in S$ such that xS=S. Then $x\in xS$, thus $x x_1=x$ for some $x_1 \in S$, so $xx_1=x=x e$, so $e=x_1 \in S$, hence $e\in xS$, then there exists $x_2 \in S$ such that $e = x x_2$, so $e = x x_2 = x x^{-1}$, which implies that $x^{-1}=x_2 \in S$. Therefore $< x >$ is cyclic group contained in S, then S is Smarandache weakly cyclic semigroup. Conversely suppose that S is Smarandache weakly cyclic semigroup, then there exist a non trivial cyclic subgroup $< x >$ of S. Let $y\in S$, then $y=x(x^{-1}y) \in xS$, thus $S\subseteq xS$ but $xS \subseteq S$, hence $xS=S$ for every $x^n\in < x >$. ♦

***Example* 1.16** $(Z\times Z_5, +)$ is a group and $Z^+\times Z_5$ is a semigroup of $Z\times Z_5$.
$(0,a)+ Z^+\times Z_5 = Z^+\times Z_5$ for all $a\in Z_5$ and $Z^+\times Z_5$ contains the cyclic subgroup $\{0\}\times Z_5$.
Hence $Z^+\times Z_5$ is a smarandache weakly cyclic semigroup.

***Remark* 1.17** Let G be a group and S is a semigroup of G with identity, then $e_s= e_G$ where e_s, e_G are the identity of G and S respectively.

***Proof*:** Since $e_s e_s = e_s = e_G e_s$, then $e_s e_s = e_G e_s$ so $e_s = e_G$. ♦

***Theorem* 1.18** If G be is a S-special definite group, then G contains an infinite countable number of semigroups which are not group.

***Proof*:** Let G be a S-special definite group. Then there exists $S \subset G$ such that S is just a semigroup. By Corollary 1.13, there exists $x \in S$ such that $x S \subset S$. We claim that x S is just a semigroup. If x S contains an identity element e_s , then by Remark 1.17, $e_s = e_G$, so $e_G \in x S$, therefore there exists $x_l \in S$ such that $x x_l = e_G$, but $x x^{-1} = e_G$ hence $x x_l = e_G = x x^{-1}$ which implies that $x^{-1} = x_l \in S$, then xS=S which is a contradiction, then xS does not contain identity consequently xS is just a semigroup .Then $S_1 = x S$ is just a semigroup and S_1 is an infinite set . By the same manner one can show the existence of a semigroup $S_2 \subset S_1$ which is not a group, then G contains an infinite countable number of semigroups which are not groups. ♦

2. *Smarandache special definite substructures*

In this section, we study many S-special definite substructures such as S-special definite subgroups, S-special definite ideals, we give characterizations of a S-special definite group using its S-special definite substructures. Moreover we study Smarandache special definite simple group. Conditions are given under which a Smarandache special definite maximal ideal of a Smarandache special definite group is Smarandache special definite prime ideal. We give many examples illustrating the results.

***Definition* 2.1,[3]** Let G be a S-special definite group and H be a subgroup of G .If H is itself a S-special definite group then we call H a Smarandache special definite subgroup of G (S-special definite subgroup).

It is clear that if G has a subgroup H which is S-special definite subgroup, then G is also S-special definite group but the converse in general is not true as it is shown in the following example.

Example 2.2 $(Z\times Z_p,+)$ is S-special definite group, because it contains the semigroup $(Z^+\times Z_p,+)$ but the Subgroup $(\{0\}\times Z_p,+)$ of $(Z\times Z_p,+)$ is not a S-special definite subgroup of G.

Theorem 2.3 Let G be a S-special definite group, Then G is a torsion free group if and only if every non trivial subgroups is S-special definite subgroup.

