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Neutrosophic Frameworks for Situation Analysis

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Abstract: *In situation analysis, an agent observing a scene receives information from heterogeneous sources of information including for example remote sensing devices, human reports and databases. The aim of this agent is to reach a certain level of awareness of the situation in order to make decisions. For the purpose of applications, this state of awareness can be conceived as a state of knowledge in the classical epistemic logic sense. Considering the logical connection between belief and knowledge, the challenge for the designer is to transform the raw, imprecise, conflictual and often paradoxical information received from the different sources into statements understandable by both man and machines. Situation analysis applications need frameworks general enough to take into account the different types of uncertainty and information present in the situation analysis context, doubled with a semantics allowing meaningful reasoning on situations. The aim of this chapter is to evaluate the capacity of neutrosophic logic and Dezert-Smarandache theory (DSmT) to cope with the ontological and epistemic problems of situation analysis.*

16.1 Introduction

The aim of Situation Analysis (SA) in a decision-making process is to provide and maintain a state of situation awareness for an agent observing a scene [1, 2]. For the purpose of applications, this state of awareness can be conceived as a state of knowledge in the classical epistemic logic sense. Considering the logical connection between belief and knowledge, the challenge for the designer is to transform the raw, imprecise, conflictual and often paradoxical information received from the different sources into statements understandable by both man and machines. Because the agent receives information from heterogeneous sources of information including for example remote sensing devices, human reports and databases, two simultaneous tasks need to be achieved: *measuring the world* and *reasoning about the structure of the world*. A great challenge in SA is the conciliation of both quantitative and qualitative information processing in mathematical and logical frameworks. As a consequence, SA applications need frameworks general enough to take into account the different types of uncertainty and information present in the SA context, doubled with a semantics allowing meaningful reasoning on belief, knowledge and situations. The formalism should also allow the possibility to encompass the case of multiagent systems in which the state of awareness can be distributed over several agents rather than localized.

A logical approach based on a possible worlds semantics for reasoning on belief and knowledge in multiagent context is proposed in [3]. This work by Halpern and Moses can be used as a blueprint considering that it allows to handle numerical evaluations of probabilities, thus treating separately but nevertheless linking belief, knowledge and uncertainty. Related works are those of Fagin and Halpern [4] but also Bundy [5] which extend the probability structure of Nilsson [6] based on possible worlds semantics to a more general one close to the evidence theory developed by Dempster [7] and Shafer [8]. The result is the conciliation of both measures and reasoning in a single framework.

Independently of these works has been introduced *Neutrosophy*, a branch of philosophy which studies neutralities and paradoxes, and relations between a concept and its opposite [9]. Two main formal approaches have emerged from Neutrosophy: *neutrosophic logic*, presented as a unified logic, of which fuzzy logic, classical logic and others are special cases [10, 11]; and *Dezert-Smarandache theory* (DSmT) that can be interpreted as a generalization of Dempster-Shafer theory. On one hand, neutrosophic logic appears as an interesting avenue for SA because (1) indeterminacy is explicitly represented by the means of an indeterminacy assignment, (2) falsity, truth and indeterminacy are represented independently (three distinct assignments), (3) it is a quantified logic, meaning that numerical evaluations of truth, falsity and indeterminacy values are allowed, (4) this quantification is allowed on hyperreals intervals, a generalization of intervals of real numbers given a broader frame for interpretations, (5) many novel connectives are defined (Neut-A, Anti-A, ...). On the other hand, being built on the hyper-power set of the universe of discourse, the DSmT allows to take into account the indeterminacy linked to the very definition of the individual elements of the universe of discourse, relaxing the mutual exclusivity hypothesis imposed by the Dempster-Shafer theory (DST). This framework extends thus the DST by allowing a wider variety of events to be considered when measures become available. Indeed, a particularity of SA is that most of the time it is impossible beforehand to list every possible situation that can occur. The elements of the corresponding universe of discourse cannot, thus, be considered as an exhaustive list of situations. Furthermore, in SA situations are not clearcut elements of the universe of discourse.

The aim of this chapter is to evaluate the potential of neutrosophic logic and Dezert-Smarandache theory (DSmT) to cope with the ontological and epistemic obstacles in SA (section 16.3), *i.e.* problems due to the nature of things and to cognitive limitations of the agents, human or artificial. Section 16.4 exposes four basic principles guiding SA systems design in practice, and highlight the capacity of both neutrosophic logic and DSmT to cope with these principles. After brief formal descriptions of neutrosophic logic and DSmT (section 16.5) we propose in section 16.6 different extensions based on Kripke structures and Demspter-Shafer structures. In particular,a Kripke structure for neutrosophic propositions is presented in section 16.6.2. In the latter section, we assess the ability of neutrosophic logic to process symbolic and numerical statements on belief and knowledge using the possible worlds semantics. Moreover, we investigate the representation of neutrosophic concepts of neutrality and opposite in the possible worlds semantics for situation modelization. In section 16.6.3, after introducing Nilsson and Dempster-Shafer structures, we present a possible extension to DSmT. We also propose an example to illustrate the benefit of using a richer universe of discourse, and thus how DSmT appears as an appropriate modelling tool for uncertainty in SA. We then propose a possible connection between DSmT and neutrosophic logic in the Kripke structures setting (section 16.6.4). Finally, in section 16.7 we conclude on possible research avenues for using DSmT and neutrosophic logic in SA.

16.2 Situation analysis

The term *situation* appears in the mid-fourteenth century derived from medieval Latin *situatio* meaning *being placed into a certain location.* By the middle of the seventeenth century situation is used to discuss the moral dispositions of a person, more specifically the set of circumstances a person lies in, the relations linking this person to its *milieu* or surrounding *environment.* As will be shown below, the latter definition is close to what is meant today in the field of High-Level Data Fusion, where the mental state of *situation awareness* is studied in interaction with the surrounding environment. Common synonyms of situation with a corresponding meaning are *setting*, *case*, *circumstances*, *condition*, *plight*, *scenario*, *state*, *picture*, *state of affairs*.

Although the notion of situation is used informally in everyday language to designate a given state of affairs, a simplified view of the world, and even the position of certain objects, situation is nowadays a central concept in High-Level Data Fusion where it has been given more or less formal definitions. For

Pew [12], a situation is "*a set of environmental conditions and system states with which the participant is interacting that can be characterized uniquely by a set of information, knowledge, and response options*".

16.2.1 Situation awareness as a mental state

For Endsley and Garland [1] *Situation awareness* (SAW) is "*the perception of the elements in the environment within a volume of time and space, the comprehension of their meaning and the projection of their status in the near future*". SAW is also defined in [13] as "the active mental *representation* of the status of current cognitive functions activated in the cognitive system in the context of achieving the goals of a specific task". In particular, SAW involves three key tasks: (1) Perception, (2) Comprehension and (3) Projection, in a general multiagent context (Fig. 16.1).

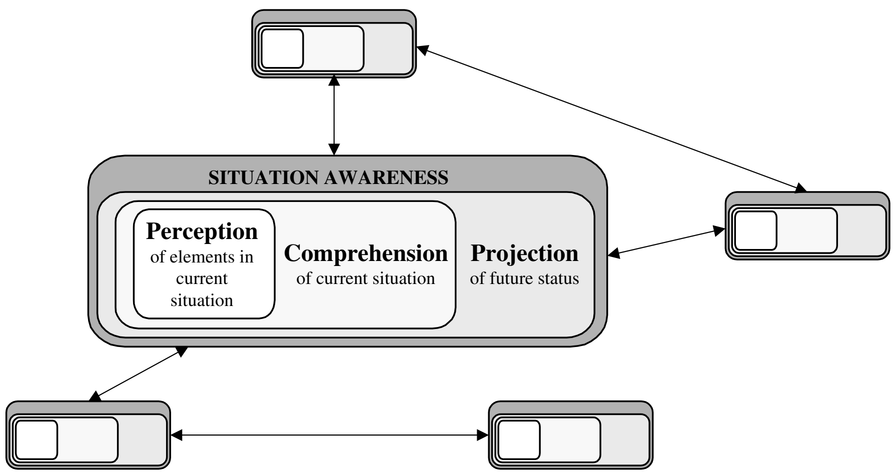


Figure 16.1: The three basic processes of situation awareness according to Endlsey and Garland (modified from [1]), in a multiagent context.

In contemporary cognitive science the concept of *mental representation* is used to study the interface between the external world and mind. Mental states are seen as relations between agents and mental representations. Formally, and following Pitt's formulation [14], for an agent to be in a psychological state Ψ with semantic property Γ is for that agent to be in a Ψ-appropriate relation to a mental representation of an appropriate kind with semantic property Γ. As far as mental states are concerned, purely syntactic approaches are not adequate for representation since semantic concepts need to be modeled.

Explicit reasoning on knowledge and the problems linked to its representation are distinctive features of situation analysis. Our position is to refer to the sources of knowledge usually considered in epistemology, namely, *Perception*, *Memory*, *Reasoning*, *Testimony* and *Consciousness* [15], and extend Endsley's model of situation awareness [1] where perception appears as the only source of knowledge.

