The Theory of n-scales

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Abstract

We give a theory of n-scale previously called as n dimensional time scale. In previous approaches to the theory of time scales, multi dimensional scales were taken as product space of two time scales [1, 2]. Here we define an n-scale as an arbitrary closed subset of $\mathbb{R}^n$. Modified forward and backward jump operators, Δ-derivatives and multiple integrals on n-scales are defined.

1 n-Scales

Previous studied in the literature considered multi-dimensional time scales as product space of two or many time scales [1, 2]. However this is a rather severe restriction. In this section we define multi-dimensional time scales and name them as n-scales. This is mainly because when multiple dimensions are introduced, other dimensions may denote space rather than time.

Just as a 1-scale (*i.e.* time scale) is a nonempty arbitrary closed subset of $\mathbb{R}$, we would like to define an n-scale as a nonempty arbitrary closed subset of $\mathbb{R}^n$. Though it may not be problematic for continuous parts of n-scales, defining neighborhood relations for discrete parts is crucial. When an n-scale is taken as a direct product of two time scales [1], the very nature of direct product gives neighborhood relations and the problem does not appear. However when generalizing the time scale structure to n-scales one must specify neighborhood relations in the form of an undirected graph. This graph can also connect boundary of discrete points to continuous parts of an n-scale.

Definition 1.1. *An n-scale $(\mathbb{T}^n, G)$ is a tuple where $\mathbb{T}^n$ is a nonempty arbitrary closed subset of $\mathbb{R}^n$ and G is a directed graph that does not contain cyclic edge appointments. The graph should cover the measure zero subsets of $\mathbb{T}$. G indicates the neighborhood structure of measure zero subsets of $\mathbb{T}^n$.*

In Figure 1 a simplistic 2-scale is drawn. This definition is much more flexible. For example the discrete meshes used to numerically solve partial differential equations are seen to be n-scales. The difference in n-scales is that it unifies discrete and continuous structures.

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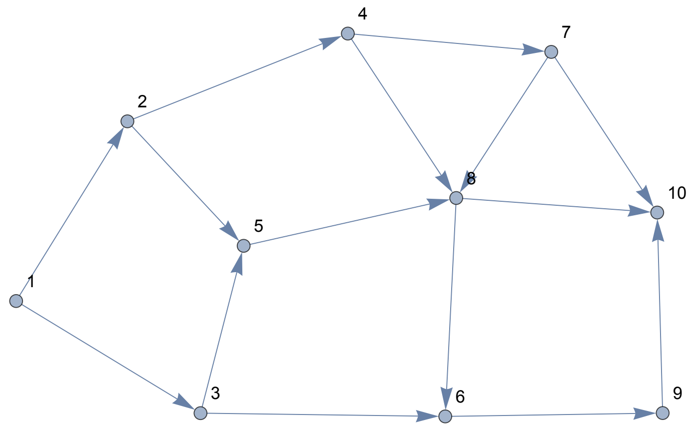


Figure 1: A 2-scale consisting of ten points. The straight lines denote the neighborhood structure.

Definition 1.2. *The graph structure lets one know the neighbors of p that can be reached through following directions indicated by the graph. The neighbors can be labelled with numbers such as $1, 2, \ldots n_p$. The directed forward jump operator σ_i yields the i^{th} neighbor of p. In a similar manner, ρ_i denotes the i^{th} neighbor of p in the reverse direction that is found in the graph.*

Theorem 1.1. *A graph G can be chosen for an n-scale such that in every cell of G one can find one point where all of its neighbors can be reached via σ operator.*

