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Classical Combination Rules Generalized to DSm Hyper-power Sets and their Comparison with the Hybrid DSm Rule

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Abstract: *Dempster's rule, non-normalized conjunctive rule, Yager's rule and Dubois-Prade's rule for belief functions combination are generalized to be applicable to hyper-power sets according to the DSm theory. A comparison of the rules with DSm rule of combination is presented. A series of examples is included.*

3.1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Belief functions were originally introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [13].

For a combination of beliefs Dempster's rule of combination is used in DST. Under strict probabilistic assumptions, its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. It is not uncommon to find examples where the assumptions are not fulfilled and where results of Dempster's rule are counter-intuitive, e.g. see [1, 2, 14], thus a rule with more intuitive results is required in such situations.

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Hence, a series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are Dubois and Prade's rule [9] and Yager's rule of belief combination [17]. Others include a wide class of weighted operators [12] and an analogous idea proposed in [11], the Transferable Belief Model (TBM) using the so-called non-normalized Dempster's rule [16], disjunctive (or dual Dempster's) rule of combination [4, 8], combination 'per elements' with its special case — minC combination, see [3], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [10].

A brand new approach performs the Dezert-Smarandache (or Dempster-Shafer modified) theory (DSmT) with its DSm rule of combination. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general; mathematically it means that belief functions are not defined on the power set of the frame, but on a so-called hyper-power set, i.e., on the Dedekind lattice defined by the frame; 2) a new combination mechanism which overcomes problems with conflict among the combined beliefs and which also enables a dynamic fusion of beliefs.

As the classical Shafer's frame of discernment may be considered the special case of a so-called hybrid DSm model, the DSm rule of combination is compared with the classic rules of combination in the publications about DSmT [7, 14].

Unfortunately, none of the classical combination rules has been formally generalized to hyper-power sets, thus their comparison with the DSm rule is not fully objective until now.

This chapter brings a formal generalization of the classical Dempster's, non-normalized conjunctive, Dubois-Prade's, and Yager's rules to hyper-power sets. These generalizations perform a solid theoretical background for a serious objective comparison of the DSm rule with the classical combination rules.

The classic definitions of Dempster's, Dubois-Prade's, and Yager's combination rules are briefly recalled in Section 3.2, basic notions of DSmT and its state which is used in this text (Dedekind lattice, hyper-power set, DSm models, and DSmC and DSmH rules of belief combination) are recalled in Section 3.3.

A generalization of Dempster's rule both in normalized and non-normalized versions is presented in Section 3.4, and a generalization of Yager's rule in Section 3.5. Both these classic rules are straightforwardly generalized as their ideas work on hyper-power sets simply without any problem.

More interesting and more complicated is the case of Dubois-Prade's rule. The nature of this rule is closer to DSm rule, but on the other hand the generalized Dubois-Prade's rule is not compatible with a dynamic fusion in general. It works only for a dynamic fusion without non-existential constraints, whereas a further extension of the generalized rule is necessary in the case of a dynamic fusion with non-existential constraints.

Section 3.7 presents a brief comparison of the rules. There is a series of examples included. All the generalized combination rules are applied to belief functions from examples from the DSmT book Vol. 1 [14]. Some open problems for a future research are mentioned in Section 3.8 and the concluding Section 3.9 closes the chapter.

3.2 Classic definitions

All the classic definitions assume an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$, the values of bba are called *basic belief masses (bbm).* The value $m(A)$ is called the *basic belief mass*[1] *(bbm) of A*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^Θ. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$. If a focal element is a one-element subset of Θ, we are referring to a *singleton.*

Let us start with the classic definition of Dempster's rule. *Dempster's (conjunctive) rule of combination* $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} Km_1(X)m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, with $\kappa = \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [13]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e. g. [16].

Yager's rule of combination Ⓨ, see [17], is given as
$(m_1 Ⓨ m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subset \Theta$,
$(m_1 Ⓨ m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$,
and $(m_1 Ⓨ m_2)(\emptyset) = 0$;

Dubois-Prade's rule of combination Ⓓ is given as
$(m_1 Ⓓ m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset, X\cup Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 Ⓓ m_2)(\emptyset) = 0$, see [9].

3.3 Introduction to the DSm theory

Because DSmT is a new theory which is in permanent dynamic evolution, we have to note that this text is related to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [14]. Rapid development of the theory is demonstrated by appearing of the current second volume of the book. For new advances of DSmT see other chapters of this volume.

3.3.1 Dedekind lattice, basic DSm notions

Dempster-Shafer modified Theory or Dezert-Smarandache Theory (DSmT) by J. Dezert and F. Smarandache [7, 14] allows mutually overlapping elements of a frame of discernment. Thus, a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example, we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[2]. DSmT allows that an object can have 2 or 3

[1] $m(\emptyset) = 0$ *is often assumed in accordance with Shafer's definition [13]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive* $m(\emptyset)$ *as it assumes* $m(\emptyset) \geq 0$.

[2] www.gallup.unm.edu/∼smarandache/DSmT.htm

colours at the same time: e.g. it can be both red and blue, or red and green and blue in the same time, it corresponds to a composition of the colours from the 3 basic ones.

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), but they are defined on a so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions, they are called generalized basic belief assignments and generalized basic belief functions.

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ now, we have $D^\Theta = \{\alpha_0, \alpha_1, ...\alpha_{18}\}$, where $\alpha_0 = \emptyset, \alpha_1 = \theta_1 \cap \theta_2 \cap \theta_3, \alpha_2 = \theta_1 \cap \theta_2, \alpha_3 = \theta_1 \cap \theta_3, ..., \alpha_{17} = \theta_2 \cup \theta_3, \alpha_{18} = \theta_1 \cup \theta_2 \cup \theta_3$, i.e., $|D^\Theta| = 19$ for $|\Theta| = 3$.

A *generalized basic belief assignment (gbba)* m is a mapping $m : D^\Theta \longrightarrow [0,1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *generalized basic belief mass (gbbm) of* A. A *generalized belief function (gBF)* Bel is a mapping $Bel : D^\Theta \longrightarrow [0,1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$, generalized belief function Bel uniquely corresponds to gbba m and vice-versa.

3.3.2 DSm models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i.e., all elements of an exhaustive frame of discernment can mutually overlap themselves, we refer to the *free DSm model* $\mathcal{M}^f(\Theta)$, i.e., about the DSm model free of constraints.

In general it is possible to add exclusivity or non-existential constraints into DSm models, we speak about *hybrid DSm models* in such cases.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows them: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case all the elements of the 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model as *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings additional information about a frame of discernment saying that θ_3 is impossible; it forces all the gbbm's of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination which is used in a dynamic fusion.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ (*vacuous DSm model* in [14]) we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only case where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_{\mathcal{M}}, \emptyset\}$, where $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of all elements of D^Θ which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set[3].

For a given DSm model we can define (in addition to [14]) $\Theta_{\mathcal{M}} = \{\theta_i | \theta_i \in \Theta, \theta_i \notin \boldsymbol{\emptyset}_{\mathcal{M}}\}$, $\Theta_{\mathcal{M}} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_{\mathcal{M}} = \bigcup_{\theta_i \in \Theta_{\mathcal{M}}} \theta_i$, i.e. $I_{\mathcal{M}} \overset{\mathcal{M}}{\equiv} I_t$, $I_{\mathcal{M}} = I_t \cap \Theta_{\mathcal{M}}$, $I_{\mathcal{M}_\emptyset} = \emptyset$. $D^{\Theta_{\mathcal{M}}}$ is a hyper-power set on the DSm frame of discernment $\Theta_{\mathcal{M}}$, i.e., on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. It holds $\Theta_{\mathcal{M}} = \Theta$, $D^{\Theta_{\mathcal{M}}} = D^\Theta$and $I_{\mathcal{M}} = I_t$ for any DSm model without non-existential constraint. Whereas *reduced (or constrained) hyper-power set* $D^\Theta_{\mathcal{M}}$ (or $D^\Theta(\mathcal{M})$) from Chapter 4 in [14] arises from D^Θ by identifying of all $\mathcal{M}$-equivalent elements. $D^\Theta_{\mathcal{M}^0}$ corresponds to classic power set 2^Θ.

3.3.3 The DSm rules of combination

The *classic DSm rule DSmC* is defined on the free DSm models as it follows[4]:
$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledplus m_2)(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X) m_2(Y)$.
Since D^Θ is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classic DSm rule guarantees that $(m_1 \circledplus m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [14].