16.2.2 Situation Analysis as a process

For Roy [2] "*Situation Analysis is a process, the examination of a situation, its elements, and their relations, to provide and maintain a product, i.e. a state of* Situation Awareness *(SAW) for the decision maker*". For a given situation the SA process creates and maintains a mental representation of the situation. Situation analysis corresponds to the levels 2, 3 and 4 of the JDL data fusion model [16, 17], hence to higher-levels of data fusion. A revisited version of the well-known model is presented on figure 16.2, with classical applications associated to the different levels. A complete situation model must take into account the following tasks of: A. Situation *perception* composed of Situation Element Acquisition, Common Referencing, Perception Origin Uncertainty Management, and Situation Element Perception Refinement as subtasks. B. Situation *comprehension* composed of Situation Element Contextual Analysis, Situation Element Interpretation, Situation Classification, Situation Recognition, and Situation Assessment as subtasks. C. Situation *projection* composed of Situation Element Projection, Impact Assessment, Situation Monitoring. Situation Watch, and Process Refinement [2].

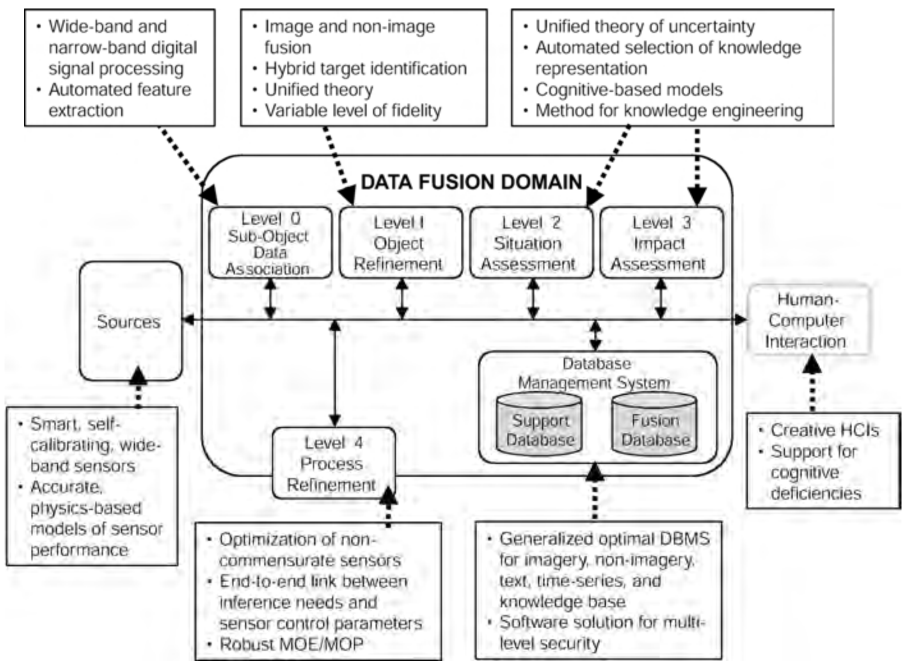


Figure 16.2: Revisited JDL data fusion model and applications [18].

The conception of a system for SA must rely on a mathematical and/or logical formalism capable of translating the mechanisms of the SAW process at the human level. The formalism should also allow the possibility to encompass the case of multiagent systems in which the state of awareness can be distributed over several agents rather than localized. A logical approach based on a possible worlds semantics for reasoning on belief and knowledge is proposed in [3]. This work by Halpern and Moses can be used as a blueprint considering that it allows to handle numerical evaluations of probabilities, thus treating separately but nevertheless linking belief, knowledge and uncertainty.

Furthermore, mathematical and logical frameworks used to model mental states should be able to represent and process autoreference such as beliefs about one's own beliefs, beliefs about beliefs about ... and so on.

16.2.3 A general model of a distributed system

In 1990, Halpern and Moses proposed a model of distributed knowledge processing [3] that can be used for the purpose of situation analysis, as stated above. Short definitions are given below for the different components of the model:

- A **distributed system** is a finite collection of two or more interacting agents $\mathcal{A}_1, \ldots, \mathcal{A}_n$ (connected by a communication network);
- The **local state** of an agent is the determined by the encapsulation of all the information an agent has access to at a given instant;
- The **state of the environment** is defined as the information relevant to the system but not contained in the state of the agents;
- The **global state** of a system is given by the sum of the agents' local states together with the state of the environment;
- A **run** is a function from time to global states;
- A **point** is a pair (r, m) consisting of a run r and a time m;
- A **system** is defined as a set of runs. A system can also be viewed as a Kripke structure supplemented with a way to assign truth values.

This model is illustrated on figure 16.3 and appears as a sufficient basis for defining the basic concepts of situation analysis. Indeed, the local state of an agent $\mathcal{A}_i$ can also be called its Knowledge-Base (denoted by KB_i) upon which an awareness function delimits these subsets, the latter being particular views of a given situation (see section 16.4.2 on contextualization). From an algebraic point of view, a same agent can generate different views of the same situation, either disjoint or overlapping or nested.

16.3 Sources of uncertainty in Situation Analysis

Situation analysis is experimental by nature. A major obstacle encountered in the process lies in the ubiquity of uncertainty. While in a previous paper [19], we highlighted four main facets of uncertainty:

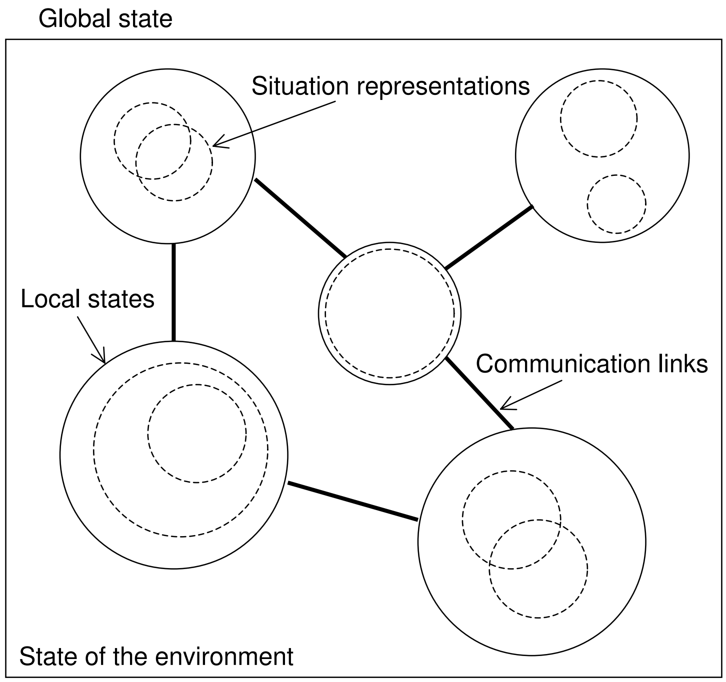


Figure 16.3: The general model of a distributed system proposed by Halpern and Moses in [3] adapted for situation representation.

(1) Meaning (mental sate or property of the information), (2) Interpretation (objective or subjective), (3) Types (fuzziness, non-specificity and discord) and (4) Mathematical representations (quantitative vs. qualitative approaches), in this section, we rather review the potential sources of uncertainty and obstacles arising in a situation analysis context.

Uncertainty has two main meanings in most of the classical dictionaries [19]: Uncertainty as a *state of mind* and uncertainty as a *physical property of information*. The first meaning refers to the state of mind of an agent, which does not possess the needed information or knowledge to make a decision; the agent is in a state of uncertainty: "I'm not sure that this object is a table". The second meaning refers to a physical property, representing the limitation of perception systems: "The length of this table is uncertain" (given the measurement device used).

Sociologists like Gérald Bronner [20] consider uncertainty as a state of mind, this state depending on our power on the uncertainty, and our capacity to avoid it. He distinguishes two types of uncertainty: *uncertainty in finality* (or material uncertainty) and *uncertainty of sense*. Uncertainty in finality is "the state of an individual that, wanting to fulfill a desire, is confronted with the open field of the possibles" ("Will my car start?"). Whereas uncertainty of sense is "the state of an individual when a part, or the whole of its systems of representation is deteriorated or can be". Uncertainty in finality corresponds to the uncertainty in which lies our understanding of the world, while uncertainty of sense bears on the representation of the world. Bronner identifies three types of uncertainty in finality, according to one's power on uncertainty, and the capacity to avoid it:

- Situation of type I: Uncertainty does not depend on the agent and can not be avoided;
- Situation of type II: Uncertainty does not depend on the agent but can be avoided;
- Situation of type III: Uncertainty is generated by the agent and can be avoided.

In situation analysis, agents are confronted to uncertainty of sense (data driven) from the bottom-up perspective and to uncertainty in finality (goal driven) from the top-down perspective. It follows that there are two kinds of limits to state estimation and prediction in Situation Analysis:

1. *Ontological limits* due to the nature of things and
2. *Epistemic limits* due to cognitive limitations of the agents, human or artificial.

Typical obstacles [21] are *anarchy* and *instability* when the situation is not governed by an identifiable law or in the absence of nomic stability. *Chance* and *chaos*, are serious obstacles to state evaluation and prediction as far as an exact estimation is sought for although regularities and determinism are observed. Another typical obstacle is the *vagueness* of concepts. Natural language concepts are inherently vague, meaning that their definition is approximate and borderline cases arise. This is true as well for properties but also for concepts.

Indeterminacy is another unavoidable obstacle. It may arise from paradoxical conclusions to a given inference (*i.e.* Russell's paradox, or sorites paradox), from impossible physical measurements (*i.e.* position *and* speed of an atomic particle) or for practical reasons (*i.e.* NP-complete problems). From a given theoretical stand point (classical vs. quantum mechanics), indeterminacy may nevertheless be proposed as a conclusion to specific unanswerable questions in order to nevertheless allow reasoning using the remaining information.