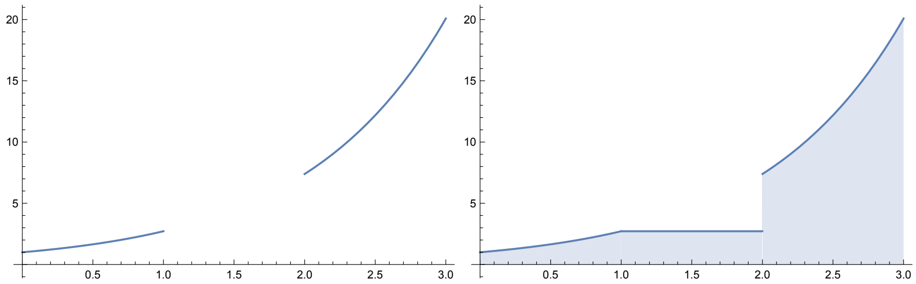
Proof. The proof is algorithmic. We begin with an undirected graph and will make it directed with the desired property.

1. Choose a random point (p) and connect it with one of its neighbors (say that it is q) via forward directed edge.
2. Choose a cell that includes the point p.
3. Connect p with its other neighbor in the cell with forward directed edge and repeat this same procedure until q is reached.
4. Choose one edge in the previous cell label the point where edge emerges as p' and reaches to q'.
5. Repeat commands 2–5 until all the graph becomes directed.

□

Now, let us move on to definition of directed derivatives. Suppose that we want to calculate a derivative of a function f at a point p. There are two cases to consider: 1) p lies in the interior of a continuous region, 2) p is an element of discrete set of points.

Definition 1.3. *In the first case, the definition Δ-derivative is usual partial derivative. However in the second case, we need the graph structure of the n-scale. It allows one to navigate through the neighboring points. The derivative directed to the i'th neighbor is just the difference equation*



(a) The plot of e^x on $\mathbb{T}$.

(b) The integral of e^x on $\mathbb{T}$.

Figure 2

$$f_i^{\Delta}(p) = \frac{f^{\sigma_i}(p) - f(p)}{\Delta x_i(p)} \tag{1}$$

where $f^{\sigma_i}(p)$ *is the function* f *evaluated at the* i^{th} *neighbor of the point* p *and* $\Delta x_i(p)$ *is the distance between the point* p *and its* i^{th} *neighbor.*

We move on to define integrals on n-scales. It would be useful to begin with an example from 1-scales. Let $\mathbb{T} = [0,1] \cup [2,3]$. We would like to integrate e^x on $\mathbb{T}$. The result of the integral is the sum of areas that lie below the function e^x on $\mathbb{T}$ plus the area of the rectangle where time scale makes a jump. See Figure 2 for an illustration.

What one observes in Figure 2 is two fold. First, the region between 1 and 2 that is excluded by the time scale gives a contribution to the integral. Hence we understand that the *connection structure* of a time scale is important in calculating integrals as well as its domain. Second, the contribution to the integral from the separation site is proportional to the value of the function on right scattered point. For example if we integrated e^{3-x} on $\mathbb{T}$ the result of the integral would be higher because the contribution to the integral from the separation site would be e^2 instead of just e. Hence we conclude time scales induce a *direction* in space. This directionality can be traced back to definition of Δ-derivatives in that whether one uses right scattered points or left scattered points in the definition of difference equations.

Integration on a n-dimensional subset of an n-scale is the usual Riemann integral. So, there is no need to much in this part. What is important, however, is integration in measure zero sets. These sets will give contribution to the integral as we have seen in the example of integration of e^x on a 1-scale. For that purpose let us illustrate the integration on discrete subsets with a simple example. The general rule can be inferred via induction.

Suppose we want to integrate a function f on a discrete subset of an n-scale. For that purpose we focus on a small subset where each vertex is connected to one another. See Figure 3. It could be of a rectangular shape instead of a triangle, it is not important. In the figure we see three points. 1 is connected to 2 and 3, 3 is connected to 2. Observe that all of the neighbors of 1 can be reached via σ. The integral of f is simple to calculate: value of f at $1\times$the area of the triangle.