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the nature of the problem under consideration, which requires us to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_\emptyset(\Theta)$, for $k \geq 2$ independent sources of information is defined as: $m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledplus m_2 \circledplus ... \circledplus m_k)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)]$, where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined for two sources (for n-ary versions see [14]) as it follows:
$S_1(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X) m_2(Y)$,
$S_2(A) = \sum_{X,Y \in\, \boldsymbol{\emptyset},\ [\mathcal{U}=A] \vee [(\mathcal{U} \in\, \boldsymbol{\emptyset}) \wedge (A = I_t)]} m_1(X) m_2(Y)$,
$S_3(A) = \sum_{X,Y \in D^\Theta,\ X \cup Y = A,\ X \cap Y \in\, \boldsymbol{\emptyset}} m_1(X) m_2(Y)$ with $\mathcal{U} = u(X) \cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are supposed to be in some canonical form, e.g. CNF. Unfortunately no mention about the canonical form is included in [14]. $S_1(A)$ corresponds to the classic DSm rule on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets in both the input gbba's, which arises due to non-existential constraints and is transferred to the total or relative ignorance; and $S_3(A)$ transfers the sum of masses of relatively and absolutely empty sets, which arise as conflicts of the input gbba's, to the non-empty union of input sets[5].

On the degenerated DSm model $\mathcal{M}_\emptyset$ it must be $m_{\mathcal{M}_\emptyset}(\emptyset) = 1$ and $m_{\mathcal{M}_\emptyset}(A) = 0$ for $A \neq \emptyset$.

The hybrid DSm rule generalizes the classic DSm rule to be applicable to any DSm model. The hybrid DSm rule is commutative but not associative. It is the reason the n-ary version

[3] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_{\mathcal{M}}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_{\mathcal{M}} \cup \{\emptyset\}$.

[4] To distinguish the DSm rule from Dempster's rule, we use $\circledplus$ instead of $\oplus$ for the DSm rule in this text.

[5] As a given DSm model $\mathcal{M}$ is used a final compression step must be applied, see Chapter 4 in [14], which is part of Step 2 of the hybrid DSm combination mechanism and "consists in gathering (summing) all masses corresponding to same proposition because of the constraints of the model". I.e., gbba's of $\mathcal{M}$-equivalent elements of D^Θ are summed. Hence the final gbba m is computed as $m(A) = \sum_{X \equiv A} m_{\mathcal{M}(\Theta)}(X)$; it is defined on the reduced hyper-power set $D^\Theta_{\mathcal{M}}$.

of the rule should be used in practical applications. For the n-ary version of $S_i(A)$, see [14]. For easier comparison with generalizations of the classic rules of combination we suppose all formulas in CNF, thus we can include the compression step into formulas $S_i(A)$ as it follows[6]:
$S_1(A) = \sum_{X\equiv A,\ X\in D^\Theta} m_{\mathcal{M}^f(\Theta)}(X) = \sum_{X\cap Y\equiv A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_2(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}\equiv A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_3(A) = \sum_{X,Y\in D^\Theta,\ (X\cup Y)\equiv A,\ X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}})} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_i(A) = 0$ for $A = \emptyset$, and for $A \notin D^\Theta_{\mathcal{M}}$ (where $\mathcal{U}$ is as it is above).

We can further rewrite the DSmH rule to the following equivalent form:
$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledast m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y) +$
$\sum_{X,Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}_{\mathcal{M}}=A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y) +$
$\sum_{X,Y\in D^\Theta,\ X\cup Y=A\ X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$ for all $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$m_{\mathcal{M}(\Theta)}(\emptyset) = 0$ and $m_{\mathcal{M}(\Theta)}(A) = 0$ for $A \in (D^\Theta \backslash D^\Theta_{\mathcal{M}})$.

3.4 A generalization of Dempster's rule

Let us assume all elements X from D^Θ to be in CNF in the rest of this contribution, unless another form of X is explicitly specified. With $X = Y$ we mean that the formulas X and Y have the same CNF. With $X \equiv Y$ ($X \overset{\mathcal{M}}{\equiv} Y$) we mean that the formulas X and Y are equivalent in DSm model $\mathcal{M}$, i.e. their DNFs are the same up to unions with some constrained conjunctions of elements of Θ.

Let us also assume non-degenerated hybrid DSm models, i.e., $\Theta_{\mathcal{M}} \neq \emptyset$, $I_{\mathcal{M}} \notin \boldsymbol{\emptyset}_{\mathcal{M}}$. Let us denote $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_{\mathcal{M}} \cup \{\emptyset\}$, i.e. set of set of all elements of D^Θ which are forced to be empty trough the constraints of DSm model $\mathcal{M}$ extended with classic empty set $\emptyset$, hence we can write $X \in \boldsymbol{\emptyset}_{\mathcal{M}}$ for all $\emptyset \neq X \overset{\mathcal{M}}{\equiv} \emptyset$ or $X \in \boldsymbol{\emptyset}$ for all $X \overset{\mathcal{M}}{\equiv} \emptyset$ including $\emptyset$.

The classic Dempster's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ (the rule adds it to $(m_1 \oplus m_2)(X\cap Y)$) whenever it is non-empty, otherwise the mass is normalized. In the free DSm model all the intersections of non-empty elements are always non-empty, thus no normalization is necessary and Dempster's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ coincides with the classic DSm rule: $(m_1 \oplus m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y=A} m_1(X)m_2(Y) = (m_1 \circledast m_2)(A) = m_{\mathcal{M}^f(\Theta)}(A)$. It follows the fact that the classic DSm rule (DSmC rule) is in fact the conjunctive combination rule generalized to the free DSm model. Hence, Dempster's rule generalized to the free DSm model is defined for any couple of belief functions.

Empty intersections can appear in a general hybrid model $\mathcal{M}$ due to the model's constraints, thus positive gbbm's of constrained elements (i.e equivalent to empty set) can appear, hence the normalization should be used to meet the DSm assumption $m(X) = 0$ for $X \overset{\mathcal{M}}{\equiv} \emptyset$. If we sum together all the gbbm's $m_{\mathcal{M}^f(\Theta)}(X)$ which are assigned to constrained elements of the

[6]We can further simplify the formulas for DSmH rule by using a special canonical form related to the used hybrid DSm model, e.g. $CNF_{\mathcal{M}}(X) = X_{\mathcal{M}} \in D^\Theta_{\mathcal{M}}$ such that $CNF(X) \equiv X_{\mathcal{M}}$. Thus all subexpressions '$\equiv A$' can be replaced with '$= A$' in the definitions of $S_i(A)$ and '$S_i(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$' can be removed from the definition. Hence we obtain a similar form to that published in DSmT book Vol. 1:
$S_1(A) = \sum_{X\cap Y=A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$,
$S_2(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$,
$S_3(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$.
Hence all the necessary assumptions of the definitions of $S_i(A)$ have been formalized.

hyper-power set ($X \in \Theta$, $X \overset{\mathcal{M}}{\equiv} \emptyset$) and assign the resulting sum to $m(\emptyset)$ (or more precisely to $m_{\mathcal{M}}(\emptyset)$), we obtain the non-normalized generalized conjunctive rule of combination. If we redistribute this sum of gbbm's among non-constrained elements of D^Θ using normalization as it is used in the classic Dempster's rule, we obtain the generalized Dempster's rule which meets DSm assumption $m(\emptyset) = 0$.

3.4.1 The generalized non-normalized conjunctive rule

The *generalized non-normalized conjunctive rule of combination* $\circledcirc$ is given as
$(m_1 \circledcirc m_2)(A) = \sum_{X,Y \in D^\Theta, X \cap Y \equiv A} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$(m_1 \circledcirc m_2)(\emptyset) = \sum_{X,Y \in D^\Theta, X \cap Y \in \emptyset} m_1(X)m_2(Y)$,
and $(m_1 \circledcirc m_2)(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$.

We can easily rewrite it as

$$(m_1 \circledcirc m_2)(A) = \sum_{X,Y \in D^\Theta, X \cap Y \equiv A} m_1(X)m_2(Y)$$

for $A \in D^\Theta_{\mathcal{M}}$ ($\emptyset$ including), $(m_1 \circledcirc m_2)(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$.

Similarly to the classic case of the non-normalized conjunctive rule, its generalized version is defined for any couple of generalized belief functions. But we have to keep in mind that positive gbbm of the classic empty set ($m(\emptyset) > 0$) is not allowed in DSmT[7].

3.4.2 The generalized Dempster's rule

To eliminate positive gbbm's of empty set we have to relocate or redistribute gbbm's $m_{\mathcal{M}^f(\Theta)}(X)$ for all $X \overset{\mathcal{M}}{\equiv} \emptyset$. The normalization of gbbm's of non-constrained elements of D^Θ is used in the case of the Dempster's rule.

The *generalized Dempster's rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \in D^\Theta, X \cap Y \equiv A} K m_1(X)m_2(Y)$$

for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X,Y \in D^\Theta, X \cap Y \in \emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \notin D^\Theta_{\mathcal{M}}$.

Similarly to the classic case, the generalized Dempster's rule is not defined in fully contradictive cases[8] in hybrid DSm models, i.e. whenever $\kappa = 1$. Specially the generalized Dempster's rule is not defined (and it cannot be defined) on the degenerated DSm model $\mathcal{M}_\emptyset$.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Dempster's rule to the following equivalent form: $(m_1 \oplus m_2)(A) = \phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)]$,

[7]The examples, which compare DSmH rule with the classic combination rules in Chapter 1 of DSmT book Vol. 1. [14], include also the non-normalized conjunctive rule (called Smets' rule there). To be able to correctly compare all that rules on the generalized level in Section 3.7 of this chapter, we present, here, also a generalization of the non-normalized conjunctive rule, which does not respect the DSm assumption $m(\emptyset) = 0$.