Ignorance of the underlying laws governing the situation is a major cause of uncertainty. For example not knowing that a given tactical maneuver is possible precludes the possibility to predict its occurrence. Especially present in human affairs *innovation* can be a major obstacle in SA. New kinds of objects (weapons), processes (courses of action) or ideas (doctrines) arise and one has no choice but to deal with it and adapt.

Myopia or data ignorance, is also a typical problem in SA. Data must be available on time in order to assess a situation, meaning that even if the information sources exist circumstances can prevent their delivery. Another case of myopia occurs when data is not available in sufficient detail, as in pattern recognition when classes are only coarsely defined or when sensors have limited spatial resolution. Data is thus accessible through estimations obtained by sampling as in surveys, by the computation of aggregates as in Data Fusion or by the modelization of rough estimates. As a consequence the available data is only imprecise and incomplete and leads most of the time to conflicting choices of decision. A major task of SA is change detection, failure prediction.

Any attempt in the conception of a system is be bounded by *inferential incapacity* of human or artificial agents. Limitations in agents can arise because of a lack of awareness. As far as knowledge is concerned, an agent cannot always give a value to a proposition, for example if it is not even aware of the existence of the concept denoted by the proposition at hand. Agents are *resource bounded* meaning that agents have only limited memorization capabilities, in some cases they have power supply limitations, etc. or have only limited cognitive and computational capabilities. Agents may also have limited visual or auditory acuity. Sometimes, these limitations come from the outside and are situation driven: electronic countermeasures, only a limited amount of time or money is available to do the job, etc. Furthermore agents cannot focus on all issues simultaneously. As Fagin and Halpern puts it in [22] "[...] Even if $\mathcal{A}$ does perfect reasoning with respect to the limited number of issues on which he is focusing in any given frame of mind, he may not put his conclusions together. Indeed, although in each frame of mind agent $\mathcal{A}$ may be consistent, the conclusions $\mathcal{A}$ draws in different frames of mind may be inconsistent." Finally, agents must work with an inconsistent set of beliefs. For example, we know that lying is amoral, but in some case we admit it could be a good alternative to a crisis.

16.4 Ontological principles in Situation Analysis

Given the limitations and the sources of uncertainty involved in Situation Analysis (section 16.3), we state in this section four main ontological principles that should guide SA systems design in practice: (1) allowing statements and reasoning about uncertainty to be made, (3) contextualization, (2) enrichment of the universe of discourse, and (4) allowing autoreference.

16.4.1 Allowing statements and reasoning about uncertainty

We begin with two observations that will guide the discussion of this section:

1. Many concepts are linked to uncertainty: Vagueness, indeterminacy, truth, belief, indiscernibility, ambiguity, non-specificity, incompleteness, imprecision to name a few. Although these concepts are *a priori* distinct, it is common to confuse them and to be unable to talk about one without any reference to the other. The recent development of new theories of uncertainty aims at separating these aspects, and bring clarifications in this direction as it is the case for probability theory and fuzzy logic. Another contribution in this direction is the axiomatization proposed by Fagin and Halpern in [4] which provides a semantical structure to reasoning about both belief and probability, and thus distinguishing these two often confused concepts.

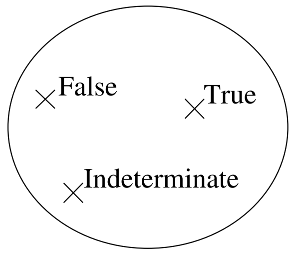
2. Although it is possible to deal with uncertainty in general using purely qualitative notions, the mixture of discrete and continuous objects composing the world has led to introduce degrees.

In a very general sense as written in the previous section (section 16.3), uncertainty is often seen as the result of indeterminacy. As far as formalization is concerned the classical means of reasoning soon exposed their limitations. Propositional Calculus (PC) relies on the principle of bivalence expressing the fact that a proposition is either True or False. Hence, only two truth values are allowed leaving no way to express indeterminacy. The most common way go beyond bivalence is to introduce supplementary truth values in the PC framework. The signification of the supplementary truth value differs from one author to another, from one logic to another. However, it is common to denote truth, falsity and indeterminacy by 1, 0 and $\frac{1}{2}$ respectively.

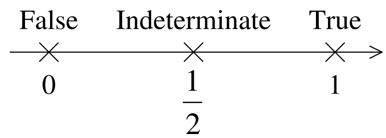
Here the problem of the meaning of the uncertainty arises. For a given type of uncertainty (contingent future events, indetermination, etc.) corresponds a particular interpretation of the set of connectives. If Łukasiewicz was primarily interested with the problem of contingent future event or possibility, Kleene in 1938 [23] proposed three value logics used in recursion theory in order to design stopping criteria and allow for indeterminacy of some propositions. Bochvar (1938) [24] proposed a logic quantifying propositions as sensible and senseless. For him true and false propositions are meaningful, the third truth-value designates meaningless or paradoxical propositions. Bochvar's system of logic, was later rediscovered by HalldÈn in 1949 [25] and used to process vague and nonsensical propositions. In fact, the different meanings of uncertainty are translated in the particular definitions given to logical connectors with respect to common intuition of the terms at hand.

It is important to note that in general the truth values are not ordered and just like in PC the truth values are purely conventional. In this sense, the so-called values of the truth tables can be considered qualitative (see Fig. 16.4-(a)). However, these three truth values can also be ordered, representing then a rough quantitative description of the world (see Fig. 16.4-(b)). But intuition also tells us that things are not always clear cut in the real world and rather appear in tones of gray. A three-valued logic can be generalized to a n-valued logic and by extension to fuzzy logic with an infinite number of truth-values ranging on the real set interval $[0; 1]$. Such an extension introduces thus an order between truth statements (see Fig. 16.4-(c)). Another consequence of this extension is that the notion of uncertainty is now expressed explicitly in terms of truth or falsity. While in a three-valued logic, indeterminacy, possibility or vagueness are expressed as neither True nor False, in Łukasiewicz's or fuzzy logic, to take a more recent example, the uncertainty is expressed by an explicit reference to truth or falsity.

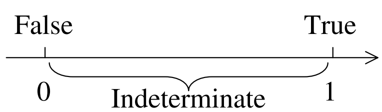
The introduction of degrees imposes then an order between values. The truth becomes then a kind of false and vice-versa, and the qualitative aspect of the three initial truth values is lost, with their independence. Yet another extension which conciliates both qualitative and quantitative aspects of indeterminacy is to consider different independent aspects of uncertainty and represent them on independent axes. This



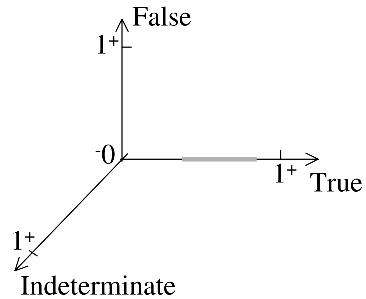
(a) Three non-ordered truth values corresponding to purely qualitative description.



(b) Three ordered truth values corresponding to a rough qualitative description.



(c) Infinite number of ordered truth values ranging from FALSE to TRUE.



(d) Three independent axes for three ordered features of the uncertainty (truth, falsity and indeterminacy).

Figure 16.4: Supplementary truth values for representing indeterminacy.

is the principle developed by Smarandache in the neutrosophic logic [10, 11], where the considered aspects of uncertainty are truth, falsity and indeterminacy (see Fig. 16.4-(d)). Hence, in neutrosophic logic both the qualitative aspect of non-ordered three-valued logics and the quantitative aspect of fuzzy logic are combined. One main benefit of neutrosophic logic is that indeterminacy can be addressed by two different manners: (1) Using the indeterminacy function independently[1] of the truth and falsity functions or (2) using the three previous functions as it is commonly done in fuzzy logic. Moreover, because of the assumed independence of the three concepts of truth, falsity and indeterminacy, NL is able to represent paradoxes, for example something that is completely true, completely false and completely indeterminate. Neutrosophy and neutrosophic logics are introduced respectively in sections 16.5.1 and 16.5.2.

[1]Note however that although truth, falsity and indeterminacy are considered independently in NL, the use of the hyperreals is a means to make them dependent. Indeed, an absolutely TRUE proposition ($T(\phi) = 1^+$) is also absolutely FALSE ($F(\phi) =^- 0$). This condition is not required for relatively TRUE propositions ($T(\phi) = 1$) [10].

Finally, we remind that although indeterminacy has been discussed from a logical point of view, indeterminacy is also represented in more quantitative approaches. Indeed, in probability theory, assigning a probability value in $]0; 1[$ to an event translates the indeterminate state of this event. It has nothing to do with the truth of the event, but rather with its potential occurrence. By extension, Dempster-Shafer theory, possibility theory or Dezert-Smarandache theory are other numerical approaches to deal with indeterminacy. Some of these approaches are briefly discussed in section 16.5.3.

16.4.2 Contextualization

In SA, the operation of contextualization serves many purposes and is at the basis of the abstract notion of *situation* itself as it is understood by defence scientists, software engineers and commanding officers as well. According to Theodorakis [26], in the context of information modelling, "*a context is viewed as a reference environment relatively to which descriptions of real world objects are given. The notion of context may be used to represent real world partitions, divisions, or in general, groups of information, such as situations, viewpoints, workspaces, or versions*". In this sense a context is a mental, thus partial, representation of a real situation. For Theodorakis [26] "*A situation records the state of the world as it is, independently of how it is represented in the mind of an agent. A situation is complete as it records all the state of the world. Whereas, contexts are partial as they represent situations and hence capture different perspectives or record different levels of detail of a particular situation*".

