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On the Smarandache LCM ratio

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Abstract Two types of the Smarandache LCM ratio functions have been introduced by Murthy [1]. Recently, the second type of the Smarandache LCM ratio function has been considered by Khainar, Vyawahare and Salunke [2]. This paper establishes the relationships between these two forms of the Smarandache LCM ratio functions, and derives some reduction formulas and interesting properties in connection with these functions.

Keywords Smarandache LCM ratio functions (of two kinds), reduction formulas.

§1. Introduction

The Smarandache LCM ratio function, proposed by Murthy [1], is as follows :

Definition 1.1. *The Smarandache LCM ratio function of degree r, denoted by $T(n, r)$, is*

$$T(n,r) = \frac{[n, n+1, n+2, \cdots, n+r-1]}{[1, 2, 3, \cdots, r]}, \qquad n, r \in \mathbb{N},$$

where $[n_1, n_2, \cdots, n_k]$ denotes the least common multiple (LCM) of the integers $n_1, n_2, \cdots, n_k$.

The explicit expressions for $T(n, 1)$ and $T(n, 2)$ are already mentioned in Murthy [1], and are reproduced in the following two lemmas.

Lemma 1.1. $T(n, 1) = n$ *for all* $n \geq 1$.

Lemma 1.2. *For* $n \geq 1$, $T(n, 2) = \frac{n(n+1)}{2}$.

The following two lemmas, due to Maohua [3], give explicit expressions for $T(n, 3)$ and $T(n, 4)$ respectively.

Lemma 1.3. *For* $n \geq 1$,

$$T(n,3) = \begin{cases} \frac{n(n+1)(n+2)}{6}, & \text{if } n \text{ is odd} \\ \frac{n(n+1)(n+2)}{12}, & \text{if } n \text{ is even} \end{cases}$$

Lemma 1.4. *For* $n \geq 1$,

$$T(n,\,4) = \begin{cases} \frac{n(n+1)(n+2)(n+3)}{72}, & \text{if } 3 \text{ divides } n \\ \frac{n(n+1)(n+2)(n+3)}{24}, & \text{if } 3 \text{ does not divide } n \end{cases}$$

Finally, the expression for $T(n, 5)$ is given by Wang Ting [4].

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Lemma 1.5. *For $n \geq 1$,*

$$T(n,\,5) = \begin{cases} \frac{n(n+1)(n+2)(n+3)(n+4)}{1440}, \; if \;\; n = 12m,\; 12m+8 \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{120}, \; if \;\; n = 12m+1,\, 12m+7 \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{720}, \; if \;\; n = 12m+2,\; 12m+6 \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{360}, \; if \;\; n = 12m+3,\, 12m+5,\; 12m+9,\, 12m+11 \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{480}, \; if \;\; n = 12m+4 \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{240}, \; if \;\; n = 12m+10 \end{cases}$$

Recently, Khairnar, Vyawahare and Salunke [2] treated the Smarandache LCM ratio function, defined as follows :

Definition 1.2. *The Smarandache LCM ratio function, denoted by $SL(n,r)$, is*

$$SL(n,r) = \frac{[n,\, n\text{-}1,\, n\text{-}2,\, \ldots,\, n\text{-}r\text{+}1]}{[1,\, 2,\, 3,\, \ldots,\, r]},\, r \leq n;\; n,\; r \in N.$$

The function $SL(n,r)$, given in Definition 1.2 above, may be called the Smarandache LCM ratio function of the second type.

In Section 2, we derive the relationships between the two functions $T(n,r)$ and $SL(n,r)$, and hence, derive the reduction formulas for $SL(n,3)$, $SL(n,4)$ and $SL(n,5)$, using the known expressions for $T(n,3)$, $T(n,4)$ and $T(n,5)$. Some more properties, together with some open problems involving these functions, are given in Section 3.

§2. Reduction formulas

The following lemma gives the relationship between $T(n,r)$ and $SL(n,r)$.

Lemma 2.1. $SL(n,r) = T(n-r+1,r)$.

Proof. This is evident from Definition 1.1 and Definition 1.2.

