Lagrangian Analysis of a Class of Quadratic Liénard-Type Oscillator Equations with Exponential-Type Restoring Force function

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Abstract

This research work proposes a Lagrangian and Hamiltonian analysis for a class of exactly integrable quadratic Liénard-type harmonic nonlinear oscillator equations and its inverted version admitting a position-dependent mass dynamics.

1. Analysis of the class of quadratic Liénard-type harmonic nonlinear oscillator equations

This section is devoted to the analysis of a class of quadratic Liénard-type nonlinear dissipative oscillator equations that admits exact analytical harmonic periodic solutions. Consider the equation [1, 2]

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 x e^{2\gamma\varphi(x)} = 0 \qquad (1)$$

that represents the class of equations under analysis. γ and ω are arbitrary parameters, and $\varphi(x)$ is an arbitrary function of x. The dot over a symbol means differentiation with respect to time, and prime holds for differentiation with respect to x. The equation (1) is of the general form

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \qquad (2)$$

for which the first integral is given by [3]

$$\mathrm{I}(\dot{x}, x) = \dot{x}^2 e^{2\int f(x)dx} + 2\int g(x) e^{2\int f(x)dx} dx \qquad (3)$$

So, a first integral of (1) may be written as

$$\mathrm{I}(\dot{x}, x) = \dot{x}^2 e^{-2\gamma\varphi(x)} + \omega^2 x^2 \qquad (4)$$

By application of the formula [4]

$$L(\dot{x}, x) = \dot{x}\int^{\dot{x}} \frac{\mathrm{I}(\dot{x}, x)}{\dot{x}^2} d\dot{x} \qquad (5)$$

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the Lagrangian of the equation (1) becomes

$$L(\dot{x},x)=\dot{x}^2e^{-2\gamma\varphi(x)}-\omega^2x^2 \tag{6}$$

Applying the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \tag{7}$$

to the equation (6), gives the equation (1). Now, using [3]

$$H(p,x)=p\dot{x}-L(x,\dot{x}) \tag{8}$$

one may deduce from (6) the Hamiltonian

$$H(p,x)=\frac{p^2}{4}e^{2\gamma\varphi(x)}+\omega^2x^2 \tag{9}$$

Let us now consider, as illustration, some specific examples of (1). Let $\varphi(x)=x$. Then (1) becomes

$$\ddot{x}-\gamma\dot{x}^2+\omega^2xe^{2\gamma x}=0 \tag{10}$$

The equation (10) admits the first integral

$$\mathrm{I}(\dot{x},x)=\dot{x}^2e^{-2\gamma x}+\omega^2x^2 \tag{11}$$

which provides the Lagrangian function

$$L(\dot{x},x)=\dot{x}^2e^{-2\gamma x}-\omega^2x^2 \tag{12}$$

The application of the Euler-Lagrange equation (7) to (12) gives, as expected, (10) .In this regard the Hamiltonian associated to (10) takes the form

$$H(p,x)=\frac{p^2}{4}e^{2\gamma x}+\omega^2x^2 \tag{13}$$

So, the Hamilton equations

$$\begin{cases}\dot{x}=\dfrac{\partial H}{\partial p}\\ \dot{p}=-\dfrac{\partial H}{\partial x}\end{cases} \tag{14}$$

yield for (13)

$$\begin{cases} \dot{x} = \dfrac{p}{2} e^{2\gamma x} \\ \dot{p} = -\dfrac{p^2}{2} \gamma x e^{2\gamma x} - 2\omega^2 x \end{cases} \tag{15}$$

The explicit expression for the canonically conjugate momentum p, as a function of x and $\dot{x}$ takes then the form

$$\dot{p} = -2e^{-2\gamma x}\left(\gamma \dot{x}^2 + \omega^2 x e^{2\gamma x}\right) \tag{16}$$