For Brézillon [27] a context can be "*a set of preferences and/or beliefs, a window on a screen, an infinite and only partially known collection of assumptions, a list of attributes, the product of an interpretation, a collection of context schemata, paths in information retrieval, slots in object-oriented languages, buttons which are functional, customizable and shareable, possible worlds, assumptions under which a statement is true or false, a special, buffer-like data structure, an interpreter which controls the system's activity, the characteristics of the situation and the goals of the knowledge use, entities (things or events) related in a certain way, the possibility that permits to listen what is said and what is not said*".

Contextualization is an operation largely applied in artificial intelligence, natural language processing, databases and ontologies, communication, electronic documentation and machine vision. The principal benefits from contextualization are the *modularity of representation*, *context dependent semantics*, and *focused information access* [27]. As far as SA is concerned, a context or if one prefers, a representation of a situation, is a means to encapsulate information while eliminating the unnecessary details, makes it possible to refer to a given representation of the world while allowing different interpretations on the meaning of this precise representation and finally gives a access to a mechanism to focus on details when required.

Using the notation defined earlier (section 16.2.3), a context or a situation s is a view on the global state of an agent $\mathcal{A}$ built on a given database KB. This view can be shared by multiple agents through communication links. As will be shown below, contexts are means to make reasoning local allowing for example an agent to hold incoherent beliefs or to deal with incomplete information and knowledge.

Contextualizations are usually based on criteria such as

- *time*: limits due to real time applications requirements or planning objectives,
- *space*: limits due to range of sensors or territorial frontiers,
- *function*: discrimination according to objects functions or agents social roles,
- *structure*: distinction between cooperative or egoistic behavior.

Agents performing situation analysis are embedded in complex and dynamically changing environments. Many problems arise (1) from the unpredictability and instability of such environments, (2) from the particularities of the SA tasks to accomplish and finally (3) from the agents own limitations, both physical and mental.

1. The unpredictability and instability of the environment will force the agent to concentrate on the most certain information available and leave unmeasured events that are not yet accessible.

   In this case, the result of contextualization is for example the constitution of the σ-algebra used in probability theory (see section 16.5.3). Similarly, the generic operation consisting in the specification of upper and lower bounds over sets of events is also a form of contextualization. This operation is present in different theories such as Demspter-Shafer theory (belief and plausibility measures or lower and upper probabilities) and rough set theory (lower and upper approximations).

2. Depending on the complexity of the environment, the different tasks involved in SA will not require the same level of attention, the same depth of reasoning and nor be subject to the same reaction delays. Consequently the agents will only consider limited time and space frames in order to efficiently answer operational requirements. These limits are imposed voluntarily by designers of SA systems, implemented by experienced game players and but also innate to many biological systems.

   Two models have been proposed for the partition of sets of possibles worlds (see section 16.6.1), the Rantala and sieve models. Rantala models [28] are a modification of the standard Kripke model semantics that incorporate the notion of *impossible worlds*, allowing to distinguish them from possible worlds. In these impossible worlds anything can hold even contradictions. The notion captures the fact that a non-ideal agent may believe in things that are not consistent, false, etc. but are nonetheless considered as epistemic alternatives. Sieve models have been proposed by Fagin and Halpern in 1988 [22] in order to prevent the problem of omniscience by introducing a function that act as a sieve. Instead of introducing nonstandard world or situations, sieve models introduce *segregation between formulas* that can be known or believed and other that cannot. The sieve function indicates in fact if the agent is **aware** of a given formula in a given situation. Being aware amounts at knowing or believing the formula in question.

3. It is a common practice in SA to consider resource bounded agents, even implicitly. In economics the notion of unbounded rationality refers to the consideration of all possible alternatives and choosing the best one often using optimization techniques. The opposite view of rational choice theory, bounded rationality, rather considers that there are finite limits to information and calculations a human brain or a mechanical memory device can hold *i.e.* Bremermann's computational limit. This view also holds that deliberation costs should be included in models, limiting furthermore rationality for the sake of economy.

According to many authors [29, 30, 31], in neutrosophy the attribution of truth values can be bound to specific circumstances making it thus a contextual theory of truth [32]. Unary neutrosophic connectives such as A', Anti-A, Neut-A (see section 16.5.1), seem particularly interesting for the manipulation of contextual concepts.

16.4.3 Enrichment of the universe of discourse

The *universe of discourse* is the set of objects (concrete or abstract) considered in a given context. It could be a set of classes, a set of targets, a set of actions to take, etc, but also a set of possible worlds (*i.e.* of possible states of the world). Let S represent the universe of discourse, the set of all possible outcomes of an experiment:

$$S = \{s_1, s_2, \ldots, s_n\} \tag{16.1}$$

The universe of discourse is in a sense, the result of a contextualization operation (section 16.4.2) since all objects existing in the world are not present in this set; a choice has been made (voluntarily or not). It is then the support for problem-solving situation and represents the objects about which we are able to talk.

However, it represents an ideal model assuming a perfect description. Unfortunately, real world is often different and more complex than expected. Indeed, on one hand the agents have a limited access to knowledge and on the other hand, objects in the real world itself are not clear cut and a perfect description is in general impossible. These features of reality cannot in general be taken into account in the modelization of the problem (*i.e.* in the definition of the universe of discourse). Hence, a solution to deal with the two different kinds of limitations we face to in SA, epistemic limitation (due to cognitive limitations of the agents, human or artificial) and ontological limitation (due to the nature of things), (section 16.3), is to artificially enrich the universe of discourse.

1. The failure of the sources of knowledge of an agent leads mainly to *indiscernibility* (see section 16.3). Indeed, an epistemic limitation implies the necessity of considering other objects than those originally present in S. In particular, the incapacity of a agent to distinguish between two objects s_1 and s_2 at a given time, in a given context is represented by $s_1 \cup s_2$ which is another object, built from S but not explicitly in S. $s_1 \cup s_2$ is then the best answer the agent can give at a given time, even if it knows that the answer is either s_1 **or** s_2.

   In probability theory, because of the axiom of additivity, we cannot refer to $s_1 \cup s_2$ independently of the rest of the universe. Indeed, $\mu(s_1 \cup s_2) = \mu(s_1) + \mu(s_2) - \mu(s_1 \cap s_2)$ if μ is a probability measure over S. Hence, to account for this limitation of the access to knowledge (epistemic limitation), we can enrich the universe of discourse and consider the power set of S, *i.e.* the set of all subsets of S:

$$2^S = \{A | A \subseteq S\} = \{\varnothing, s_1, s_2, \ldots, s_n, (s_1, s_2), \ldots, (s_{n-1}, s_n), \ldots, S\}^2 \tag{16.2}$$

   where $\varnothing$ denotes the empty set. This enrichment of the universe of discourse allows ignorance and uncertainty to be best represented, as well as a supplementary types of conflict to be taken into account. If probability theory is based on the classical set notion, the notion of power set is the basis for Dempster-Shafer theory (see section 16.5.3 for a brief description), possibility theory and rough sets theory. In this context, we can assign measures to every subset of S, independently of the others. Note finally that Dempster-Shafer theory is based on the assumption of a universe of discourse composed by an exhaustive list of mutually exclusive elements [33], a very restrictive constraint in practice.

2. Another limitation is due to the fact that the observable world is more complex than we can describe. This ontological limitation is linked to the properties of the objects and has nothing to do with our perception means. For example, $s_1 \cap s_2$ represents another object composed by both s_1 **and** s_2. It is neither s_1 nor s_2 but something between them. Hence, yet another extension is the construction of the hyper-power set constituted of all the combinations of the union and intersection operators applied to the elements of S:

$$D^S = \{\varnothing, s_1, \ldots, s_n, (s_1 \cup s_2), \ldots, S, (s_1 \cap s_2), \ldots, (s_1 \cap s_2) \cup s_3, \ldots\} \tag{16.3}$$

   If the elements of S are mutually exclusive ($s_i \cap s_j = \varnothing$, for all $i \neq j$), then $D^S = 2^S$. However, considering D^S is a more general case allowing $s_i \cap s_j \neq \varnothing$, *i.e.* allowing objects of the universe of discourse to overlap. An example, is an universe constituted of vague concepts. Extending the definition of the probability measure the hyper-power set is the principle of Dezert-Smarandache theory [33]. In this framework, no initial assumption on the mutually exclusivity on S is imposed,

[2] Here, (s_1, s_2) is used to denote $(s_1 \cup s_2)$.

and the exhaustivity is somewhat delayed since new objects can be constructed on those of S. A brief description of DSmT is proposed in section 16.5.3.

Therefore, we can say that while Dempster-Shafer theory of evidence is an *epistemic theory* since it only represents epistemic limitations, Dezert-Smarandache is basically an *epistemic and ontological theory* since this framework combines both epistemic and ontological view points.