Proof: Suppose that every non trivial subgroup of G is a S-special definite subgroup.
Let $a\in G$. Then <a> is S-special definite subgroup, so by Theorem 1.4, for some $k\in Z^+$, a^k is a torsion free element. This imply that a is a torsion free element because if $a^m=e$ for some $k\in Z^+$,then $(a^k)^m=(a^m)^k=(e)^k=e$, which is a contradiction .Conversely suppose that G is a torsion free group . Then every non trivial subgroup contains a torsion free element. By Theorem 1.4 , every non trivial subgroup is a S-special definite subgroup. ♦

The following theorem gives a characterization of a S-special definite group.

***Theorem* 2.4** Let G be a group. If G is a S-special definite group, then G has a proper subset H which is a S-special definite group.

Proof: Suppose that G is a S-special definite group. Then G contains a torsion free element a. Let $H= \{ a^{2n} ; n \in Z\}$, then $a\notin H$,so H is a proper subgroup of G and $a^2 \in H$ is a torsion free element, so H is S-special definite group. Consequently G has a proper subset H such that H is S-special definite group. ♦

***Proposition* 2.5** Let G be a group. If G is a S-special definite group, then G contains an infinite countable number of proper S-special definite subgroups.

Proof: Suppose that G is S-special definite group. By Theorem 2.4, G contains a proper subset H_1 such that H_1 is S-special definite group, since H_1 is S-special definite group, for the same reason H_1 contains a proper subset H_2 such that H_2 is S-special definite group. Then for all $n \in N$ we get a proper subset H_{n+1} of H_n such that H_{n+1} is S-special definite group. Since H_n is S-special definite subgroup of G for all $n \in N$, then G contains an infinite countable number of proper S-special definite subgroups. ♦

The following result is a direct consequence of Proposition 2.5.

***Corollary* 2.6** If G is S-special definite group, then G contains an infinite countable number of subgroups.

The converse of this corollary is not true in general for example.

***Example* 2.7** The infinite direct sum $\oplus$ Zp, p runs over all prime numbers is not a S-special definite group but it contains an infinite number of subgroups.

***Theorem* 2.8** Let H be a subgroup of a group G, then H is a S-special definite subgroup if and only if aHa^{-1} is S-special definite subgroup for each $a \in G$.

Proof: Suppose that H is a S-special definite group, then there exist $x \in H$ such that x is a torsion free element. Then axa^{-1} is a torsion free element of aHa^{-1}, because if $(axa^{-1})^n=e$ for some $n \in Z^+$, then $a x^n a^{-1}=e$ then $x^n =e$ which is a contradiction with x is a torsion free element. So aHa^{-1} is a S-special definite group. Conversely suppose that aHa^{-1} is a S-special definite group for each $a \in G$. Take a=e , so $H=eHe^{-1}$ is a S-special definite group. ♦

Definition 2.9,[3] Let G be a S-special definite group, H be a normal subgroup of G, we call H to be a Smarandache Special definite normal subgroup of G if H is itself a S-special definite group. If G has no S-special definite normal subgroups but G is a S-special definite group then we call G to be a Smarandache definite Special simple group(S-special definite simple group).

***Proposition* 2.10** If G is a commutative S-special definite group, then G can not be a S-special definite simple group.

Proof: Let G be a commutative S-special definite group. Then by Theorem 2.4, G contains a proper subset H such that H is S-special definite subgroup, since G is commutative then H is S-special definite normal subgroup, so G can not be S-special definite simple group. ♦

***Theorem* 2.11** Let $a \in G$ be a torsion free element and the centralizer of a, C (a) =G, then G can not be S-special definite simple group.

***Proof* :** Let $H =\{ a^{2n};\ n \in Z\}$ .Then H is a proper subset of G and H is a S-special definite subgroup, since C(a)=G , then gHg^{-1} =H for every $g \in G$, that is H is a S-special definite normal subgroup, then G can not be S-special definite simple group. ♦

The following example illustrated Theorem 2.11.

