Is Dirichlet's proof of Fermat's Last Theorem for n = 5 flawed

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Abstract

We give an illogical point in Dirichlet's proof, therefore the used infinite descent is not powered in his proof

1 Dirichlet's proof for n = 5

First, we rewrite a proof in the case z is odd and divisible by 5 (summary only, for details, please see:$x^5 + y^5 = z^5$ (Dirichlet's proof) in [1], [2]), which was proven by Dirichlet as follows:

Lemma. *if the equation $x^5 + y^5 = z^5$ is satisfied in integers, then one of the numbers x, y, and z must be divisible by 5 (corollary of Sophie Germain's theorem)*

Since x and y are both odd, their sum and difference are both even numbers.

$$2p = x + y$$

$$2q = x - y$$

Where the non-zero integers p and q are coprime and have different parity (one is even, the other odd). Since x = p + q and y = p - q, $z = 2^m 5^n z'$ it follows that

$$2^m 5^n z' = x^5 + y^5 = (p+q)^5 + (p-q)^5 = 2p(p^4 + 10p^2q^2 + 5q^4) \tag{1}$$

Since 5 divides $2p(p^4 + 10p^2q^2 + 5q^4)$, then there must be r such that p = 5r
$2p(p^4 + 10p^2q^2 + 5q^4) = 2(5r)[(5r)^4 + 10(5r)^2q^2 + 5q^4] =$
$2.5^2r(125r^4 + 50r^2q^2 + q^4)$
$2.5^2r(q^4 + 50r^2q^2 + 125r^4)$
Define three values u, v, t to be the following:
$t = q^4 + 50r^2q^2 + 125r^4$
$u = q^2 + 25r^2$
$v = 10r^2$
And note that $t = u^2 - 5v^2$
and t is a fifth power since $z^5 = 2.5^2r.t$, two factors 2.5^2r, and t are relatively prime, so t is a fifth power and 2.5^2r is a fifth power.
By using the infinite descent, Dirichlet claimed that if t is a fifth power, then there must be a smaller solution.
Setting:
$u = c(c^4 + 50c^2d^2 + 125d^4)$
$v = 5d(c^4 + 10c^2d^2 + 5d^4)$
now 2.5^2r is a fifth power, so $(2.5^2r)^2$ is a fifth power
$(2.5^2r)^2 = 2.5^3.10r^2 = 2.5^3.v = 2.5^3.5d(c^4 + 10c^2d^2 + 5d^4)$
since $\gcd 2.5^4d, c^4 + 10c^2d^2 + 5d^4 = 1$, then 2.5^4d and $c^4 + 10c^2d^2 + 5d^4$ are fifth power.
in other hand, $c^4 + 10c^2d^2 + 5d^4 = (c + 5d^2)^2 - 5(2d^2)^2 = u'^2 - 5v'^5$

Setting:
$u' = c'(c'^4 + 50c'^2d'^2 + 125d'^4)$
$v' = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4)$
Since 2.5^4d is a fifth power, so $(2.5^4d)^2$ is also a fifth power
$(2.5^4d)^2 = 2.5^8 2d^2 = 2.5^8v' = 2.5^9d'(c'^4 + 10c'^2d'^2 + 5d'^4)$
So $2.5^9d'$, and $c'^4 + 10c'^2d'^2 + 5d'^4$ are also fifth power. $c'^4 + 10c'^2d'^2 + 5d'^4$ and $c^4 + 10c^2d^2 + 5d^4$ are the same form, and $d' < d$, by infinite descent, the original equation $t = u^2 - 5v^2$ has no solution.

2 Disproof

We consider u and v:
$u = c(c^4 + 50c^2d^2 + 125d^4)$
$u^2 = c^2(c^4 + 50c^2d^2 + 125d^4)^2$
$u^2 = c^2(c^8 + 50^2c^4d^4 + 125^2d^8 + 2.50c^6d^2 + 2.125c^4d^4 + 2.50.125c^2d^6)$
$u^2 = c^{10} + 50^2c^6d^4 + 125^2c^2d^8 + 2.50c^8d^2 + 2.125c^6d^4 + 2.50.125c^4d^6$
$v = 5d(c^4 + 10c^2d^2 + 5d^4)$
$v^2 = 5^2d^2(c^4 + 10c^2d^2 + 5d^4)^2$
$v^2 = 5^2d^2(c^8 + 10^2c^4d^4 + 5^2d^8 + 2.10c^6d^2 + 2.5c^4d^4 + 2.10.5c^2d^6)$
$5v^2 = 5^3c^8d^2 + 10^25^3c^4d^6 + 5^5d^{10} + 2.10.5^3c^6d^4 + 2.5^4c^4d^6 + 2.10.5^4c^2d^8$
So $u^2 - 5v^2 = c^{10} + 125^2c^2d^8 - 2.10.5^4c^2d^8 + 2.50.125c^4d^6 - 10^2.5^3c^4d^6 - 2.5^4c^4d^6 + 50^2c^6d^4 + 2.125c^6d^4 - 2.10.5^3c^6d^4 + 2.50c^8d^2 - 5^3c^8d^2 - 5^5d^{10}$
$u^2 - 5v^2 = c^{10} - 5c^85d^2 + 10c^65^2d^4 - 10c^45^3d^6 + 5c^25^4d^8 - 5^5d^{10}$