[8]Note that in a static combination it means a full conflict/contradiction between input BF's. Whereas in the case of a dynamic combination it could be also a full conflict between mutually non-conflicting or partially conflicting input BF's and constraints of a used hybrid DSm model. E.g. $m_1(\theta_1 \cup \theta_2) = 1$, $m_2(\theta_2 \cup \theta_3) = 1$, where θ_2 is constrained in a used hybrid model.

where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise, $S_1^{\oplus}(A)$, $S_2^{\oplus}(A)$, and $S_3^{\oplus}(A)$ are defined by
$S_1^{\oplus}(A) = S_1(A) = \sum_{X,Y \in D^\Theta,\, X\cap Y \equiv A} m_1(X)m_2(Y)$,
$S_2^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^\Theta \quad Z\notin \boldsymbol{\emptyset}} S_1(Z)} \sum_{X,Y \in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$,
$S_3^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^\Theta \quad Z\notin \boldsymbol{\emptyset}} S_1(Z)} \sum_{X,Y \in D^\Theta,\ X\cup Y \notin\ \boldsymbol{\emptyset},\ X\cap Y \in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$.

For proofs see Appendix 3.11.1.

$S_1^{\oplus}(A)$ corresponds to a non-conflicting belief mass, $S_3^{\oplus}(A)$ includes all classic conflicting masses and the cases where one of X, Y is excluded by a non-existential constraint, and $S_2^{\oplus}(A)$ corresponds to the cases where both X and Y are excluded by (a) non-existential constraint(s).

It is easy verify that the generalized Dempster's rule coincides with the classic one on Shafer's model $\mathcal{M}^0$, see Appendix 3.11.1. Hence, the above definition of the generalized Dempster's rule is really a generalization of the classic Dempster's rule. Similarly, we can notice that the rule works also on the free DSm model $\mathcal{M}^f$ and its results coincide with those by DSmC rule. We can define n-ary version of the generalized Dempster's rule, analogically to n-ary versions of DSm rules, but because of its associativity it is not necessary in the case of the Dempster's rule.

3.5 A generalization of Yager's rule

The classic Yager's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ whenever it is non-empty, otherwise the mass is added to $m(\Theta)$. As all the intersections are non-empty in the free DSm model, nothing should be added to $m_1(\Theta)m_2(\Theta)$ and Yager's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ also coincides with the classic DSm rule.

$$(m_1 \circledY m_2)(A) = \sum_{X,Y \in D^\Theta,\, X\cap Y = A} m_1(X)m_2(Y) = (m_1 \circledcirc m_2)(A).$$

The *generalized Yager's rule of combination* $\circledY$ for a general hybrid DSm model $\mathcal{M}$ is given as

$$(m_1 \circledY m_2)(A) = \sum_{X,Y \in D^\Theta,\, X\cap Y \equiv A} m_1(X)m_2(Y)$$

for $A \notin\ \boldsymbol{\emptyset},\ \Theta_{\mathcal{M}} \neq A \in D^\Theta_{\mathcal{M}}$,

$$(m_1 \circledY m_2)(\Theta_{\mathcal{M}}) = \sum_{\substack{X,Y\in D^\Theta \\ X\cap Y \equiv \Theta_{\mathcal{M}}}} m_1(X)m_2(Y) + \sum_{\substack{X,Y\in D^\Theta \\ X\cap Y \in\ \boldsymbol{\emptyset}_{\mathcal{M}}}} m_1(X)m_2(Y)$$

and $(m_1 \circledY m_2)(A) = 0$ otherwise, i.e. for $A \in\ \boldsymbol{\emptyset}$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}})$.

It is obvious that the generalized Yager's rule of combination is defined for any couple of belief functions which are defined on hyper-power set D^Θ.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Yager's rule to an equivalent form: $(m_1 \circledY m_2)(A) = \phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)]$, where $S_1^{\circledY}(A)$, $S_2^{\circledY}(A)$, and $S_3^{\circledY}(A)$ are defined by:

$$S_1^{\circledY}(A) = S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y)$$

$$S_2^{\textcircled{Y}}(\Theta_{\mathcal{M}}) = \sum_{X,Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$$

$$S_2^{\textcircled{Y}}(A) = 0 \quad \text{for}\ A \neq \Theta_{\mathcal{M}}$$

$$S_3^{\textcircled{Y}}(\Theta_{\mathcal{M}}) = \sum_{\substack{X,Y\in D^\Theta,\\ X\cup Y\notin\ \boldsymbol{\emptyset},\\ X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}}} m_1(X)m_2(Y)$$

$$S_3^{\textcircled{Y}}(A) = 0 \quad \text{for}\ A \neq \Theta_{\mathcal{M}}.$$

For proofs see Appendix 3.11.2.

Analogically to the case of the generalized Dempster's rule, $S_1^{\textcircled{Y}}(A)$ corresponds to non-conflicting belief mass, $S_3^{\textcircled{Y}}(A)$ includes all classic conflicting masses and the cases where one of X, Y is excluded by a non-existential constraint, and $S_2^{\textcircled{Y}}(A)$ corresponds to the cases where both X and Y are excluded by (a) non-existential constraint(s).

It is easy to verify that the generalized Yager's rule coincides with the classic one on Shafer's model $\mathcal{M}^0$. Hence the definition of the generalized Yager's rule is really a generalization of the classic Yager's rule, see Appendix 3.11.2.

Analogically to the generalized Dempster's rule, we can observe that the formulas for the generalized Yager's rule work also on the free DSm model and that their results really coincide with those by DSmC rule. If we admit also the degenerated (vacuous) DSm model $\mathcal{M}_\emptyset$, i.e., $\Theta_{\mathcal{M}_\emptyset} = \emptyset$, it is enough to modify conditions for $(m_1 \textcircled{Y} m_2)(A) = 0$, so that it holds for $\Theta_{\mathcal{M}} \neq A \in\ \boldsymbol{\emptyset}$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}})$. Then the generalized Yager's rule works also on $\mathcal{M}_\emptyset$; and because of the fact that there is the only bba $m_\emptyset(\emptyset) = 1,\ m_\emptyset(X) = 0$ for any $X \neq \emptyset$ on $\mathcal{M}_\emptyset$, the generalized Yager's rule coincides with the DSmH rule there.

3.6 A generalization of Dubois-Prade's rule

The classic Dubois-Prade's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ whenever it is non-empty, otherwise the mass $m_1(X)m_2(Y)$ is added to $X \cup Y$ which is always non-empty in the DST.

In the free DSm model all the intersections of non-empty elements are always non-empty, thus nothing to be added to unions and Dubois-Prade's rule generalized to the free model $\mathcal{M}^f(\Theta)$ also coincides with the classic DSm rule

$$(m_1 \textcircled{D} m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y = A} m_1(X)m_2(Y) = (m_1 \textcircled{\#} m_2)(A).$$

In the case of a static fusion, only exclusivity constraints are used, thus all the unions of $X_i \in D^\Theta$, $X \notin\ \boldsymbol{\emptyset}$ are also out of $\boldsymbol{\emptyset}$. Thus we can easily generalize Dubois-Prade's rule as $(m_1 \textcircled{D} m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y) + \sum_{X,Y\in D^\Theta,\, X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\, X\cup Y\equiv A} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, and $(m_1 \textcircled{D} m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ or $A \notin D^\Theta_{\mathcal{M}}$.

The situation is more complicated in the case of a dynamic fusion, where non-existential constraints are used. There are several sub-cases how $X \cap Y \in\ \boldsymbol{\emptyset}$ arises.

There is no problem if both X, Y are out of $\boldsymbol{\emptyset}$, because their union $X \cup Y \notin \boldsymbol{\emptyset}$. Similarly if at the least one of X, Y is out of $\boldsymbol{\emptyset}$ then their union is also out of $\boldsymbol{\emptyset}$.

On the other hand if both X, Y are excluded by a non-existential constraint or if they are subsets of elements of D^Θ excluded by non-existential constraints then their union is also excluded by the constraints and the idea of Dubois-Prade's rule is not sufficient to solve this case. Thus the generalized Dubois-Prade rule should be extended to cover also such cases.

Let us start with a simple solutions. As there is absolutely no reason to prefer any of non-constrained elements of D^Θ, the mass $m_1(X)m_2(Y)$ should be either normalized as in Dempster's rule or added to $m(\Theta_\mathcal{M})$ as in Yager's rule. Another option — division of $m_1(X)m_2(Y)$ to k same parts — does not keep a nature of beliefs represented by input belief functions. Because $m_1(X)m_2(Y)$ is always assigned to subsets of X, Y in the case of intersection or to supersets of X, Y in the case of union, addition of $m_1(X)m_2(Y)$ to $m(\Theta)$ is closer to Dubois-Prade's rule nature as $X, Y \subset \Theta$. Whereas the normalization assigns parts of $m_1(X)m_2(Y)$ also to sets which can be disjoint with both of X, Y.