Note that, in Lemma 2.1 above, the condition $n-r+1 \geq 1$ requires that $SL(n,r)$ is defined only for $r \leq n$.

The explicit expressions for the functions $SL(n,1)$, $SL(n,2)$, $SL(n,3)$ and $SL(n,4)$ are given in Theorems 2.1 - 2.4 below.

Theorem 2.1. *For any $n \geq 1$, $SL(n,\, 1) = n$.*

Theorem 2.2. *For any $n \geq 2$, $SL(n,\, 2) = \frac{n(n\text{-}1)}{2}$.*

Proof. By Lemma 1.2 and Lemma 2.1,

$$\mathrm{SL(n,\, 2) = T(n - 1,\, 2) = \frac{(n\text{-}1)n}{2}}.$$

Theorem 2.3. *For any $n \geq 3$,*

$$SL(n,3) = \begin{cases} \frac{n(n\text{-}1)(n\text{-}2)}{6},\; if\, n\, is\, odd \\ \frac{n(n\text{-}1)(n\text{-}2)}{12},\; if\, n\, is\, even \end{cases}$$

Proof. Using Lemma 1.3, together with Lemma 2.1,

$$SL(n,\,3)=T(n-2,\,3)=\begin{cases}\frac{\text{(n-2)(n-1)n}}{6}, & \text{if } n-2 \text{ is odd}\\ \frac{\text{(n-2)(n-1)n}}{12}, & \text{if } n-2 \text{ is even}\end{cases}$$

Now, since n is odd or even according as $n-2$ is odd or even respectively, the result follows.

Theorem 2.4. *For any $n \geq 4$,*

$$SL(n,4)=\begin{cases}\frac{n(n-1)(n-2)(n-3)}{72}, & \textit{if 3 divides } n\\ \frac{n(n-1)(n-2)(n-3)}{24}, & \textit{if 3 does not divide } n\end{cases}$$

Proof. By Lemma 1.4 and Lemma 2.1,

$$SL(n,\,4)=T(n-3,\,4)=\begin{cases}\frac{\text{(n-3)(n-2)(n-1)n}}{72}, & \text{if 3 divides } n-3\\ \frac{\text{(n-3)(n-2)(n-1)n}}{24}, & \text{if 3 does not divide } n-3\end{cases}$$

But, 3 divides n – 3 if and only if 3 divides n. This, in turn, establishes the theorem.

Theorem 2.5. *For any $n \geq 5$,*

$$SL(n,\,5)=\begin{cases}\frac{n(n-1)(n-2)(n-3)(n-4)}{1440}, & \textit{if } n=12m,\ 12m+4\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{120}, & \textit{if } n=12m+1,\ 12m+3,\ 12m+7,\ 12m+9\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{720}, & \textit{if } n=12m+2\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{360}, & \textit{if } n=12m+5,\ 12m+11\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{480}, & \textit{if } n=12m+6,\ 12m+10\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{240}, & \textit{if } n=12m+8\end{cases}$$

Proof. By virtue of Lemma 1.5 and Lemma 2.1,

$$SL(n,\,5)=T(n-4,\,5)$$

$$=\begin{cases}\frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{1440}, & \text{if } n-4=12m,\ 12m+8\\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{120}, & \text{if } n-4=12m+1,\ 12m+7\\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{720}, & \text{if } n-4=12m+2,\ 12m+6\\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{360}, & \text{if } n-4=12m+3,\ 12m+5,\ 12m+9,\ 12m+11\\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{480}, & \text{if } n-4=12m+4\\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{240}, & \text{if } n-4=12m+10\end{cases}$$

Now, since n – 4 is of the form 12m if and only if n is of the form 12m + 4, n – 4 is of the form 12m + 8 if and only if n is of the form 12m, n – 4 is of the form 12m + 9 if and only if n is of the form 12m + 1, n – 4 is of the form 12m + 11 if and only if n is of the form 12m + 3, n – 4 is of the form 12m + 10 if and only if n is of the form 12m + 2, etc., the result follows.

§3. Some open problems

In this section, we give some open problems involving the functions $SL(n, r)$.