***Example* 2.12**

1) $GL_2(R)$ under multiplication is an infinite non Commutative group and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in GL_2(R)$ is a torsion free element and $C\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = GL_2(R)$, then $GL_2(R)$ can not be S-special definite simple group.
2) Let G={a+bi+cj +dk ; a,b, c,d∈R} then (G-{0},.) can not be S-special definite simple group because 2 is a torsion free element of G and C(2)=G .

The condition that there exist a torsion free element a of G such that C (a) =G in Theorem 2.11, is not necessary as it is shown in the following example and G is not a S-special definite simple group and C (a) ≠ G for all a∈G.

***Example* 2.13** Let G={ (a,b) ; a,b∈R and a≠0 where R the set of real numbers } and define (a,b) *(c,d) =(a c,bc+d) . Then G is an infinite non commutative group and (1,0) is identity the element of G and $(a,b)^{-1} = (1/a , -b/a)$ [8] . Now C ((a,b)) ≠ G for every (a,b) ≠ (1,0) , because if a≠1 then (a,b) (a, b+1) ≠ (a, b+1) (a,b) , then (a, b+1) ∉ C((a,b)) and if $b\neq 0$ then $(a,b)*(a+1,b)\neq(a+1,b)*(a,b)$, so $(a+1,b)\notin C((a,b))$. So $C((a,b))\neq G$. Clearly $H=\{(1,a);\ a\in R\}$ is a subgroup of G. Since $(1,1)\in H$ and (1,1) is a torsion free element , then H is a S-special definite subgroup. Let $(a,b)\in G$ and $(1,c)\in H$, then $(a,b)*(1,c)*(a,b)^{-1}=(a,b+c)*(1/a,-b/a)=(1,c/a)\in H$. Then H is normal subgroup, consequently H is S-special definite normal subgroup. Then by Theorem 2.11, G is not a S-special definite simple group.

***Proposition* 2.14** Let G be a group and H and K be subgroups of G. If $H\cap K$ is S-special definite subgroup, then H and K are S-special definite subgroup.

***Proof*:** Let $H\cap K$ be a S-special definite subgroup, By Theorem 1.4, there exist $a\in H\cap K$ such that a is a torsion free element, then both of H and K contain a which is a torsion free element, so both of H and K are S-special definite subgroup . ♦

The converse of Proposition 2.14, need not be true in general for example.

***Example* 2.15** Let $<2>=\{2^n, n\in Z\}$ and $<3>=\{3^n, n\in Z\}$ are two S-special definite subgroup of (Q-{0},.) But $<2>\cap<3>=\{1\}$ is not S-special definite group.

In the following Theorem a necessary and sufficient condition is given under which the product of two subgroup is a S-special definite subgroup.

***Theorem* 2.16** Let G be a commutative group, H and K be two subgroups of G. Then HK is a S-special definite subgroup of G if and only if at least one of H or K is a S-special definite subgroup.

Proof: Suppose that H is a S-special definite subgroup, then by Theorem 1.4, there exist a torsion free element $a\in H$, then $a=ae\in HK$ Consequently H K is S-special definite subgroup. Similarly if K is a S-special definite subgroup. Conversely suppose that HK is a S-special definite subgroup. Since $HK\cong H\times K$, then by Proposition 1.11, and Proposition 1.8, we get at least one of H or K is a S-special definite subgroup.

At the end of the paper we discuss some types S-special definite ideals.

***Definition* 2.17, [9]** A nonempty subset T of a semigroup S is said to be an ideal of S if $s\in S$, $t\in T$ imply that st, ts $\in T$.

***Definition* 2.18, [3]** Let G is a S-special definite group. A proper subset P of G is said to be Smarandache special definite ideal (S-special definite ideal) of G if P is an ideal with respect to some semigroup T of G.

***Proposition* 2.19.** Let G be a S-special definite group. Then every S-special definite ideal of G contains a torsion free element.