16.4.4 Autoreference

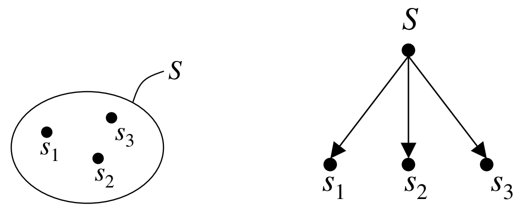
By *autoreference* we mean the capacity of an agent for introspection or selfreference. For example, an agent should be granted the capacity of holding beliefs belief about its own declarations, and not only about the declarations of the other agents.

1. A classical mean for modelling autoreference is by the way of hypersets. The notion of hyperset has been first introduced by Aczel [34] and Barwise and Etchemendy [35] to overcome Russell's paradox[3]. A recursive definition extends the notion of classical set, allowing hypersets to contain themselves, leading to infinitely deep sets (for example, $x = 1 + 1/x$). A *well-founded set* is a set without infinite descending membership sequence, whereas the others are called *non-well-founded sets*.

2. In modal logics, Kripke structures are used as a semantics (see section 16.6.1). In a Kripke structure, an accessibility relation is defined over a set of possible worlds which models either the structure of the world or the agent properties. The desired properties of an agent are then modeled by imposing some properties to the accessibility relation. In particular, if the relation is reflexive and transitive, then the agent possesses the capacity of *positive introspection* (the agent knows that it knows). Also if the relation is an equivalence relation, the agent is capable of formulating declarations about its ignorance (*negative intropection*).

Although these two models, hypersets and Kripke models, are presented here as distinct ones, both are semantics of (multi-agent) modal logics. In [37, 38], it has been proven the equivalence of both semantics. Indeed, with the notion of hyperset comes the graph metaphor which replaces the "container" metaphor used in classical set theory (see figure 16.5). By definition, a *graph* G is a pair (S, R), where S is a set of nodes and R is a relation over S. A *labeled graph* is a triple $\mathcal{S} = \langle S, \mathsf{R}, \pi \rangle = \langle G, \pi \rangle$ where G is a graph and π is a valuation function from P to 2^S, with P being a set of propositional variables, that assigns to each p of P a subset of S. However, a Kripke model can be viewed as a *directed labeled graph*, whose

[3] "*Russell's paradox is the most famous of the logical or set-theoretical paradoxes. The paradox arises within naive set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself, hence the paradox*" [36].



(a) Using "container" metaphor.

(b) Using graph metaphor.

Figure 16.5: Representation of classical sets. Arrows in figure (b) mean that s_i is a member of S.

nodes are the possible worlds, the link between nodes representing the accessibility relation, labeled by truth assignments[4].

First introduced for modal logics and knowledge logics, the model proposed by Kripke appears as an elegant structure for reasoning about knowledge in a multi-agent context. Moreover, it is based on the notion of *possible world*, which is close to the intuitive notion of situation. Hence, we choose it as the basic structure for situation analysis. In section 16.6, we develop our argumentation to connect Kripke structures with neutrosophic frameworks. After a more formal description of Kripke structures (section 16.6.1), we first extend this structure to neutrosophic logic (section 16.6.2). Then, considering mainly the notion of possible worlds, we extend probability structures to DSm structures (section 16.6.3). And finally, we make the connection between DSmT and neutrosophic logic through Kripke structures (section 16.6.4).

16.5 Neutrosophic frameworks for Situation Analysis

16.5.1 Neutrosophy

Neutrosophy is presented by F. Smarandache as "*a new branch of philosophy, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra*" [9]. It is formalized as follows:

> Let A be an idea, a proposition, a theory, an event, a concept, an entity. Then, using different unary operators, we define
>
> - A', a version of A;
> - Anti-A, the opposite of A;

[4] Although the demonstration proposed in [37, 38] is more complex (!) it lies on the previous remark.

- Non-A, what is not A;
- Neut-A, what is neither A nor Anti-A.

Neut-A represents a neutrality in between the two extremes, A and Anti-A. Hence, between A and Anti-A there is a continuum-power spectrum of neutralities Neut-A, A – Neut-A – Anti-A. Note that Non-A is different from Anti-A (Non-$A \neq$ Anti-A), but also that Anti-$A \subset$ Non-A, Neut-$A \subset$ Non-A, $A \cap \text{Anti-}A = \varnothing$, $A \cap \text{Non-}A = \varnothing$.

We give below an example for multi-agent situation analysis:

> Let's assume a system composed of n agents $\mathcal{A}_1, \ldots, \mathcal{A}_n$. Let call KB_i the Knowledge-Base of agent i, $i = 1, \ldots, n$. Then,
>
> - KB_1 is all the information agent $\mathcal{A}_1$ has access to;
> - KB'_1 is another version of KB_1: for example, an update of KB_1, or KB_1 issued from a partition of the sources of information of $\mathcal{A}_1$, hence another view of KB;
> - Anti-KB_1 is all the information agent $\mathcal{A}_1$ has not access to (or the information it did not use for a given representation of the situation);
> - Non-KB_1 is all the information agents $\mathcal{A}_2, \ldots, \mathcal{A}_n$ have access to, but not shared with $\mathcal{A}_1$ plus the information nobody has access to;
> - Neut-KB_1 is all the information agents $\mathcal{A}_2, \ldots, \mathcal{A}_n$ have access to, but not shared with $\mathcal{A}_1$.

The only formal approaches derived from neutrosophy that will be studied in this chapter are: The neutrosophic logic introduced by Smarandache [10, 11] and the Dezert-Smarandache theory proposed by Dezert and Smarandache [33, 39]. In sections 16.5.2 and 16.5.3 we review the basics of these approaches.

16.5.2 Neutrosophic logic

Neutrosophic logic (NL) is a method for neutrosophic reasoning. This non-classical logic is a multiple-valued logic which generalizes, among others, the fuzzy logic. It is the "*(first) attempt to unify many logics in a single field*" [10].

While in classical logic, a concept (proposition) A is either True or False, while in fuzzy logic A is allowed to be more or less True (and consequently more or less False) using truth degrees, in neutrosophic logic, a concept A is T% True, I% Indeterminate and F% False, where $(T, I, F) \subset \|^-0, 1^+\|^3$. The interval $\|^-0, 1^+\|$ is an *hyperreal interval*[5], the heigh part of this notation refering to a

[5]Hyperreals - Non-standard reals (hyperreals) have been introduced in 1960. Let $[0, 1]$ be the real standard interval *i.e.* , the set of real numbers between 0 and 1. An extension of this interval is to replace the lower and lower bounds by the non-standard counterparts $^-0$ and 1^+, being respectively $0 - \epsilon$ and $1 + \epsilon$, where $\epsilon > 0$ is an infinitesimal number (*i.e.* such that for all integer $n > 0$, $\epsilon < \frac{1}{n}$).

three-dimensional space. As a general framework, neutrosophic logic corresponds to an extension in three distinct directions:

1. With A, are considered Non-A, Anti-A, Neut-A, and A';
2. The semantics is based on **three independent** assignments, not a single one as it is commonly used in the other logics;
3. These three assignments take their values as **subsets** of the **hyperreal** interval $\|^-0, 1^+\|$, instead in $[0, 1]$.

A is thus characterized by a triplet of truth-values, called the *neutrosophical value*:

$$\mathrm{NL}(A) = (T(A), I(A), F(A)) \tag{16.4}$$

where $(T(A), I(A), F(A)) \subset \|^-0, 1^+\|^3$.

16.5.3 Dezert-Smarandache theory (DSmT)

Because the theory proposed by Dezert and Smarandache is presented as a generalization of Dempster-Shafer theory, the latter being itself interpreted as a generalization of probability theory, we briefly review the basics of these two theories before introducing DSmT.

A *probability space* is a 3-tuple $\mathcal{P} = \langle S, \chi, \mu \rangle$ where:

- $S = \{s_1, s_2, \ldots, s_n\}$ is the sample space, the set of the elementary events, the set of all outcomes for a given experiment;
- χ is a σ-algebra of S;
- μ is a probability assignment from χ to $[0, 1]$.

To each element of χ is assigned a non-negative real number $\mu(A)$, a *probability measure of* A (or simply probability of A) that must satisfy the following axioms: (1) $\mu(A) \geq 0$; (2) $\mu(S) = 1$; (3) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j = \varnothing$ for $A_i \neq A_j$.

Axiom 3 is also known as the condition of σ-additivity, or simply *axiom of additivity* and plays a crucial role in the theory of probability. Indeed, it imposes a restriction on the measurable sets (*i.e.* the set to which we are able to assign probability measures), since one direct consequence is $\mu(A) = 1 - \mu(A)$, where $A = S \backslash A$. In other words, $\mu(A)$ does not depend on any $\mu(B)$ such that $B \subset A$.

The theory of evidence has been originally developed by Dempster in 1967 in his work on upper and lower probabilities [7], and later on by Shafer in its famous book *A Mathematical Theory of Evidence* [8], published in 1976. Often interpreted as an extension of the Bayesian theory of probabilities, the theory of evidence offers the main advantage of better representing uncertainty because the measures are defined on the power set of the universe of discourse, instead of the universe itself as the probability theory does. This particularity leads to the relaxation of the additivity axiom of the probability theory by a less restrictive one, a super-additivity axiom.