Putting now $\varphi(x) = \dfrac{1}{2} x^2$, into (1), one may obtain as equation

$$\ddot{x} - \gamma \dot{x}^2 x + \omega^2 x e^{\gamma x^2} = 0 \tag{17}$$

A first integral of (17) takes then the form

$$\mathrm{I}(\dot{x}, x) = \dot{x}^2 e^{-\gamma x^2} + \omega^2 x^2 \tag{18}$$

The associated Lagrangian becomes

$$L(\dot{x}, x) = \dot{x}^2 e^{-\gamma x^2} - \omega^2 x^2 \tag{19}$$

The application of the Euler-Lagrange equation (7) to (19) gives with satisfaction (17). So, the associated Hamiltonian may be written as

$$H(p, x) = \frac{p^2}{4} e^{\gamma x^2} + \omega^2 x^2 \tag{20}$$

Such that the Hamilton equations take the form

$$\begin{cases} \dot{x} = \dfrac{p}{2} e^{\gamma x^2} \\ \dot{p} = -\dfrac{p^2}{2} x e^{\gamma x^2} - 2\omega^2 x \end{cases} \tag{21}$$

The relation between $\dot{x}$ and $\dot{p}$ reads in this perspective

$$\dot{p} = -2xe^{-\gamma x^2} (\gamma \dot{x}^2 + \omega^2 e^{\gamma x^2}) \tag{22}$$

2. Analysis of inverted versions

Consider now the inverted version of (1)

$$\ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 x e^{2\gamma \varphi(x)} = 0 \tag{23}$$

which gives for $\varphi(x) = x$, the following equation

$$\ddot{x}+\gamma\dot{x}^2+\omega^2 x e^{2\gamma x}=0 \tag{24}$$

The first integral of (24) may be then deduced from (3) as

$$\mathrm{I}(\dot{x},x)=\dot{x}^2 e^{2\gamma x}+\frac{\omega^2}{2\gamma}x e^{4\gamma x}-\frac{\omega^2}{8\gamma^2}e^{4\gamma x} \tag{25}$$

Therefore, the Lagrangian for (24) may be written in the form

$$L(\dot{x},x)=\dot{x}^2 e^{2\gamma x}+\frac{\omega^2}{8\gamma^2}e^{4\gamma x}-\frac{\omega^2}{2\gamma}x e^{4\gamma x} \tag{26}$$

In this regard, it may be verified that the application of the Euler-Lagrange equation (7) to (26) yields, as expected, (24). The Hamiltonian for (24) may also be computed as

$$H(p,x)=\frac{p^2}{4}e^{-2\gamma x}+\frac{\omega^2}{2\gamma}x e^{4\gamma x}-\frac{\omega^2}{8\gamma^2}e^{4\gamma x} \tag{27}$$

which gives the Hamiltonian equations

$$\begin{cases} \dot{x}=\dfrac{p}{2}e^{-2\gamma x} \\ \dot{p}=\dfrac{p^2}{2}\gamma x e^{-2\gamma x}-2\omega^2 x e^{4\gamma x} \end{cases} \tag{28}$$

from which the canonically conjugate momentum becomes

$$\dot{p}=2e^{2\gamma x}\left(\gamma\dot{x}^2-\omega^2 x e^{2\gamma x}\right) \tag{29}$$

By analysis, other forms of equations are also suggested by the previous studied equations. So, the following equations may also be considered in the perspective of this study, that is

$$\ddot{x}+\gamma\dot{x}^2 x+\omega^2 x e^{\gamma x^2}=0 \tag{30}$$

or in general

$$\ddot{x}+\gamma\varphi'(x)\dot{x}^2+\omega^2 x e^{\gamma\varphi(x)}=0 \tag{31}$$

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+\omega^2 x e^{\gamma\varphi(x)}=0 \tag{32}$$

Finally one may consider the following more generalizations

$$\ddot{x}+\gamma\varphi'(x)\dot{x}^2+\omega^2 h(x) e^{\gamma\varphi(x)}=0 \tag{33}$$

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+\omega^2 h(x) e^{\gamma\varphi(x)}=0 \tag{34}$$

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{2\gamma\varphi(x)} = 0 \tag{35}$$

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{2\gamma\varphi(x)} = 0 \tag{36}$$

These equations will be investigated in a subsequent work.

References

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