***Proof* :** Let I be a S-special definite ideal of G . Then there exists $S\subset G$ such that S is just a semigroup of G and I is ideal of S. By Lemma 1.3, S contain torsion a free element x. Let $a\in I$, then $a\,x\in I$. If ax is torsion free element, then the proof is complete. Otherwise is a $(a\,x)^n=e$ for some $n\in Z^+$, then $(a\,x)^{n-1}a\,x=e$, so $x^{-1}=(a\,x)^{n-1}a\in I$, then $x^{-1}\in I$ and x^{-1} torsion free element. ♦

***Definition* 2.20 , [3]** Let G be a S-special definite group, T be a semigroup of G. We say P is a Smarandache special definite maximal ideal (S-special definite maximal ideal) of G related to T if for any other S-special definite ideal M related to T such that $P\subseteq M\subseteq T$ then either P = M or M =T. We say U is a S-special definite minimal ideal (S-special definite minimal ideal) related to T if for any other ideal V related to T with $V\subset U$ then V= U. We call an S -Special definite ideal W related to T to be a S- Special definite prime ideal (S-special definite prime ideal) if $a\,b\in W$ implies $a\in W$ or $b\in W$ where $a,b\in T$.

***Lemma* 2.21** If I is S- special definite ideal of a S-special definite group G related to a semigroup S and I contain identity element e, then I=S.

Proof: Since $I\subseteq S$ and for each $a\in S$ implies that $a=a\,e\in I$, then I=S. ♦

***Theorem* 2.22** Let G be a commutative S-special definite group, then G has no S- special definite minimal ideal.

Proof: Let I be a S-special definite ideal of G related to the semigroup S. Then I is not a subgroup of G, since if I is subgroup of G, then $e\in I$ by Lemma 2.21, I=S which is a contradiction. By Corollary 1.13, there exists $x\in I$ such that $xI \subset I$.

Let $s\in S$ and $x i\in x I$ where $i\in I$, thus $i s\in I$. Then $(x i) s = x (i s) \in x I$, so $x I \subset I$ is a S-special definite ideal of G related to semigroup S. Therefore I is not S-special definite minimal ideal, so G contains no S-special definite minimal ideal. ♦

***Theorem* 2.23** Let G be a S-special definite group and $T \subset G$ be a commutative semigroup with identity, then every S-special definite maximal ideal of G related to T is a S-special definite prime ideal of G related to T.

***Proof*:** Let M be a S-special definite maximal ideal of G related to T. Let $a b\in M$ such that $a\notin M$ and $a,b\in T$. Put $J=aT\cup M$, since if $t\in T$ and $r\in J$, then r=m or $r=at_2$ for some $m\in M$ and $t_2\in T$, thus $rt=mt\in M$ or $rt=a(t_2t)\in aT$, then $rt\in aT\cup M$ but T is a commutative semigroup, hence $tr=rt\in aT\cup M$, so J is a S-special definite ideal of G related to T. Now $a=ae\in aT\cup M$, then $M\subset J \subseteq T$ but M is a S-special definite maximal ideal of G related to T, so J=T , then $e\in aT\cup M$ but $e\notin M$, then $e\in aT$, $e=at_3$ for some $t_3\in T$. Since $ba\in M$ and $t_3\in T$ and M is a S-special definite maximal ideal of G related to T, then $b=b\,e=b\,(at_3) = (ba)\,t_3\in M$. Then M is a S-special definite prime ideal of G related to T. ♦

Finally, we show by an example the condition that T contains an identity in Theorem 2.23 is necessary.

***Example* 2.24** $(\mathbb{Q}-\{0\},.)$ is a S-special definite group and $(9Z-\{0\},.)$ is S-special definite maximal ideal of G related to $(3Z-\{0\},.)$,but not S-special definite prime ideal of G related to $(3Z-\{0\},.)$ by $3.3\in 9Z-\{0\}$ but $3\notin 9Z-\{0\}$.

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