To find a more sophisticated solution, we have to turn our attention to the other cases, where $X \cap Y, X \cup Y \in \boldsymbol{\emptyset}$, and where a simple application of the idea of Dubois-Prade's rule also does not work. Let us assume a fixed hybrid DSm model $\mathcal{M}(\Theta)$ now. Let us further assume that neither X nor Y is a part of a set of elements which are excluded with a non-existential constraint, i.e., $X \cup Y \not\subseteq \bigcup Z_i$ where Z_is are excluded by a non-existential constraint[9]. Let us transfer both X and Y into disjunctive normal form (a union of intersections / a disjunction of conjunctions). Thus, $X \cup Y$ is also in disjunctive form (DNF we obtain by simple elimination of repeating conjuncts/intersections) and at the least one of the conjuncts, let say $W = \theta_{1w} \cap \theta_{2w} \cap ... \cap \theta_{iw}$, contains θ_{jw} non-equivalent to empty set in the given DSm model $\mathcal{M}(\Theta)$. Thus it holds that $\theta_{1w} \cup \theta_{2w} \cup ... \cup \theta_{jw} \notin \boldsymbol{\emptyset}$. Hence we can assign belief masses to $\theta_{1w} \cup \theta_{2w} \cup ... \cup \theta_{jw}$ or to some of its supersets. This idea fully corresponds to Dubois-Prade's rule as the empty intersections are substituted with unions. As we cannot prefer any of the conjuncts — we have to substitute $\cap$s with $\cup$s in all conjuncts of the disjunctive normal form of $X \cup Y$ — we obtain a union $U_{X \cup Y}$ of elements of Θ. The union $U_{X \cup Y}$ includes θ_{jw}; thus it is not equivalent to the empty set and we can assign $m_1(X)m_2(Y)$ to $U_{X \cup Y} \cap I_\mathcal{M} \notin \boldsymbol{\emptyset}$ [10].

Thus we can now formulate a definition of the generalized Dubois-Prade rule. We can distinguish three cases of input generalized belief functions: (i) all inputs satisfy all the constraints of a hybrid DSm model $\mathcal{M}(\Theta)$ which is used (a static belief combination), (ii) inputs do not satisfy the constraints of $\mathcal{M}(\Theta)$ (a dynamic belief combination), but no non-existential constraint is used, (iii) completely general inputs which do not satisfy the constraints, and non-existential constraints are allowed (a more general dynamic combination). According to these three cases, we can formulate three variants of the generalized Dubois-Prade rule.

[9]Hence $X \cup Y$ has had to be excluded by dynamically added exclusivity constraints, e.g. $X = \theta_1 \cap \theta_2$, $Y = \theta_2 \cap \theta_3 \cap \theta_4$ $X \cup Y = (\theta_1 \cap \theta_2) \cup (\theta_2 \cap \theta_3 \cap \theta_4)$, and all $\theta_1, \theta_2, \theta_3, \theta_4$ are forced to be exclusive by added exclusivity constraints, thus $X \cap Y, X \cup Y \in \boldsymbol{\emptyset}_\mathcal{M}$.

[10]We obtain $(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) \cap I_\mathcal{M}$ in the example from the previous footnote.

The *simple generalized Dubois-Prade rule of combination* $\textcircled{\tiny DP}$ is given as[11]

$$(m_1 \textcircled{\tiny DP} m_2)(A) \;=\; \sum_{X\cap Y\equiv A} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)\, m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and $(m_1 \textcircled{\tiny DP} m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$.

The *generalized Dubois-Prade rule of combination* $\textcircled{\tiny DP}$ is given as

$$(m_1 \textcircled{\tiny DP} m_2)(A) = \sum_{X\cap Y\equiv A} m_1(X)m_2(Y) + \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)m_2(Y) + \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv A}} m_1(X)m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and
$(m_1 \textcircled{\tiny DP} m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$,
where $U_{X\cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

The *extended generalized Dubois-Prade rule of combination* $\textcircled{\tiny DP}$ is given as

$$(m_1 \textcircled{\tiny DP} m_2)(A) \;=\; \sum_{X\cap Y\equiv A} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)\, m_2(Y)$$
$$+ \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv A}} m_1(X)\, m_2(Y)$$

for $\emptyset \neq A \neq \Theta_{\mathcal{M}},\; A \in D^{\Theta}_{\mathcal{M}}$,

$$(m_1 \textcircled{\tiny DP} m_2)(\Theta_{\mathcal{M}}) \;=\; \sum_{X\cap Y\equiv \Theta_{\mathcal{M}}} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv \Theta_{\mathcal{M}}}} m_1(X)\, m_2(Y)$$
$$+ \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv \Theta_{\mathcal{M}}}} m_1(X)\, m_2(Y) \;+\; \sum_{U_{X\cup Y}\in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)\, m_2(Y),$$

and

$$(m_1 \textcircled{\tiny DP} m_2)(A) = 0$$

otherwise, i.e., for $A \in \boldsymbol{\emptyset}$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$,
where $U_{X\cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

In the case (i) there are positive belief masses assigned only to the $X_i \in D^{\Theta}$ such that $X \notin \boldsymbol{\emptyset}$, hence the simple generalized Dubois-Prade rule, which ignores all the belief masses assigned to $Y \in \boldsymbol{\emptyset}$, may be used. The rule is defined for any couple of BF's which satisfy the constraints.

[11] We present here 3 variants of the generalized Dubois-Prade rule, formulas for all of them include several summations over $X, Y \in D^{\Theta}$, where X, Y are more specified with other conditions. To simplify the formulas in order to increase their readability, we do not repeat the common condition $X, Y \in D^{\Theta}$ in sums in all the following formulas for the generalized Dubois-Prade rule.

In the case (ii) there are no $U_{X\cup Y} \in \boldsymbol{\emptyset}$, hence the generalized Dubois-Prade rule, which ignores multiples of belief masses $m_1(X)m_2(Y)$, where $U_{X\cup Y} \in \boldsymbol{\emptyset}$, may be used.

In the case (iii) the extended generalized Dubois-Prade rule must be used, this rule can handle all the belief masses in any DSm model, see 1a) in Appendix 3.11.3.

It is easy to verify that the generalized Dubois-Prade rule coincides with the classic one in Shafer's model $\mathcal{M}^0$, see 2) in Appendix 3.11.3.

The classic Dubois-Prade rule is not associative, neither the generalized one is. Similarly to the DSm approach we can easily rewrite the definitions of the (generalized) Dubois-Prade rule for a combination of k sources.

Analogically to the generalized Yager's rule, the formulas for the generalized Dubois-Prade's rule work also on the free DSm model $\mathcal{M}^f$ and their results coincide with those of DSmC rules there, see 1b) in Appendix 3.11.3. If we admit also the degenerated (vacuous) DSm model $\mathcal{M}_\emptyset$, i.e., $\Theta_{\mathcal{M}_\emptyset} = \emptyset$, it is enough again to modify conditions for $(m_1 \textcircled{\scriptsize DP} m_2)(A) = 0$, so that it holds for $\Theta_\mathcal{M} \neq A \in \boldsymbol{\emptyset}$ and for $A \in (D^\Theta \setminus D^\Theta_\mathcal{M})$. Then the extended generalized Dubois-Prade's rule works also on $\mathcal{M}_\emptyset$ and it trivially coincides with DSmH rule there.

To be easily comparable with the DSm rule, we can rewrite the definitions of the generalized Dubois-Prade rules to an equivalent form similar to that of DSm:

the generalized Dubois-Prade rule:

$$(m_1 \textcircled{\scriptsize DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_2^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where

$$S_1^{\textcircled{\tiny DP}}(A) = S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y),$$

$$S_2^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_\mathcal{M},\, U_{X\cup Y}\equiv A} m_1(X)m_2(Y),$$

$$S_3^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\in \boldsymbol{\emptyset}_\mathcal{M},\, (X\cup Y)\equiv A} m_1(X)m_2(Y).$$

the simple generalized Dubois-Prade rule:

$$(m_1 \textcircled{\scriptsize DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where $S_1^{\textcircled{\tiny DP}}(A), S_3^{\textcircled{\tiny DP}}(A)$ as above;
the extended generalized Dubois-Prade rule:

$$(m_1 \textcircled{\scriptsize DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_2^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where $S_1^{\textcircled{\tiny DP}}(A), S_3^{\textcircled{\tiny DP}}(A)$ as above, and

$$S_2^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_\mathcal{M},\, [U_{X\cup Y}\equiv A]\vee[U_{X\cup Y}\in\boldsymbol{\emptyset}\wedge A=\Theta_\mathcal{M}]} m_1(X)m_2(Y).$$

For a proof of equivalence see 3) in Appendix 3.11.3.