First, we state and prove the following two results.

Lemma 3.1. *For any integer $n \geq 1$, $SL(n, n) = 1$.*

Proof. This is evident from Definition 1.2.

Lemma 3.2. *If p is a prime, then p divides SL(p, r) for all r <p.*

Proof. By definition,

$$SL(p,\, r) = \frac{[\text{p, p-1, p-2, ..., p-r+1}]}{[1, 2, 3, ..., \text{r}]},\ r \leq p.$$

Now, p divides [p, p − 1, p − 2, . . . , p − r + 1] for all r <p, while p does not divide [1, 2, 3, . . . , r]. Thus, p divides SL(p, r).

Using the values of SL(n, r), the following table, called the Smarandache-Amar LCM triangle, is formed as follows :

The 1$^{\text{st}}$ column contains the elements of the sequence $\{SL(n,\, 1)\}_{n=1}^{\infty}$, the 2$^{\text{nd}}$ column is formed with the elements of the sequence $\{SL(n,\, 2)\}_{n=2}^{\infty}$, and so on, and in general, the k-th column contains the elements of the sequence $\{SL(n,\, \mathrm{k})\}_{n=k}^{\infty}$,

Note that, the 1$^{\text{st}}$ column contains the natural numbers, and the 2$^{\text{nd}}$ column contains the triangular numbers.

The Smarandache-Amar LCM triangle is

	1-st column SL(n, 1)	2-nd column SL(n, 2)	3-rd column SL(n, 3)	4-th column SL(n, 4)	5-th column SL(n, 5)	6-th column SL(n, 6)	7-th column SL(n, 7)	8-th column SL(n, 8)	9-th column SL(n, 9)
1-st row	1								
2-nd row	2	1							
3-rd row	3	3	1						
4-th row	4	6	2	1					
5-th row	5	10	10	5	1				
6-th row	6	15	10	5	1	1			
7-th row	7	21	35	35	7	7	1		
8-th row	8	28	28	70	14	14	2	1	
9-th row	9	36	84	42	42	42	6	3	1
10-th row	10	45	60	210	42	42	6	3	1
11-th row	11	55	165	330	462	462	66	33	55
12-th row	12	66	440	165	66	462	66	33	11
13-th row	13	78	286	715	429	858	858	429	143

Note that, by Lemma 3.1, the leading diagonal contains all unity.

Lemma 3.3. *If p is a prime, then sum of the elements of the p-th row ≡1 (mod p).*

Proof. The sum of the p-th row is

$$
\begin{aligned}
&\mathrm{SL(p,\ 1) + SL(p,\ 2) + \ldots + SL(p,\ p - 1) + SL(p,\ p)} \\
&= \mathrm{[SL(p,\ 1) + SL(p,\ 2) + \ldots + SL(p,\ p - 1)] + 1} \\
&\equiv \ \ 1\ (\mathrm{mod\ p}),
\end{aligned}
$$

by virtue of Lemma 3.2.

Lemma 3.4. *If p is a prime, then p does not divide SL(2p, r) for any* $p \le r \le 2p$. *Moreover, if q is the prime next to p, then q divides SL(2p, r) for all* $p \le r \le q - 1$.

Proof. If p is a prime, then p divides [2p, 2p − 1, . . . , 2p − r + 1] for all r ≤ 2p, and p divides [1, 2, . . . , r] for all r ≥ p. Hence, p does not divide

$$SL(2p,\ r) = \frac{[2p,\ 2p-1,\ ...,\ 2p-r+1]}{[1,\ 2,\ ...,\ r]},\ \mathrm{p} \le r \le 2p. \tag{1}$$

To prove the remaining part of the lemma, first note that, by Bertrand's postulate (see, for Example, Hardy and Wright [5]), there is at least one prime, say, q, such that p <q <2p. Now, from (1), q divides the numerator if p ≤ r ≤ q − 1, but q does not divide the denominator. As such, q divides SL(2p, r) for all p ≤ r ≤ q − 1.

Open Problem # 1 : Is it possible to find a congruence property for the sum of the elements of the k-th row when k is a composite?