Definition 1.4 (Integration on n-scales)**.** *Let* $\mathbb{T}$ *be an* n*-scale, and* f *be a*

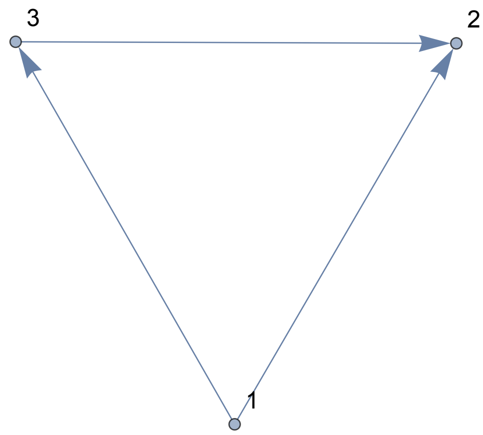


Figure 3: A simple 2-scale consisting of three points. The integral of a function f equals its value at the point 1 times the area of the triangle.

piecewise continuous function on each subset of $\mathbb{T}$ *with respect to dimension of each subset. The integral of* f *is the sum of Riemann integrals on* n*-dimensional subsets of* $\mathbb{T}$ *and integrals on measure zero subsets that are calculated using the triangulation rule.*

We illustrate how to take Δ-integrals on n-scales using a simple example of the 2-scale drawn in Figure 4. The edge length is a where as the points are positioned at vertices of equilateral triangles.

Suppose we want to integrate $f(x, y)$. As one looks carefully to the connection structure of the 2-scale, one sees that the points that contributes to the integral are $1, 2, 3, 4, 5, 6, 7$. However note that point 1 contributes with multiplicity six. The are of each equilateral triangle with side length of a is $a^2\sqrt{3}/4$. Let $p(n)$ be the coordinates of the point n. Then the result of the integral is as follows:

$$\int_{\mathbb{T}} f(x, y)\Delta x \Delta y = \frac{a^2\sqrt{3}}{4} \sum_{i=1}^{7} m_i f(p(i)) \tag{2}$$

where m_i is the number of triangles considered with $p(i)$ as the starting point. As it is seen on Figure 4 $m_1 = 6$ where as $m_i = 1$ for $i \neq 1$. The point 1 seems to have a special role but it is illusory. Its multiplicity being six is because of the special way that the graph is chosen. It is also important to show that by choosing a suitable graph, one can favor some points over the others.

References

[1] Martin Bohner and GS Guseino. Multiple integration on time scales. *Dynamic systems and applications*, 14(3/4):579, 2005.

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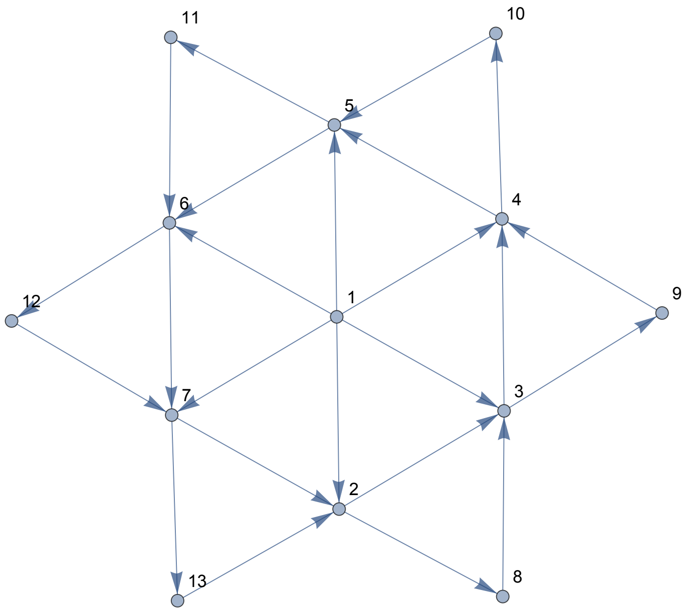

Figure 4: A highly symmetrical 2-scale consisting of thirteen points. Elements of 2-scale are located at vertices of equilateral triangles. Each edge has the length a.