Functions $S_1^{\textcircled{DP}}, S_2^{\textcircled{DP}}, S_3^{\textcircled{DP}}$ have interpretations analogical to $S_i^{\oplus}$ and $S_i^{\textcircled{Y}}$ for $\oplus$ and $\textcircled{Y}$. $S_2^{\textcircled{DP}}$ is principal for distinguishing of the variants of the Dubois-Prade rule. In the case (i) no positive belief masses are assigned to $X \in \boldsymbol{\emptyset}$ thus $S_2^{\textcircled{DP}}(A) \equiv 0$, in the case (ii) $S_2^{\textcircled{DP}}(A)$ sums $m_1(X)m_2(Y)$ only for $U_{X\cup Y} \equiv A$, whereas in the case (iii) also $U_{X\cup Y} \in \boldsymbol{\emptyset}_{\mathcal{M}}$ must be included.

In general, some θ_is can repeat several times in $U_{X\cup Y}$, they are eliminated in DNF. Hence we obtain a union of elements of Θ which are contained in X, Y. Let us note that this union $U_{X\cup Y}$ of elements of Θ coincides with $\mathcal{U} = u(X) \cup u(Y)$, more precisely $U_{X\cup Y} \cap I_{\mathcal{M}}$ coincides with $\mathcal{U} \cap I_{\mathcal{M}} = u(X) \cup u(Y) \cap I_{\mathcal{M}}$. Thus the generalized Dubois-Prade rule gives the same results as the hybrid DSmH rule does. Let us further note that the extension of Dubois-Prade's rule, i.e. addition of $m_1(X)m_2(Y)$ to $m(\Theta_{\mathcal{M}})$ for $X, Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ also coincides with the computation with the DSmH rule in the case, where $\mathcal{U} \in \boldsymbol{\emptyset}_{\mathcal{M}}$. Hence, the extended generalized Dubois-Prade rule is fully equivalent to the DSmH rule.

3.7 A comparison of the rules

As there are no conflicts in the free DSm model $\mathcal{M}^f(\Theta)$ all the presented rules coincide in the free DSm model $\mathcal{M}^f(\Theta)$. Thus the following statement holds:

Statement 1. *Dempster's rule, the non-normalized conjunctive rule, Yager's rule, Dubois-Prade's rule, the hybrid DSmH rule, and the classic DSmC rule are all mutually equivalent in the free DSm model* $\mathcal{M}^f(\Theta)$.

Similarly the classic Dubois-Prade rule is equivalent to the DSm rule for Shafer's model. But in general all the generalized rules $\oplus, \textcircled{Y}, \textcircled{DP}$, and DSm rule are different. A very slight difference comes in the case of Dubois-Prade's rule and the DSm rule. A difference appears only in the case of a dynamic fusion where some belief masses of both (of all in an n-ary case) source generalized basic belief assignments are equivalent to the empty set (i.e. $m_1(X), m_2(Y) \in \boldsymbol{\emptyset}_{\mathcal{M}}$ or $m_i(X_i) \in \boldsymbol{\emptyset}_{\mathcal{M}}$). The generalized Dubois-Prade rule is not defined and it must be extended by adding $m_1(X)m_2(Y)$ or $\Pi_i\ m_i(X_i)$ to $m(\Theta_{\mathcal{M}})$ in this case. The generalized Dubois-Prade rule coincides with the DSm rule in all other situations, i.e., whenever all input beliefs fit the DSm model, which is used, and whenever we work with a DSm model without non-existential constraints, see the previous section. We can summarize it as it follows:

Statement 2. *(i) If a hybrid DSm model $\mathcal{M}(\Theta)$ does not include any non-existential constraint or if all the input belief functions satisfy all the constraints of $\mathcal{M}(\Theta)$, then the generalized Dubois-Prade rule is equivalent to the DSm rule in the model $\mathcal{M}(\Theta)$. (ii) The generalized Dubois-Prade rule extended with addition of $m_1(X)m_2(Y)$ (or $\Pi_i\ m_i(X_i)$ in an n-ary case) to $m(\Theta_{\mathcal{M}})$ for $X, Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ (or for $X_i \in \boldsymbol{\emptyset}_{\mathcal{M}}$ in an n-ary case) is fully equivalent to the hybrid DSmH rule on any hybrid DSm model.*

3.7.1 Examples

Let us present examples from Chapter 1 from DSm book 1 [14] for an illustration of the comparison of the generalized rules with the hybrid DSm rule.

Example 1. The first example is defined on $\Theta = \{\theta_1, \theta_2, \theta_3\}$ as Shafer's DSm model $\mathcal{M}^0$ with the additional constraint $\theta_3 \equiv \emptyset$, i.e. $\theta_1 \cap \theta_2 \equiv \theta_3 \equiv \emptyset$ in DSm model $\mathcal{M}_1$, and subsequently $X \equiv Y \equiv \emptyset$ for all $X \subseteq \theta_1 \cap \theta_2$ $Y \subseteq \theta_3$. We assume two independent source belief assignments m_1, m_2, see Table 3.1.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}_1$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{D}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}_1}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{D}}$
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0	$\emptyset$					
$\theta_1 \cap \theta_2$	0	0	0.21	$\emptyset$					
$\theta_1 \cap \theta_3$	0	0	0.13	$\emptyset$					
$\theta_2 \cap \theta_3$	0	0	0.14	$\emptyset$					
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0	0	0	$\emptyset$					
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0	0	0	$\emptyset$					
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0	0	0.11	$\emptyset$					
$\Box$	0	0	0	$\emptyset$					
θ_1	0.10	0.50	0.21	θ_1	0.34	0.600	0.21	0.21	0.34
θ_2	0.40	0.10	0.11	θ_2	0.25	0.314	0.11	0.11	0.25
θ_3	0.20	0.30	0.06	$\emptyset$					
$\Box\theta_1$	0	0	0	θ_1	0	0	0	0	0
$\Box\theta_2$	0	0	0	θ_2	0	0	0	0	0
$\Box\theta_3$	0	0	0	$\emptyset$					
$\theta_1 \cup \theta_2$	0.30	0.10	0.03	$\theta_1 \cup \theta_2$	0.41	0.086	0.03	0.68	0.41
$\theta_1 \cup \theta_3$	0	0	0	θ_1	0	0	0	0	0
$\theta_2 \cup \theta_3$	0	0	0	θ_2	0	0	0	0	0
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	$\theta_1 \cup \theta_2$	0	0	0	0	0
$\emptyset$				$\emptyset$			0.65		

Table 3.1: Example 1 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{D}$ on hybrid DSm model $\mathcal{M}_1$.

A description of Table 3.1. As DSm theory admits general source basic belief assignments defined on the free DSm model $\mathcal{M}^f$, all elements of D^Θ are presented in the first column of the table. We use the following abbreviations for 4 elements of D^Θ: $\Box$ for $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap (\theta_2 \cup \theta_3)$, $\Box\theta_1$ for $\theta_1 \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3)$, $\Box\theta_2$ for $\theta_2 \cup (\theta_1 \cap \theta_3)$, and $\Box\theta_3$ for $\theta_3 \cup (\theta_1 \cap \theta_2)$. Thus $\Box$ is not any operator here, but just a symbol for abbreviation; it has its origin in the papers about minC combination [3, 5, 6], see also Chapter 10 in DSm book Vol. 1 [14].

Source gbba's m_1, m_2 follow in the second and the third column. The central part of the table contains results of DSm combination of the beliefs: the result obtained with DSmC rule, i.e. resulting gbba m_{DSmC}, is in the 4th column and the result obtained with DSmH is in the 6th column. Column 5 shows equivalence of elements of the free DSm model $\mathcal{M}^f$ to those of the assumed hybrid DSm model $\mathcal{M}_1$. Finally, the right part of the table displays the results of combination of the source gbba's with the generalized combination rules (with the generalized Dempster's rule $\oplus$ in the 7-th column, with the generalized non-normalized Dempster's rule $\textcircled{\cap}$ in column 8, etc.). The resulting values are always cumulated, thus the value for $m(\theta_1)$ is only in the row corresponding to θ_1, whereas all the other rows corresponding to sets equivalent to θ_1 contain 0s. Similarly, all the fields corresponding to empty set are blank with the exception that for $m_{\textcircled{\cap}}(\emptyset)$, i.e. the only one where positive $m(\emptyset)$ is allowed. The same structure of the table is used also in the following examples.