Open Problem # 2 : Is it possible to find the sum of the elements of the k-th row?

Note that, by Lemma 3.2 and Lemma 3.4, some of the elements of the (2p)-th row is divisible by p, and some elements are not divisible by p but are divisible by q, where q is the next larger prime to p.

Looking at the 9^{th} row of the triangle, we observe that the number 42 appears in three consecutive places. Note that, 42 is divisible by the prime next to 7 in the interval (p, 2p) with p = 5.

Open Problem # 3 : In the Smarandache-Amar triangle, is it possible to find (in some row) repeating values of arbitrary length?

Note that, the above problem is related to the problem of finding the solutions of the equation

$$SL(n, r) = SL(n, r + 1). \tag{2}$$

A necessary and sufficient condition that (2) holds is

$$([n,\ n-1,\ ...,\ n-r+1],\ n-r)(r+1) = ([1,\ 2,\ ...,\ r],\ r+1)(n-r). \tag{3}$$

The proof is as follows : The equation (2) holds for some n and r if and only if

$$\frac{[n,\ n-1,\ ...,\ n-r+1]}{[1,\ 2,\ ...,\ r]} = \frac{[n,\ n-1,\ ...,\ n-r]}{[1,\ 2,\ ...,\ r,\ r+1]}$$

that is, if and only if

$$[n,\ n-1,\ ...,\ n-r+1].[1,\ 2,\ ...,\ r,\ r+1] = [n,\ n-1,\ ...,\ n-r].[1,\ 2,\ ...,\ r]$$

that is, if and only if

$$[n,\ n-1,\ ...,\ n-r+1].\frac{[1,\ 2,\ ...,\ r](r+1)}{([1,\ 2,\ ...,\ r],\ r+1)} = \frac{[n,\ n-1,\ ...,\ n-r+1](n-r)}{([n,\ n-1,\ ...,\ n-r+1],\ n-r)}.[1,\ 2,\ ...,\ r],$$

which reduces to (3) after simplification.

Lemma 3.5. *If n is an odd (positive) integer, then the equation (2) has always a solution.*

Proof. We show that

$$\mathrm{SL}(2\mathrm{r} + 1, \mathrm{r}) = \mathrm{SL}(2\mathrm{r} + 1, \mathrm{r} + 1) \text{ for any integer r} \geq 1.$$

In this case, the necessary and sufficient condition (3) takes the form

$$([2r+1,\, 2r,\, ...,\, r+2],\, r+1)(r+1) = ([1,\, 2,\, ...,\, r],\, r+1)(r+1).$$

Now, since

$$([2r+1,\, 2r,\, ...,\, r+2],\, r+1) = ([1,\, 2,\, ...,\, r],\, r+1) \,\text{for any integer r} \geq 1,$$

we see that (3) is satisfied, which, in turn, establishes the result.

If n is an even integer, the equation (2) may not have a solution. A counter-example is the case when n = 4. However, we have the following result.

Lemma 3.6. *If n is an integer of the form n = 2p + 1, where p is a prime, then*

$$SL(2p,\, p) = SL(2p,\, p + 1)$$

if and only if

$$([1,\, 2,\, ...,\, p],\, p+1) = p+1.$$

Proof. If n = 2p + 1, then the l.h.s. of the condition (3) is

$$([2p,\, 2p-1,\, ...,\, p+1],\, p)(p+1) = p(p+1),$$

which, together with the r.h.s. of (3), gives the desired condition.

Conjecture 3.1. *The equation SL(n, r) = SL(n, r + 1) has always a solution for any n ≥ 5.*

In the worst case, SL(n, n − 1) = SL(n, n) = 1, and the necessary and sufficient condition is that n divides [1, 2, . . . , n − 1].