A *belief function* is defined from 2^S to $[0,1]$, satisfying the following axioms: (1) $\mathrm{Bel}(\varnothing) = 0$; (2) $\mathrm{Bel}(S) = 1$; (3) For every positive integer n, and for every collection $A_1, \ldots, A_n$ of subsets of S, $\mathrm{Bel}(A_1 \cup \ldots \cup A_n) \geq \sum_i \mathrm{Bel}(A_i) - \sum_{i<j} \mathrm{Bel}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathrm{Bel}(A_1 \cap \ldots \cap A_n)$. Contrary to the probability measure, the belief measure is non-additive and the axiom of additivity for probability theory is replaced by an axiom of *superadditivity*. The main consequence of this axiom is that **every** element of the power set of S is measurable. Hence, we can have $\mathrm{Bel}(A) > \mathrm{Bel}(B)$ if $B \subset A$.

A belief function is often defined using a *basic probability assignment* (or basic belief assignment) m from 2^S to $[0,1]$ that must satisfy the following conditions: (1) $m(\varnothing) = 0$ and (2) $\sum_{A \in 2^S} m(A) = 1$. Then we have $\mathrm{Bel}(A) = \sum_{B \subseteq A, B \in 2^S} m(B)$.

Dezert-Samrandache theory (DSmT) [33, 39] is another extension in this direction since all the elements of the hyper-power set are measurable. Then a *general basic belief mass* is defined from D^S to $[0,1]$, satisfying the following conditions:

$$m(\varnothing) = 0 \text{ and } \sum_{A \in D^S} m(A) = 1 \tag{16.5}$$

Hence, for example elements of the type of $s_i \cap s_j$, $i \neq j$ are allowed to be measured. The *general belief function* is then defined by:

$$\mathrm{Bel}'(A) = \sum_{B \subseteq A, B \in D^S} m(B) \tag{16.6}$$

We note Bel' to distinguish between the belief function in the Shafer sense, Bel.

DSmT is thus a more general framework that deals with both ontological and epistemic uncertainty. However, as most of quantitative approaches it lacks a formal structure for reasoning. In the following section, we propose a way to add such semantics to DSmT.

16.6 Possible worlds semantics for neutrosophic frameworks

The possible world semantics provides an intuitive means for reasoning about situations. It delivers a general approach to providing semantics to logical approaches with applicability to neutrosophic logic (section 16.6.2). Moreover, possible worlds semantics is often borrowed from logical approaches to fill the lack of semantics of numerical approaches, as it will be detailed below (section 16.6.3).

16.6.1 Kripke model

A Kripke model [40] is a mathematical structure that can be viewed as a *directed labeled graph*. The graph's nodes are the possible worlds s belonging to a set S of possible worlds, labeled by truth assignments π. More formally,

> A *Kripke model* is a triple structure $\mathcal{S}_K$ of the form $\langle S, \mathsf{R}, \pi \rangle$ where
>
> - S is a non-empty set (the set of possible worlds);
> - $\mathsf{R} \subseteq S \times S$ is the accessibility relation;
> - $\pi : (S \longrightarrow P) \longrightarrow \{0; 1\}$ is a truth assignment to the propositions per possible world.
>
> where $P = \{p_1, \ldots, p_n\}$ is a set of propositional variables, and $\{0; 1\}$ stands for $\{$True; False$\}$.

A world s is considered *possible* with respect to another world s' whenever there is an edge linking s and s'. This link is defined by an arbitrary binary relation, technically called the *accessibility relation* Figure Fig. 16.6 illustrates the following example:

> An agent is wondering if "it is raining in New York" (ϕ) and if "it is raining in Los Angeles" (ψ). Since this agent has no information at all about the situation, it will consider possible situations (worlds)$S = \{s_1, s_2, s_3, s_4\}$:
>
> - A situation s_1 in which it is both raining in New York and in Los Angeles, *i.e.* $\pi(s_1)(\phi) =$ True and $\pi(s_1)(\psi) =$ True.
> - A situation s_2 in which it is raining in New York but not in Los Angeles, *i.e.* $\pi(s_2)(\phi) =$ True and $\pi(s_2)(\psi) =$ False.
> - A situation s_3 in which it is not raining in New York and raining in Los Angeles, *i.e.* $\pi(s_3)(\phi) =$ False and $\pi(s_3)(\psi) =$ True.
> - A situation s_4 in which it is neither raining in New York nor in Los Angeles, *i.e.* $\pi(s_4)(\phi) =$ False and $\pi(s_4)(\psi) =$ False.

16.6.1.1 modelling the structure of the world

A very interesting feature of Kripke model semantics, is that it is possible to generate axioms for the different systems of modal logic by expressing conditions on the accessibility function defined on $\mathcal{S}_K$. These conditions can be used to express properties or limitations of agents (according to a given model of the world). For example, any epistemic system built upon a Kripke model satisfying a *reflexive* accessibility relation satisfies also the *true knowledge axiom* (T). If the model satisfies a *reflexive* and *transitive* accessibility relation, it satisfies also the axiom of *positive introspection* (4). Satisfaction of the

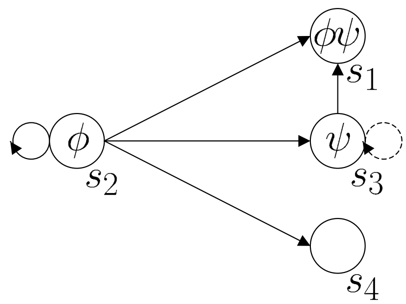


Figure 16.6: Example of a set of possible worlds and their accessibility relations.

axiom of *negative introspection* (5) is given by an *equivalence* relation (see table 16.1). System **K45** is obtained by making *transitive* and *Euclidian* the accessibility function, whereas **KD45** which is sometimes used to model evidential reasoning on Dempster-Shafer structures (see section 16.6.3.2) is obtained by making R *transitive*, *Euclidian* and *serial*. This is summarized in , and explained below.

Table 16.1: Axioms, epistemic logic systems and accessibility relations between possible worlds.

Accessibility relation (R)	Axioms
Reflexive	(T) $K\phi \rightarrow \phi$ (True knowledge)
Reflexive + Transitive	(4) $K\phi \rightarrow KK\phi$ (Positive introspection)
Equivalence	(5) $K\phi \rightarrow K\neg K\phi$ (Negative intropection)

16.6.1.2 Truth assignment

As previously said, to each world $s \in S$, there is an associated truth assignment $\pi(s)$ defined from P to $\{0; 1\}$ such that:

$$\pi(s)(p) = \begin{cases} 1 \text{ if } s \vDash p \\ 0 \text{ if } s \nvDash p \end{cases} \tag{16.7}$$

where $p \in P$. $s \vDash p$ means that the world s entails the proposition p, or in other words, that p is TRUE in s.

The assignments $\pi(s)$ are expected to obey to the classical definitions of the connectives so that for example $\pi(s)(p) = S \backslash \pi(s)(p)$, $\pi(s)(p \wedge q) = \pi(s)(p) \cap \pi(s)(q)$, etc.

A *formula* is any composition of some elements of P with the basic connectives $\neg$ and $\wedge$. Let call Φ the set of formulae and ϕ an element of Φ. For example, $\phi_1 = p_1 \wedge \neg p_2$, $\phi_2 = \neg p_1$, $\phi_3 = p_1 \wedge \ldots p_n$, $\phi_i \in \Phi$, $i = 1, \ldots, n$. Hence, the truth assignments $\pi(s)$ are also defined for any formula of Φ, $\pi(s)(\phi)$ being equal to 1 if ϕ is TRUE in s.

To each p of P, there is an associated *truth set* A_p of all the elements of S for which $\pi(s)(p)$ is TRUE:

$$A_p = \{s \in S | \pi(s)(p) = 1\} \tag{16.8}$$

A_p is then the set of possible worlds in which p is TRUE, and can also be noted $A_p = \{s \in S | s \vDash p\}$. By extension, to each formula ϕ is associated a truth set, A_ϕ.

Note that the elements of P are not necessarily mutually exclusive. A way to obtain mutually exclusive elements is to build the set $\mathcal{A}_t$, the set of basic elements, where a basic element[6] is a formula of the (conjunctive) form $\delta = p'_1 \wedge \ldots \wedge p'_n$ with p'_i being either p_i or $\neg p_i$, $p_i \in P$. Any formula $\phi \in \Phi$ can then be written in a disjunctive form as $\phi = \delta_1 \vee \ldots \vee \delta_k$, with $\delta_i \in \mathcal{A}_t$.

To each world s, there is an associated basic element δ of $\mathcal{A}_t$ describing thus the truth values of the propositions of P in S. Whereas many worlds can be associated to the same basic element, a basic element can be associated with any world (see example of section 16.6.3.4). The basic elements are just an alternate way to specify the truth assignment π.

16.6.1.3 Multi-agent context

The definition of $\mathcal{S}_K$ can easily be extended to the multi-agent case. Indeed, if we consider a set of agents $\mathcal{A}_1, \ldots, \mathcal{A}_n$, then on the same set of possible worlds S, and with the same truth assignment π, we can define n accessibility relations R_i, $i = 1, \ldots, n$, one per agent.

The different conditions on the R_is will characterize then the different properties of the $\mathcal{A}_i$s, facing to the same situation.

16.6.2 Kripke structure for neutrosophic propositions

We introduced in section 16.5.2, the basics of neutrosophic logic.