Example 2. Le us assume, now, two independent sources m_1, m_2 over 4-element frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, where Shafer's model $\mathcal{M}^0$ holds, see Table 3.2.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2$	0	0	0.9604	$\emptyset$					
$\theta_1 \cap \theta_4$	0	0	0.0196	$\emptyset$					
$\theta_2 \cap \theta_3$	0	0	0.0098	$\emptyset$					
$\theta_2 \cap \theta_4$	0	0	0.0098	$\emptyset$					
$\theta_3 \cap \theta_4$	0	0	0.0002	$\emptyset$					
θ_1	0.98	0	0	θ_1	0	0	0	0	0
θ_2	0	0.98	0	θ_2	0	0	0	0	0
θ_3	0.01	0	0	θ_3	0	0	0	0	0
θ_4	0.01	0.02	0.0002	θ_4	0.0002	1	0.0002	0.0002	0.0002
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.9604	0	0	0	0.9604
$\theta_1 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_4$	0.0196	0	0	0	0.0196
$\theta_2 \cup \theta_3$	0	0	0	$\theta_2 \cup \theta_3$	0.0098	0	0	0	0.0098
$\theta_2 \cup \theta_4$	0	0	0	$\theta_2 \cup \theta_4$	0.0098	0	0	0	0.0098
$\theta_3 \cup \theta_4$	0	0	0	$\theta_3 \cup \theta_4$	0.0002	0	0	0	0.0002
$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0.9998	0
$\emptyset$				$\emptyset$			0.9998		

Table 3.2: Example 2 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on Shafer's DSm model $\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

The structure of Table 3.2 is the same as in the case of Table 3.1. Because of the size of the full table for DSm combination on a 4-element frame of discernment, rows which contain only 0s and blank fields are dropped.

Note, that input values are shortened by one digit here (i.e. 0.98, 0.02, and 0.01 instead of 0.998, 0.002, and 0.001) in comparison with the original version of the example in [14]. Nevertheless the structure and features of both the versions of the example are just the same.

Example 3. This is an example for Smet's case, for the non-normalized Dempster's rule. We assume Shafer's model $\mathcal{M}^0$ on a simple 2-element frame $\Theta = \{\theta_1, \theta_2\}$. We assume $m(\emptyset) \geq 0$, in this example, even if it is not usual in DSm theory, see Table 3.3.

Example 4. Let us assume Shafer's model $\mathcal{M}^0$ on $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ in this example, see Table 3.4.

Example 5. Let us assume again Shafer's model $\mathcal{M}^0$ on a simple 2-element frame $\Theta = \{\theta_1, \theta_2\}$, see Table 3.5.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{\mathcal{P}}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{\mathcal{P}}}$
$\theta_1 \cap \theta_2$	0	0	0.28	$\emptyset$					
θ_1	0.40	0.60	0.24	θ_1	0.48	0.143	0.24	0.24	0.48
θ_2	0.40	0.10	0.04	θ_2	0.18	0.857	0.04	0.04	0.18
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.34	0	0	0.72	0.34
$\emptyset$	0.20	0.30	0.44	$\emptyset$			0.72		

Table 3.3: Example 3 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{\mathcal{P}}$ on Shafer's DSm model $\mathcal{M}^0$.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{\mathcal{P}}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{\mathcal{P}}}$
$\theta_1 \cap \theta_2$	0	0	0.9702	$\emptyset$					
$\theta_1 \cap (\theta_3 \cup \theta_4)$	0	0	0.0198	$\emptyset$					
$\theta_2 \cap (\theta_3 \cup \theta_4)$	0	0	0.0098	$\emptyset$					
θ_1	0.99	0	0	θ_1	0	0	0	0	0
θ_2	0	0.98	0	θ_2	0	0	0	0	0
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.9702	0	0	0	0.9702
$\theta_3 \cup \theta_4$	0.01	0.02	0.0002	$\theta_3 \cup \theta_4$	0.0002	1	0.0002	0.0002	0.0002
$\theta_1 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_3 \cup \theta_4$	0.0198	0	0	0	0.0198
$\theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_2 \cup \theta_3 \cup \theta_4$	0.0098	0	0	0	0.0098
$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0.9998	0
$\emptyset$				$\emptyset$			0.9998		

Table 3.4: Example 4 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{\mathcal{P}}$ on Shafer's DSm model $\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

$\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{\mathcal{P}}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{\mathcal{P}}}$
$\theta_1 \cap \theta_2$	0.40	0.30	0.89	$\emptyset$					
θ_1	0.50	0.10	0.05	θ_1	0.24	$0.\overline{45}$	0.05	0.05	0.24
θ_2	0.10	0.60	0.06	θ_2	0.33	$0.\overline{54}$	0.06	0.06	0.33
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.43	0	0	0.89	0.43
$\emptyset$				$\emptyset$			0.89		

Table 3.5: Example 5 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{\mathcal{P}}$ on Shafer's DSm model $\mathcal{M}^0$.

Example 6. As all the above examples are quite simple, usually somehow related to Shafer's model, we present also one of the more general examples (Example 3) from Chapter 4 DSm book Vol. 1; it is defined on the DSm model $\mathcal{M}_{4.3}$ based on 3-element frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with constraints $\theta_1 \cap \theta_2 \equiv \theta_2 \cap \theta_3 \equiv \emptyset$; and subsequently $\theta_1 \cap \theta_2 \cap \theta_3 \equiv \theta_2 \cap (\theta_1 \cup \theta_3) \equiv \emptyset$, see Table 3.6.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}_{4.3}$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}_{4.3}}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0.16	$\emptyset$					
$\theta_1 \cap \theta_2$	0.10	0.20	0.22	$\emptyset$					
$\theta_1 \cap \theta_3$	0.10	0	0.12	$\theta_1 \cap \theta_3$	0.17	0.342	0.12	0.12	0.17
$\theta_2 \cap \theta_3$	0	0.20	0.19	$\emptyset$					
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0	0	0	$\theta_1 \cap \theta_3$	0	0	0	0	0
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0	0	0.05	$\emptyset$					
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0	0	0.01	$\theta_1 \cap \theta_3$	0	0	0.01	0.01	0
$\square$	0	0	0	$\theta_1 \cap \theta_3$	0	0	0	0	0
θ_1	0.10	0.20	0.08	θ_1	0.16	0.263	0.08	0.08	0.16
θ_2	0.20	0.10	0.03	θ_2	0.12	0.079	0.03	0.03	0.12
θ_3	0.30	0.10	0.10	θ_3	0.23	0.263	0.10	0.10	0.23
$\square\theta_1$	0	0	0.02	θ_1	0	0	0.02	0.02	0
$\square\theta_2$	0	0	0	$\square\theta_2$	0.01	0	0	0	0.01
$\square\theta_3$	0	0	0	θ_3	0	0	0	0	0
$\theta_1 \cup \theta_2$	0.10	0	0	$\theta_1 \cup \theta_2$	0.11	0	0	0	0.11
$\theta_1 \cup \theta_3$	0.10	0.20	0.02	$\theta_1 \cup \theta_3$	0.08	0.053	0.02	0.02	0.08
$\theta_2 \cup \theta_3$	0	0	0	$\theta_2 \cup \theta_3$	0.05	0	0	0	0.05
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3$	0.07	0	0	0.62	0.07
$\emptyset$				$\emptyset$			0.62		

Table 3.6: Example 6 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on hybrid DSm model $\mathcal{M}_{4.3}$.

3.7.2 A summary of the examples

We can mention that all the rules are defined for all the presented source generalized basic belief assignments. In the case of the generalized Dempster's rule it is based on the fact that no couple of source gbba's is in full contradiction. In the case of the generalized Dubois-Prade's rule we need its extended version in Examples 1, 3, 5, and 6.

In Example 1, it is caused by constraint $\theta_3 \equiv \emptyset$ and positive values $m_1(\theta_3) = 0.20$ and $m_2(\theta_3) = 0.30$, see Table 3.1, hence we have $m_1(\theta_3)m_2(\theta_3) = 0.06 > 0$ and $\theta_3 \cap \theta_3 = \theta_3 \cup \theta_3 = \theta_3 \equiv \emptyset$ in DSm model $\mathcal{M}_1$ in question. In Example 3, it is caused by admission of positive input values for $\emptyset$: $m_1(\emptyset) = 0.20$, $m_2(\emptyset) = 0.30$. In Example 5, it is because both m_1 and m_2 have positive input values for $\theta_1 \cap \theta_2$ which is constrained. We have $m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) = 0.12$ and $(\theta_1 \cap \theta_2) \cap (\theta_1 \cap \theta_2) = \theta_1 \cap \theta_2 \equiv \emptyset \equiv (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_2)$, hence 0.12 should be added to Θ by the extended Dubois-Prade's rule. We have to distinguish this case from different cases such as e.g. $m_1(\theta_1)m_2(\theta_2)$ or $m_1(\theta_1 \cap \theta_2)m_2(\theta_2)$, where values are normally assigned to union of arguments $(\theta_1) \cup (\theta_2)$ or $(\theta_1 \cap \theta_2) \cup \theta_2 = \theta_2$ respectively. In Example 6, it is analogically caused by couples of positive inputs $m_1(\theta_1 \cap \theta_2),\ m_2(\theta_1 \cap \theta_2)$ and $m_1(\theta_1 \cap \theta_2),\ m_2(\theta_2 \cap \theta_3)$.

In Examples 2 and 4, the generalized Dubois-Prade's rule without extension can be used because all the elements of D^Θ which are constrained (prohibited by the constraints) have 0 values of gbbm's.