$$u^2 - 5v^2 = (c^2 - 5d^2)^5 \tag{2}$$

That means, the equation $u^2 - 5v^2 = n^5$ really has a solution in integer, such as $u = c(c^4 + 50c^2d^2 + 125d^4)$, $v = 5d(c^4 + 10c^2d^2 + 5d^4)$, and $n = c^2 - 5d^2$, hence, the infinite descent is not powered.
In fact, we can deny an assumption, but we can not deny a verity
Note that, the equation $u^2 - 5v^2 = n^5$ has a solution in integer, it does not imply that the equation $x^5 + y^5 = z^5$ also has a solution in integer.
Why did this illogical point happen? We begin from step:

$$v = 5d(c^4 + 10c^2d^2 + 5d^4) \tag{3}$$

v was determined by c and d, if c and d are integers, then v is also a integer as (3). There was no problem .
By using method of infinite descent, Dirichlet did the same progress :
Let $u' = c + 5d^2, v' = 2d^2$, then $c^4 + 10c^2d^2 + 5d^4 = (c + 5d^2)^2 - 5(2d^2)^2 = u'^2 - 5v'^5$
And claimed that, there exist two integers c', d' such that:
$u' = c'(c'^4 + 50c'^2d'^2 + 125d'^4)$
$v' = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4)$
It was such a mistake as follows:
Since $u' = c^5 + 5d^2$ and $v' = 2d^2$, so u' and v' was determined by them.

$$u' = c'(c'^4 + 50c'^2d'^2 + 125d'^4) \tag{4}$$

$$v' = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4) \tag{5}$$

c' and d' will be determined by (4), and (5) , and by (4) and (5) c', d' are not always integers with appointed u',v', it is such an illogical point.

However, above argument leads to solve this problem, if we can prove that:
a. The system of equations:
$q^2 + 25r^2 = c(c^4 + 50c^2d^2 + 125d^4)$
$10r^2 = 5d(c^4 + 10c^2d^2 + 5d^4)$
has no solution in integers with 2.5^2r is a fifth power
b.And $u = c(c^4 + 50c^2d^2 + 125d^4)$, $v = 5d(c^4 + 10c^2d^2 + 5d^4)$ are the only way for $u^2 - 5v^2$ to be expressed as a fifth power must be proven .
For a. if we mean using the infinite descent, the reasonable logic is that :
Assume the system of equations:
$q^2 + 25r^2 = c(c^4 + 50c^2d^2 + 125d^4)$
$10r^2 = 5d(c^4 + 10c^2d^2 + 5d^4)$
has a solution in integer, then c and d can be expressed as:

$$c^5 + 5d^2 = c'(c'^4 + 50c'^2d'^2 + 125d'^4) \tag{6}$$

$$2d^2 = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4) \tag{7}$$

($u' = c^5 + 5d^2$ and $u' = c'(c'^4 + 50c'^2d'^2 + 125d'^4)$; $v' = 2d^2$ and $v' = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4)$ is not separated). However, we can not affirm that (6) and (7) hold in integer, using the infinite descent is impossible in the case above.
In Euler's proof of FLT for n = 3, we have seen a similar formula (lemma) such as:

$$a^2 + 3b^2 = (c^2 + 3d^2)^3$$

Here: $a = c(c^2 - 9d^2)$, $b = 3d(c^2 - d^2)$ with gcd(c,d) = 1, and c, d are nonezero.
Euler also used the technique of infinite descent, but by other way in modified version, unfortunately, his proof is incorrect [3].

References

[1] Fermat's Last theorem: Proof for n = 5 http://fermatslasttheorem.blogspot.com

[2] Paulo Ribenboim's *Fermat's last theorem for Amateurs*, Springer 1999

[3] Quang N V, Euler's proof of Fermat Last's Theorem for n = 3 is incorrect Vixra:1605.0123(NT)

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