Another interesting problem is to find the solution of the equation

$$SL(n+1, r) = SL(n, r). \tag{4}$$

The equation (4) holds for some n and r if and only if

$$\frac{[n,\, n-1,\, ...,\, n-r+1]}{[1,\, 2,\, ...,\, r]} = \frac{[n+1,\, n,\, ...,\, n-r+2]}{[1,\, 2,\, ...,\, r]}$$

that is, if and only if

$$\frac{[n,\, n-1,\, ...,\, n-r+2].(n-r+1)}{([n,\, n-1,\, ...,\, n-r+2],\, n-r+1)} = \frac{(n+1).[n,\, \mathrm{n}-1,\, ...,\, n-r+2]}{([n,\, \mathrm{n}-1, ...,\, n-r+2],\, n+1)},$$

which, after simplification, leads to

$$(n-r+1).([n,\, \mathrm{n}-1, ...,\, n-r+2],\, n+1) = (n+1).([n,\, n-1,\, ...,\, n-r+2],\, n-r+1), \tag{5}$$

which is the necessary and sufficient condition for (4).

From (5), we observe the following facts :

1. n + 1 cannot be prime, for otherwise,

$$([n, \mathrm{n}-1, ..., n-r+2], n+1) = 1,$$

which leads to a contradiction.

2. In (5),

$$[n, \mathrm{n}-1, ..., n-r+2], n+1) = n+1 \;\Leftrightarrow\; ([n, n-1, ..., n-r+2], n-r+1) = n-r+1.$$

3. In (5), if n − r + 1 = 2, then

$$([n, n-1, ..., n-r+2], n-r+1) = 2 \;\Rightarrow\; [n, \mathrm{n}-1, ..., n-r+2], n+1) = n+1.$$

4. If n − r + 1 $\neq$ 2 is prime, then

$$\begin{aligned} &([n, n-1, ..., n-r+2], n-r+1) = 1 \\ &\Rightarrow \; \mathrm{n}+1 = (n-r+1).([n, \mathrm{n}-1, ..., n-r+2], n+1) \\ &\Rightarrow \; \mathrm{n}+1 = \tfrac{([n, n-1, ..., n-r+2], n+1)}{([n, n-1, ..., n-r+2], n+1)-1} r, \end{aligned} \tag{6}$$

after simplification, showing that $([n, \mathrm{n}-1, ..., n-r+2], n+1)-1$ must divide r.

5. In (5), if $([n, \mathrm{n}-1, ..., n-r+2], n+1) = n+1$, then $n-r+1$ cannot be an odd prime, for otherwise, by (6),

$$\mathrm{n}+1 = \frac{n+1}{(n+1)-1} r \;\Rightarrow\; \mathrm{n} = r,$$

which leads to a contradiction.

Conjecture 3.2. *The equation SL(n + 1, r) = SL(n, r) has always a solution for any r $\geq$ 3.*

In the worst case, SL(r + 1, r) = SL(r, r) = 1, and the necessary and sufficient condition is that $([1, 2, ..., r], r+1) = r+1$.

Remark 3.1. In [2], Khairnar, Vyawahare and Salunke mention some identities involving the ratio and sum of reciprocals of two consecutive LCM ratios. The validity of these results depends on the fact that SL(n, r) can be expressed as

$$SL(n, r) = \frac{n(n-1) \dots (n-r+1)}{r!}. \tag{7}$$

If SL(n, r) can be represented as in (7), it can be deduced that

$$\frac{SL(n, r+1)}{SL(n, r)} = \frac{n-r}{r+1}, \; \frac{1}{SL(n, r)} + \frac{1}{SL(n, r+1)} = \frac{n+1}{(r+1).SL(n, r+1)}.$$

However, the above results are valid only under certain conditions on n and r. For example, for r = 2, the above two identities are valid only for odd (positive) integers n.

Thus, the next question is : What are the conditions on n and r for (7)?

If r = p, where p is a prime, then SL(p! − 1, p) can be expressed as in (7), because in such a case

$$SL(p! - 1, p) = \frac{[\mathrm{p}!\text{-}1, \mathrm{p}!\text{-}2, ..., \mathrm{p}!\text{-}\mathrm{p}]}{[1, 2, ..., \mathrm{p}]} = \frac{(\mathrm{p}!\text{-}1)(\mathrm{p}!\text{-}2) \dots (\mathrm{p}!\text{-}\mathrm{p})}{\mathrm{p}!}.$$

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