We can observe that, $m(\emptyset) > 0$ only when using the generalized conjunctive rule $\odot$, where $m_{\odot}(\emptyset) = \sum_{Z \equiv \emptyset} m(Z)$ and $m_{\odot}(X) = m_{DSmC}(X)$ for $X \not\equiv \emptyset$. If we distribute $m_{\odot}(\emptyset)$ with normalization, we obtain the result $m_\oplus$ of the generalized Dempster's rule $\oplus$; if we relocate (add) $m_{\odot}(\emptyset)$ to $m_{\odot}(\Theta)$ we obtain $m_{\circledY}$, i.e. the result of the generalized Yager's rule.

The other exception of $m(\emptyset) > 0$ is in Example 3, where $m_{DSmC}(\emptyset) = 0.44 > 0$ because there is $m_1(\emptyset) > 0$ and $m_2(\emptyset) > 0$ what is usually not allowed in DSmT. This example was included into [14] for comparison of DSmH with the classic non-normalized conjunctive rule used in TBM.

In accordance with theoretical results we can verify, that the DSmH rule always gives the same resulting values as the generalized Dubois-Prade rule produces in all 6 examples.

Looking at the tables we can observe, that DSmH and Dubois-Prade's generate more specified results (i.e. higher gbbm's are assigned to smaller elements of D^Θ) than both the generalized non-normalized conjunctive rule $\odot$ and the generalized Yager's rule $\circledY$ produce. There is some lost of information when the generalized $\oplus$ or $\circledY$ are applied. Nevertheless, there is some lost of information also within the application of the DSmH rule. Considering the rules investigated and compared in this text we obtain the most specific results when the generalized Dempster's rule $\oplus$ is used. Another rules, which produce more specified results than the DSmH rule and the generalized Dubois-Prade's rule do, are the generalized minC combination rule [5] and PCR combination rules [15], which are out of scope of this chapter.

3.8 Open problems

As an open question remains commutativity of a transformation of generalized belief functions to those which satisfy all the constraints of a used hybrid DSm model with the particular combination rules. Such a commutation may significantly simplify functions S_2 and hence the entire definitions of the corresponding combination rules. If such a commutation holds for some combination rule, we can simply transform all input belief functions to those which satisfy constraints of the DSm model in question at first; and perform static fusion after. No dynamic fusion is necessary in such a case.

A generalization of minC combination rule, whose computing mechanism (not a motivation nor an interpretation) has a relation to the conjunctive rules on the free DSm model $\mathcal{M}^f(\Theta)$ already in its classic case [3], is under recent development. And it will also appear as a chapter of this volume.

We have to mention also the question of a possible generalization of conditionalization, related to particular combination rules to the domain of DSm hyper-power sets.

And we cannot forget for a new family of PCR rules [15], see also a chapter in this volume. Comparison of these rules, rules presented in this chapter, generalized minC combination and possibly some other belief combination rules on hyper-power sets can summarize the presented topic.

3.9 Conclusion

The classic rules for combination of belief functions have been generalized to be applicable to hyper-power sets, which are used in DSm theory. The generalization forms a solid theoretical background for full and objective comparison of the nature of the classic rules with the nature of the DSm rule of combination. It also enables us to place the DSmT better among the other approaches to belief functions.

3.10 References

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3.11 Appendix - proofs

3.11.1 Generalized Dempster's rule

1) *Correctness of the definition:*
1a) $\sum_{X,Y\in D^\Theta} m_1(X)m_2(Y) = 1$ for any gbba's m_1, m_2; multiples $0 \le m_1(X)m_2(Y) \le 1$ are summed to $m(A)$ for $X \cap Y \equiv A, \emptyset \neq A \in D^\Theta_\mathcal{M}$, all the other multiples (i.e., for $X \cap Y = \emptyset$ and for $X \cap Y = A \notin D^\Theta_\mathcal{M}$) are normalized among $\emptyset \neq A \in D^\Theta_\mathcal{M}$. Hence the formula for the generalized Dempster's rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2.
1b) It holds $\kappa = \sum_{X,Y\in D^\Theta, X\cap Y\in \boldsymbol{\emptyset}} m_1(X)m_2(X) = 0$ and $K = \frac{1}{1-\kappa} = 1$ in the free DSm model $\mathcal{M}^f$. Hence we obtain the formula for the free model $\mathcal{M}^f$ as a special case of the general formula.
2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \emptyset_{\mathcal{M}^0}$ iff $\{\theta_i|\theta_i \subseteq X\} \cap \{\theta_j|\theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$ in both the classic and the generalized Dempster's rule on Shafer's DSm model, and the same multiples are normalized by both of the rules. Thus, the results are the same for any m_1, m_2 on $\mathcal{M}^0$ and for any $A \subseteq \Theta$ and other $A \in D^\Theta$. Hence the generalized Dempster's rule is really a generalization of the classic Dempster's rule.

3) *Equivalence of expressions:* $(m_1 \oplus m_2)(A) \stackrel{?}{=} \phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)]$

$$\phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)] = \phi(A) \sum_{X\cap Y\equiv A} m_1(X)m_2(Y) +$$
$$\phi(A)[\frac{S_1(A)}{\sum_{Z\in D^\Theta\; Z\notin\boldsymbol{\emptyset}} S_1(Z)} \sum_{X,Y\in\ \boldsymbol{\emptyset}_\mathcal{M}} m_1(X)m_2(Y) +$$
$$\frac{S_1(A)}{\sum_{Z\in D^\Theta\; Z\notin\boldsymbol{\emptyset}} S_1(Z)} \sum_{X\cup Y\notin\ \boldsymbol{\emptyset},\ X\cap Y\in\ \boldsymbol{\emptyset}_\mathcal{M}} m_1(X)m_2(Y)]$$

For $A \notin \boldsymbol{\emptyset}$ we obtain the following (as $m_i(\emptyset) = 0$):

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y) + [\frac{S_1(A)}{\sum_{Z\in D^\Theta \ Z\notin\boldsymbol{\emptyset}} S_1(Z)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)] =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y) + \frac{\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)}{\sum_{X\cap Y\notin\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(1 + \frac{1}{\sum_{X\cap Y\notin\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(\frac{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} + \frac{\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(\frac{1}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)\frac{1}{1-\kappa} = \sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} K m_1(X)m_2(Y) = (m_1 \oplus m_2)(A).$$

For $A \in \boldsymbol{\emptyset}$ we obtain:

$$\phi(A)[S_1^{\oplus}(A) + S_2^{\oplus}(A) + S_3^{\oplus}(A)] = 0 \cdot [S_1^{\oplus}(A) + S_2^{\oplus}(A) + S_3^{\oplus}(A)] = 0 = (m_1 \oplus m_2)(A).$$

Hence the expression in DSm form is equivalent to the definition of the generalized Dempster's rule.

3.11.2 Generalized Yager's rule

1) *Correctness of the definition:*
1a) $\sum_{X,Y\in D^\Theta, X\cap Y=A} m_1(X)m_2(Y) = 1$ for any gbba's m_1, m_2; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cap Y = A \in \boldsymbol{\emptyset}$) are summed to $\Theta_{\mathcal{M}}$. Hence the formula for the generalized Yager's rule produces correct gbba $m_1 \circledY m_2$ for any input gbba's m_1, m_2.
1b) It holds $\sum_{X,Y\in D^\Theta, X\cap Y\in\emptyset} m_1(X)m_2(X) = 0$ in the free DSm model $\mathcal{M}^f$. Thus $(m_1 \circledY m_2)(\Theta) = m_1(\Theta)m_2(\Theta)$. Hence we obtain the formula for the free model $\mathcal{M}^f$ as a special case of the general formula.

2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ iff $\{\theta_i|\theta_i \subseteq X\} \cap \{\theta_j|\theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$, $A \neq \Theta$ in both the classic and the generalized Yager's rule on Shafer's DSm model, and the same multiples are summed to Θ by both of the rules. Thus, the results are the same for any m_1, m_2 on $\mathcal{M}^0$ and any $A \subseteq \Theta$ $(A \in D^\Theta)$. Hence the generalized Yager's rule is a correct generalization of the classic Yager's rule.

3) *Equivalence of expressions:* $(m_1 \circledY m_2)(A) \stackrel{?}{=} \phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)]$

For $\Theta_{\mathcal{M}} \neq A \notin \boldsymbol{\emptyset}$ we obtain the following:

$$\phi(A)[S_1^{\textcircled{Y}}(A) + S_2^{\textcircled{Y}}(A) + S_3^{\textcircled{Y}}(A)] = \phi(A)[\sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + 0 + 0]$$
$$= \sum_{X \cap Y \equiv A \notin \boldsymbol{\emptyset}} m_1(X)m_2(Y) = (m_1 \textcircled{Y} m_2)(A).$$

For $A = \Theta_{\mathcal{M}}$ we obtain the following:

$$\phi(\Theta_{\mathcal{M}}) \sum_{X \cap Y \equiv \Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \phi(\Theta_{\mathcal{M}})[\sum_{X,Y \in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$$
$$+ \sum_{X \cup Y \notin \boldsymbol{\emptyset},\ X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)]$$
$$= \sum_{X \cap Y \equiv \Theta_{\mathcal{M}}} m_1(X)m_2(Y) + [\sum_{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)] = (m_1 \textcircled{Y} m_2)(\Theta_{\mathcal{M}}).$$

For $A \in \boldsymbol{\emptyset}$ we obtain $\phi(A)[S_1^{\textcircled{Y}}(A) + S_2^{\textcircled{Y}}(A) + S_3^{\textcircled{Y}}(A)] = 0[S_1^{\textcircled{Y}}(A) + 0 + 0] = 0 = (m_1 \oplus m_2)(A)$. Hence the expression in DSm form is equivalent to the definition of the generalized Yager's rule.