While is classical logic, a formula ϕ is simply characterized by its truth value $\pi(\phi)$ being either 0 or 1 (TRUE or FALSE), in neutrosophic logic ϕ is allowed to be $T\%$ TRUE and $F\%$ FALSE, and $I\%$ INDETERMINATE. ϕ is thus characterized by a triplet of truth-values, called the *neutrosophical value*:

$$\mathrm{NL}(\phi) = (T(\phi), I(\phi), F(\phi)) \tag{16.9}$$

where $(T(\phi), I(\phi), F(\phi)) \subset \|^-0, 1^+\|^3$, $\|^-0, 1^+\|$ being an interval of hyperreals.

In an equivalent manner as it is done in quantum logic, where Kripke structures are extended to deal with fuzzy propositions [41], we propose here to extend the Kripke structure to deal with neutrosophic assignments. Hence, we have,

> A *Kripke model for neutrosophic propositions* is a triple structure $\mathcal{S}_K^{NL}$ of the form $\langle S, \mathsf{R}, \vec{\pi} \rangle$ where

[6] A basic element is sometimes called an atom.

- S is a non-empty set (the set of possible worlds);
- $\mathsf{R} \subseteq S \times S$ is the accessibility relation;
- $\vec{\pi} = (\pi_T, \pi_I, \pi_F)$ is a neutrosophic assignment to the propositions per possible world, *i.e.* $\pi : (S \longrightarrow P) \longrightarrow \|^-0, 1^+\|$ with π being either π_T or π_I or π_F.

where $P = \{p_1, \ldots, p_n\}$ is a set of propositional variables.

The "truth" assignment π of a classical Kripke model becomes then $\vec{\pi} = (\pi_T, \pi_F, \pi_I)$, a three-dimensional assignment, where π_T is the truth assignment, π_F is the falsity assignment and π_I is the indeterminacy assignment. Hence, in each possible world s of S, a proposition ϕ can be evaluated as $\pi_T(s)(\phi)$ TRUE, $\pi_F(s)(\phi)$ FALSE and $\pi_I(s)(\phi)$ INDETERMINATE. It follows that to ϕ is associated a truth-set A^T_ϕ, a falsity-set A^F_ϕ and an indeterminacy-set A^I_ϕ:

$$A^T_\phi = \{s \in S | \pi_T(s)(\phi) \neq 0\}$$
$$A^F_\phi = \{s \in S | \pi_F(s)(\phi) \neq 0\}$$
$$A^I_\phi = \{s \in S | \pi_I(s)(\phi) \neq 0\}$$

Note that A^T_ϕ, A^T_ϕ and A^T_ϕ are (1) no longer related, (2) fuzzy sets and may overlap.

16.6.2.1 Knowledge and belief

Halpern in [42] gives the following definitions for knowledge and belief in PWS:

- ϕ is **known** if it is TRUE in **all** the possible worlds s of S
- ϕ is **believed** if it is TRUE in **at least one** possible world s of S

On the other hand, Smarandache [10] uses the notion of world and states that $T(\phi) = 1^+$ if ϕ is TRUE in **all** the possible worlds s of S (absolute truth) and $T(\phi) = 1$ if ϕ is TRUE in **at least one** possible world s of S (relative truth) (see Tab. 16.2). Hence, in the neutrosophical framework, we can state the following definitions for knowledge and belief: ϕ is **known** if $T(\phi) = 1^+ \equiv F(\phi) =^- 0$ and ϕ is **believed** if $T(\phi) = 1 \equiv F(\phi) = 0$. Table 16.2 shows several special cases.

Furthermore, one can consider the unary operators of neutrosophic logic (Non-ϕ, Anti-ϕ, Neut-ϕ, ϕ') to model new epistemic concepts but also as a means to represent situational objects, such as neutral situation, environment (to be detailed in the final version).

16.6.3 Probability assignments and structures

Let S be the frame of discernment, s a singleton of S and A any subset of S. In probability theory, measurable objects are singletons s of S. The measures assigned to any subsets A of S are guided by the additivity axiom. Hence, measurable elements belong to a σ-algebra χ of 2^S. In Dempster-Shafer theory,

Table 16.2: Neutrosophical values for special cases (adapted from [10]).

ϕ is ...	in ... poss. world(s)	Neutrosophical value
true		$T(\phi) = 1^+ \equiv F(\phi) =^- 0$
false	**all**	$F(\phi) = 1^+ \equiv T(\phi) =^- 0$
indet.		$I(\phi) = 1^+$
true		$T(\phi) = 1 \equiv F(\phi) = 0$
false	**at least one**	$F(\phi) = 1 \equiv T(\phi) = 1$
indet.		$I(\phi) = 1$
indet.	**no**	$I(\phi) =^- 0$
not indet.	**at least one**	$I(\phi) = 0$

any element of the power set of S, 2^S is measurable. Finally, Dezert-Smarandache theory allows any element of the hyper-power set of S, D^S, to be measured. Apart these extensions to probability theory that rely on the definition set of the probability measure, there exists a clear interest for giving a better semantics to these numerical approaches. For its probabilistic logic, Nilsson uses the possible worlds semantics to build a “semantical generalization of logic”, combining logic with probability theory [6] (see section 16.6.3.1). Later on, Fagin and Halpern [4] and also Bundy [43] extend Nilsson's structure for probabilities allowing all elements of the power set to be measurable, leading to a general structure just as Dempster-Shafer theory generalizes probability theory, the Dempster-Shafer structure (see section 16.6.3.2).

In the following, after a brief review of Nilsson and Demspter-Shafer structures, we extend the latter and propose a Dezert-Smarandache structure (section 16.6.3.3), combining the DSmT framework and the possible worlds semantics. To end this part, we propose in section 16.6.3.4 an example of the potential interest of such a structure.

16.6.3.1 Nilsson structure

A *Nilsson structure* is a tuple $\mathcal{S}_N = \langle S, \chi, \mu, \pi \rangle$ where

- $S = \{s_1, s_2, s_3, \ldots\}$, the set of all possible worlds;
- χ, a σ-algebra of subsets of S;
- μ, a probability measure defined on χ;
- $\pi : (S \longrightarrow P) \longrightarrow \{0; 1\}$, is a truth assignment to the propositions per possible world.

with P being a set of propositional variables.

Note that $\langle S, \chi, \mu \rangle$ is a probability space, and Nilsson structure is also called a probabilistic structure. In this kind of structure, the only measurable elements are those of χ. However, if we are interested in any other formula of Φ, the best thing we can do is to compute the *inner* and *outer measures* [4] defined respectively by

$$\mu_*(A) = \sup\{\mu(B)|B \subseteq A, B \in \chi\} \text{ and } \mu^*(A) = \inf\{\mu(B)|B \supseteq A, B \in \chi\}$$

The unknown value $\mu(A_\phi)$ is replaced by the interval:

$$\mu_*(A_\phi) \leq \mu(A_\phi) \leq \mu^*(A_\phi) \tag{16.10}$$

Hence, instead of a single probability measure μ from χ to $[0,1]$, we can compute a pair of probability measures μ_* and μ^*.

Because in a Nilsson structure, μ is defined on χ (the set of measurable subsets) means that χ_π (the image of χ by π) is a sub-algebra of χ to ensure that $\mu(\phi) = \mu(A_\phi)$, for all $\phi \in \Phi$. Dropping this condition is a means to extend μ to 2^S (hence Nilsson structure) and leads to Dempster-Shafer structure as formalized in [4][7] and detailed below. The probability measure μ is then replaced by its inner measure μ_*.

16.6.3.2 Dempster-Shafer structure

Nilsson structure can be extended using the inner measure, *i.e.* allowing all the elements of 2^S to be measurable. Because the inner measure turns to be the belief measure introduced by Shafer in its theory of evidence [8], the resulting structure is called *Dempster-Shafer structure*. Note that χ and π are no longer required to be related in any sense.

A *Dempster-Shafer structure* [4] is a tuple $\mathcal{S}_{DS} = \langle S, 2^S, \text{Bel}, \pi\rangle$ in which

- $S = \{s_1, s_2, s_3, \ldots\}$, the set of all possible worlds;
- 2^S, the powerset of S;
- Bel, a belief measure on 2^S;
- $\pi : (S \longrightarrow P) \longrightarrow \{0; 1\}$, is a truth assignment to the propositions per possible world.

with P being a set of propositional variables.

Note that we can simply write $\mathcal{S}_{DS} = \langle S, \text{Bel}, \pi\rangle$, where Bel is a belief function $\text{Bel} : 2^S \longrightarrow [0,1]$, in the Shafer sense (see section 16.5.3).

A Nilsson structure is then a special case of Dempster-Shafer structures, in which

$$\mu_*(A_\phi) = \mu^*(A_\phi) = \mu(A_\phi) \tag{16.11}$$

for any $\phi \in \Phi$.

[7] Another way is to consider a partial mapping π, leading to Bundy's structure of incidence calculus [43].

16.6.3.3 Dezert-Smarandache structure

In [33], the authors propose a generalization of Dempster-Shafer theory defining a belief function on the hyper-power set instead of the power set as Shafer. This theory is called Dezert-Smarandache theory or simply DSmT. In an equivalent manner to the extension of Nilsson's structure to DS structure, the definition of μ can be extended to D^S, allowing all elements of the hyper-power set to be measurable. We obtain then what we can call a *Dezert-Smarandache structure* (DSm structure), an extension of the DS structure in an equivalent way as DSmT is an extension of Dempster-Shafer theory.