3.11.3 Generalized Dubois-Prade rule

1) *Correctness of the definition:*

1a) $\sum_{X,Y \in D^\Theta m_1(X)m_2(Y)} = 1$ for any gbba's m_1, m_2; Let us assume that m_1, m_2 satisfy all the constraints of DSm model $\mathcal{M}$, thus $m_1(X) \cup m_2(Y) \notin \boldsymbol{\emptyset}$ for any $X, Y \in D^\Theta_{\mathcal{M}}$; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cap Y = A \in \boldsymbol{\emptyset}$) are summed and added to $m(A)$, where $A = X \cup Y$, with the simple generalized Dubois-Prade rule. Hence the simple generalized Dubois-Prade rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2 which satisfy all the constraints of the used DSm model $\mathcal{M}$.

Let us assume a DSm model $\mathcal{M}$ without non-existential constraints, now, thus $U_{X \cup Y} \notin \boldsymbol{\emptyset}$ for any $\emptyset \neq X, Y \in D^\Theta_{\mathcal{M}}$; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed and added to $m(A)$ for $X \cap Y = A \notin \boldsymbol{\emptyset}$, other multiples are summed to $m(A)$ for $X \cup Y = A \notin \boldsymbol{\emptyset}$, $X \cap Y = A \in \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cup Y = A \in \boldsymbol{\emptyset}$) are summed and added to $m(A)$ where $A = U_{X \cup Y}$, with the generalized Dubois-Prade rule. Hence the generalized Dubois-Prade rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2 on DSm model $\mathcal{M}$ without non-existential constraints.

For a fully general dynamic belief fusion on any DSm model the following holds:

multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin \boldsymbol{\emptyset}$, other multiples are summed and added to $m(A)$ for $X \cup Y = A \notin \boldsymbol{\emptyset}$, $X \cap Y = A \in \boldsymbol{\emptyset}$, other multiples are summed and added to $m(A)$ for $U_{X \cup Y} = A \notin \boldsymbol{\emptyset}$, $X \cup Y = A \in \boldsymbol{\emptyset}$, all the other multiples (i.e., for $U_{X \cup Y} = A \in \boldsymbol{\emptyset}$) are summed and added to $\Theta_{\mathcal{M}}$. Hence the extended generalized Dubois-Prade rule produces correct gbba $m_1 \textcircled{D} m_2$ for any input gbba's m_1, m_2 on any hybrid DSm model.

1b) It holds since $\sum_{X,Y \in D^\Theta, X \cap Y \in \emptyset} m_1(X)m_2(X) = 0 = \sum_{X \cup Y \in \emptyset} m_1(X)m_2(X)$ and one has also $\sum_{X \cup Y \in \emptyset} m_1(X)m_2(X) = \sum_{U_{X \cup Y} \in \emptyset} m_1(X)m_2(X)$ in the free DSm model $\mathcal{M}^f$. Hence, the Dubois-Prade rule for the free model $\mathcal{M}^f$ is a special case of all the simple generalized Dubois-Prade rule, the generalized Dubois-Prade rule, and the extended generalized Dubois-Prade rule.

2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$ and input BF's on $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}^0}$ iff $\{\theta_i | \theta_i \subseteq X\} \cap \{\theta_j | \theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$, $A \neq \Theta$ in both the classic and the generalized Dubois-Prade rule on Shafer's DSm model, and the same multiples are summed and added to $X \cup Y = A \notin \boldsymbol{\emptyset}$ by both of the rules. $X \cup Y \notin \boldsymbol{\emptyset}$ for any couple $X, Y \in D^\Theta$ in Shafer's model, thus the 3rd sum in the generalized Dubois-Prade rule and the 4th sum in the extended rule for $\Theta_\mathcal{M}$ are always equal to 0 in Shafer's DSm model. Thus, the results are always the same for any m_1, m_2 on $\mathcal{M}^0$ and any $A \subseteq \Theta$ (and $A \in D^\Theta$). Hence all the simple generalized Dubois-Prade rule, the generalized Dubois-Prade rule, and the extended generalized Dubois-Prade rule are correct generalizations of the classic Dubois-Prade rule.

3) *Equivalence of expressions:* $(m_1 \circledast m_2)(A) \stackrel{?}{=} \phi(A)[S_1^{\circledast}(A) + S_2^{\circledast}(A) + S_3^{\circledast}(A)]$

$$\phi(A)[S_1^{\circledast}(A) + S_2^{\circledast}(A) + S_3^{\circledast}(A)] = \phi(A)[\sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + \sum_{X \cup Y \in \boldsymbol{\emptyset}_\mathcal{M},\ U_{X\cup Y} \equiv A} m_1(X)m_2(Y) + \sum_{X \cap Y \in \boldsymbol{\emptyset}_\mathcal{M},\ (X\cup Y) \equiv A} m_1(X)m_2(Y)]$$

For $A \notin \boldsymbol{\emptyset}$ we simply obtain the following:

$$1 \cdot [\sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + \sum_{X \cup Y \in \boldsymbol{\emptyset}_\mathcal{M},\ U_{X\cup Y} \equiv A} m_1(X)m_2(Y) + \sum_{X \cap Y \in \boldsymbol{\emptyset}_\mathcal{M},\ (X\cup Y) \equiv A} m_1(X)m_2(Y)] = (m_1 \circledast m_2)(A),$$

and for $A \in \boldsymbol{\emptyset}$, one gets

$$0 \cdot [S_1^{\circledast}(A) + S_2^{\circledast}(A) + S_3^{\circledast}(A)] = 0 = (m_1 \circledast m_2)(\emptyset).$$

The proof for the simple generalized Dubois-Prade rule is a special case of this proof with $S_2^{\circledast}(A) \equiv 0$.

The same holds for the extended generalized Dubois-Prade rule for $A \in \boldsymbol{\emptyset}$ and for $\Theta_\mathcal{M} \neq A \notin \boldsymbol{\emptyset}$.
For $A = \Theta_\mathcal{M}$ we obtain the following:

$$1\cdot[\sum_{X\cap Y\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \sum_{X\cup Y\in\emptyset_{\mathcal{M}},\,[U_{X\cup Y}\equiv\Theta_{\mathcal{M}}]\vee[U_{X\cup Y}\in\emptyset_{\mathcal{M}}]} m_1(X)m_2(Y)$$

$$+\sum_{X\cap Y\in\emptyset_{\mathcal{M}},\,(X\cup Y)\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y)] =$$

$$[\sum_{X\cap Y\equiv A} m_1(X)m_2(Y) + \sum_{X\cup Y\in\emptyset_{\mathcal{M}},\,U_{X\cup Y}\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \sum_{U_{X\cup Y}\in\emptyset_{\mathcal{M}}} m_1(X)m_2(Y)$$

$$+\sum_{X\cap Y\in\emptyset_{\mathcal{M}},\,(X\cup Y)\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y)] = (m_1 \circledcirc m_2)(\Theta_{\mathcal{M}})$$

Hence all three versions of the expression in DSm form are equivalent to the corresponding versions of the definition of the generalized Dubois-Prade rule.

3.11.4 Comparison statements

Statement 1: trivial.

Statement 2(ii): Let us compare definitions of DSmH rule and the generalized Dubois-Prade rule in DSm form. We have $S_1^{\circledcirc}(A) = S_1(A)$, we can simply observe that $S_3^{\circledcirc}(A) = S_3(A)$. We have already mentioned that $U_{X\cup Y} = \mathcal{U} = u(X)u(Y)$, thus also $S_2^{\circledcirc}(A) = S_2(A)$. Hence $(m_1 \circledcirc m_2)(A) = (m_1 \circledast m_2)(A)$ for any A and any m_1, m_2 in any hybrid DSm model.

Statement 2(i): If all constraints are satisfied by all input beliefs, we have $m_1(X) = m_2(Y) = 0$ for any $X, Y \in \emptyset_{\mathcal{M}}$ and $S_2(A) = 0 = S_2^{\circledcirc}(A)$. If some constraints are not satisfied, but there is no non-existential constraint in model $\mathcal{M}$, then $\mathcal{U} = U_{X\cup Y} \notin \emptyset_{\mathcal{M}}$, and $S_2(A) = \sum_{X,Y\in\emptyset_{\mathcal{M}},\,\mathcal{U}_{\mathcal{M}}=A} m_1(X)m_2(Y) = \sum_{X,Y\in\emptyset_{\mathcal{M}},\,U_{X\cup Y}\cap I_{\mathcal{M}}=A} m_1(X)m_2(Y) = S_2^{\circledcirc}(A)$ again.