A *Dezert-Smarandache structure* is a tuple $\mathcal{S}_{DSm} = \langle S, D^S, \mathrm{Bel}', \pi \rangle$ where

- $S = \{s_1, s_2, s_3, \ldots\}$, the set of all possible worlds;
- D^S, the hyper-power set of S;
- Bel', a general belief measure on D^S;
- $\pi : (S \longrightarrow P) \longrightarrow \{0; 1\}$, is a truth assignment to the propositions per possible world.

with P being a set of propositional variables.

Note that we can simply write $\mathcal{S}_{DSm} = \langle S, \mathrm{Bel}', \pi \rangle$ where Bel' is the generalized belief function defined on D^S, as defined by Dezert and Smarandache (see section 16.5.3).

16.6.3.4 Example: Ron suits

This example is proposed in [4] as *Example 2.4*:

> "*Ron has two blue suits and two gray suits. He has a very simple method for deciding what color suit to wear on any particular day: he simply tosses a (fair) coin. If it lands heads, he wears a blue suit and if it lands tails, he wears a gray suit. Once he's decided what color suit to wear, he just chooses the rightmost suit of that color on the rack. Both of Ron's blue suits are single-breasted, while one of Ron's gray suit is single-breasted and the other is double-breasted. Ron's wife, Susan, is (fortunately for Ron) a little more fashion-conscious than he is. She also knows how Ron makes his sartorial choices. So, from time to time, she makes sure that the gray suit she considers preferable is to the right (which depends on current fashions and perhaps on other whims of Susan). Suppose we don't know about the current fashion (or about Susan's current whims). What can we say about the probability of Ron's wearing a single-breasted suit on Monday?* [4]"

Let P be a set of primitive propositions, $P = \{p_1, p_2\}$. Let p_1="The suit is gray" and let p_2="The suit is double-breasted". Then $\mathcal{A}_t$, the corresponding set of basic elements is:

$$\mathcal{A}_t = \{p_1 \wedge p_2, p_1 \wedge \neg p_2, \neg p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$$

$\mathcal{A}_t$ is thus a set of mutually exclusive hypotheses: "Ron chooses a gray double-breasted suit", ..., "Ron chooses a blue single-breasted suit".

S is the set of possible states of the world, *i.e.* the set of possible worlds, where a state corresponds in this example to a selection of a particular suit by Ron. To fix the ideas, let number the suits from 1 to 4. Hence, $S = \{s_1, s_2, s_3, s_4\}$, s_i being the world in which Ron chooses the suit i. Table 16.3 lists the possible worlds and their associated meaning and atom[8]. Table 16.4 give some sets of worlds of interest and their associated formula. An alternative to describe the state of a world (*i.e.* the truth values of each

Table 16.3: The 4 states of the worlds and their associated basic element.

World	Meaning	Basic element
s_1	Blue single-breasted suit nb 1	$\neg p_1 \wedge \neg p_2$
s_2	Blue single-breasted suit nb 2	$\neg p_1 \wedge \neg p_2$
s_3	Gray single-breasted suit	$p_1 \wedge \neg p_2$
s_4	Gray double-breasted suit	$p_1 \wedge p_2$

propositions in P) is by using π is a truth assignment defined from P to 2^S. For each s in S, we have a truth assignment $\pi(s)$ defined from P to $\{0; 1\}$, such that $\pi(s)(p) = 0$ if p is false in s, and $\pi(s)(p) = 1$ if p is true in s.

Table 16.4: Some subsets of possible worlds of interest and their associated formula.

World(s)	Meaning	Formula
(s_1, s_2)	A blue suit	$\neg p_1$
(s_3, s_4)	A gray suit	p_1
(s_1, s_2, s_3)	A single-breasted suit	$\neg p_2$

Here, we have only 4 measurable events: $\mu(s_1, s_2) = \mu(s_3, s_4) = \frac{1}{2}$, $\mu(\varnothing) = 0$ and $\mu(S) = 1$. The question of interest here (What is the probability of Ron's wearing a single-breasted suit?) concerns another non-measurable event, *i.e.* (s_1, s_2, s_3). In [4], the authors gave this example to illustrate the utility of attributing values to non-measurable events, and then introduce Demspter-Shafer structures. Their conclusion for this example is then that the best we can say is that $\frac{1}{2} \leq \mu(s_1, s_2, s_3) \leq 1$, based on the inner and outer measures.

modelling the problem with 4 states means that given our prior knowledge, these states correspond to the only possible situations after Ron's selection: He will select one and only one suit among the 4 available. However, suppose that the two parts of the suits may have been mixed so we have two pieces (trousers and jacket) on the same coat-hanger. The 4 possible worlds correspond then to the 4 coat-hangers, and no longer to the 4 distinct suits. Imagining that the trousers is inside the jacket, Ron

[8]Note that the basic element $\neg p_1 \wedge p_2$ is associated with any state, while $\neg p_1 \wedge \neg p_2$ is associated with two states, s_1 and s_2.

will select his suit only on the basis of the color of the jacket. Suppose for example, that the coat-hanger he selects supports a blue jacket and gray trousers. Then, waht is the corresponding state of the world? Clearly, this situation has not been considered in the modelisation of the problem, based on a DS structure. However, using a DSm structure allow the elements of the hyper-power set of S to be measurable. Hence, the state resulting of a selection of a mixed suit corresponds to $s_i \cap s_j$, with $i \neq j$. This means that we are in both worlds s_i and s_j, and that with a single selection, Ron selected in fact two suits. So, we allow other events than those forecast to overcome.

One benefit of the resulting structure for situation analysis, is that it provides an interesting framework for dealing with both vagueness and conflict, combining the logical, semantical and reasoning aspect through the possible worlds semantics, and the measuring, combination aspect through the DSmT.

16.6.4 Connection between DSmT and neutrosophic logic in Kripke structures

Here we describe informally a possible connection between Dezert-Smarandache theory and the neutrosophic logic.

Let $\mathcal{S}_{DSm} = \langle S, \mathrm{Bel}', \pi \rangle$ be a DSm structure, and let $\mathcal{S}_K^{NL} = \langle S, \mathsf{R}, \vec{\pi} \rangle$ be the corresponding Kripke structure for neutrosophic propositions. Hence, we define a general neutrosophic structure to be $\mathcal{S}_N = \langle S, \mathrm{Bel}', \mathsf{R}, \vec{\pi} \rangle$, where:

- $S = \{s_1, s_2, s_3, \ldots\}$, the set of all possible worlds;
- Bel', a general belief measure on D^S, the hyper-power set of S;
- $\mathsf{R} \subseteq S \times S$ is the accessibility relation[9];
- $\vec{\pi} = (\pi_T, \pi_I, \pi_F)$ is a neutrosophic assignment to the propositions per possible world, *i.e.* $\pi : (S \longrightarrow P) \longrightarrow \|^-0, 1^+\|$ with π being either π_T or π_I or π_F.

where $P = \{p_1, \ldots, p_n\}$ is a set of propositional variables.

In order to reason on this structure, we need a set of axioms (as it is for example done in [4] for belief and probability) characterizing valid formulae. This can be achieved by imposing conditions on the accessibility relation R, conditions yielding hopefully to neutrosophic behaving agents.

Hence, the aim of this general structure is to conciliate (1) DSmT as a tool for modelling both epistemic and ontological uncertainty, (2) possible worlds for the representation of situations, (3) neutrosophic logic as a general logical approach to deal independently with truth, falsity and indeterminacy, and (4) Kripke structures as a support for reasoning and modelling the properties of a collection of interacting agents.

We finally note, that although a connection can be found or stated, there is *a priori* no trivial link between the neutrosophic assignments $(\pi_T(s)(\phi),\ \pi_F(s)(\phi),\ \pi_I(s)(\phi))$ that quantify truth, falsity and

[9]We consider here the monoagent case, although the extension to the multiagent case is trivial.

indeterminacy of formulae, and the belief granted to the corresponding sets of possible worlds through the general belief function proposed in DSmT, Bel'.

16.7 Conclusion

In this chapter, we proposed a discussion on neutrosophy and its capacity to tackle the situation analysis challenges. In particular, we underlined and connected to neutrosophy four basic ontological principles guiding the modelization in Situation Analysis: (1) allowing statements about uncertainty to be made, (2) contextualization, (3) enrichment of the universe of discourse, (4) allowing autoreference. The advantages of DSmT and neutrosophic logic were studied with these principles in mind. In particular, we highlighted the capacity of neutrosophic logic to conciliate both qualitative and quantitative aspects of uncertainty. Distinguishing ontological from epistemic obstacles in SA we further showed that being based on the power set, Dempster-Shafer theory appears in fact as an epistemic theory whereas Dezert-Smarandache theory, based on the richer hyper-power set, appears capable to deal with both epistemic and ontological aspects of SA. Putting forward the connection between hypersets and Kripke structures as means to model autoreference, we then focused on Kripke structures as an appropriate device for reasoning in SA. In particular, we showed that it is feasible to build a DSm structure upon the possible worlds semantics, an extension of the classical probabilistic and Dempster-Shafer structures. Considering neutrosophic logic, we showed that is could be possible to extend Kripke structures in order to take into account neutrosophic propositions, *i.e.* triplets of assignments on intervals of hyperreal numbers. We also showed how to represent the concepts of belief and knowledge with hyperreal truth (resp. falsity, indeterminacy) assignments on possible worlds. This allows one to introduce a clear qualitative distinction between *certain belief* and *knowledge*, a distinction that is not clear in traditional epistemic logic frameworks. Finally, we proposed a connection between neutrosophic logic and DSmT in the Kripke semantics setting.

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