Radical

Anthony J. Browne

April 23, 2016

ABSTRACT

Approximations of n^{th} roots are discussed. A close approximation to their decimal expansion is derived. Their relationship with the AKS test is also discussed.

There are many approximations of n^{th} roots. One common approach is Newton's method for example. It is a fairly easy process to find a working and very close approximation to square roots, or any root for that matter. I explain here an approach of my own, in which I approximate the decimal expansion itself.

If you list the values of square roots of numbers from 1 to some number, the whole numbers place in this list can be seen to follow the sequence $2n + 1$, where n is the whole numbers place value of $\sqrt{x}$. Meaning there are 3 ones, 5 twos, 7 threes,...ect. So, if you wanted to know how many square roots of whole numbers begin with the number 3, the equation would yield $2 * 3 + 1 = 7$, which is correct. Further, because each first new whole number is also the square root of the perfect square in the beginning of each interval of numbers, $n = \lfloor \sqrt{x} \rfloor$, which is the perfect square before any number x. So, n^2 is this perfect square.

It is also intuitive to note that in each interval of values, for example (4,8), the values of the decimal places nearly have a uniform distribution with respect to the interval. It will become more uniform for higher intervals. Using this assumption, its possible to estimate the decimal place values with a fraction. We can represent the location of any number x in an interval using the perfect square before it as, $n - x$. Knowing the location of x and the distance in the interval, this fraction can take the form of location over distance as follows,

$$\frac{x - n^2}{2n + 1}$$

This is a beginning approximation to the decimal expansion of $\sqrt{x}$. Consequently, in knowing that the whole number value will be n, a beginning approximation for $\sqrt{x}$ can be formulated as follows,

$$\sqrt{x} \approx \frac{x-n^2}{2n+1}+n \;=\; \frac{x+n+n^2}{2n+1} \;=\; \frac{x+n(n+1)}{2n+1}.$$

The error of this approximation can be seen in a graph of $\sqrt{x} - \frac{x+n(n+1)}{2n+1}$. You can find from this that the maximum error in this function occurs at $x = n(n+1)$. If you divide the distance $2n+1$ in each interval by the maximum error on the interval, you will get a sequence that is rapidly approaching integers, because the decimal expansion of the terms are approaching 0. This sequence truncated is, {37,101,197,325,...}. This sequence can be represented as $4(2n+1)^2+1$. So, if the maximum error divided by $2n+1$ rapidly approaches this sequence, then this sequence divided by $2n+1$ must rapidly approach the maximum error. Thus, the error is closely approximated by,

$$\frac{2n+1}{4(2n+1)^2+1}.$$

Now, if we make x equal to $n(n+1)$, the maximum points of error, and then add the approximation of this error, we get a very accurate approximation of $\sqrt{n(n+1)}$. And so,

$$\sqrt{n(n+1)} \approx \frac{2n(n+1)}{2n+1} + \frac{2n+1}{4(2n+1)^2+1}.$$

This is very accurate. By simple manipulation of the above equation, these other forms can be derived as well.

$$\sqrt{n^2+1} \approx \frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}$$

$$\sqrt{n^2-1} \approx \frac{2n(n^2-1)}{2n^2-1} + \frac{2n^2-1}{n(4(2n^2-1)^2+1)}$$

$$\sqrt{n}\sqrt{2n+1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n+1)}{4n+1} + \frac{4n+1}{4(4n+1)^2+1}\right)$$

$$\sqrt{n}\sqrt{2n-1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n-1)}{4n-1} + \frac{4n-1}{4(4n-1)^2+1}\right)$$

These are extremely accurate. As an example, for $n = 35$, the calculation of $\sqrt{35^2 + 1}$ using the second approximation formula above is accurate to 19 decimal places. And as discussed, this accuracy increases for higher n. The error reaches a limit of 0 at infinity. From this information one can derive approximations to other types of functions, as an example:

$$|n| \approx \frac{1}{\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

$$Sgn(n) \approx \frac{1}{n\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

There are some properties of roots that seem to be fundamental in the understanding of prime numbers. As an example, $(x+1)^n - x^n$ gives the number of n^{th} roots that have x in the integer position. If you expand this for several n, you can find that there are two separate representations for odd and even n.

$$(x+1)^{2n-1} - x^{2n-1} = (2n-1)\sum_{j=1}^{n} \frac{\binom{n-2+j}{2j-2}}{2j-1}(x^2+x)^{n-j}$$

$$(x+1)^{2n} - x^{2n} = (2x+1)\sum_{j=1}^{n} \binom{n-1+j}{2j-1}(x^2+x)^{n-j}$$

If we evaluate this at $x = 1$, we will find

$$2^{2n-1} - 1 = (2n-1)\sum_{j=1}^{n} \frac{\binom{n-2+j}{2j-2}}{2j-1}2^{n-j}$$

$$2^{2n} - 1 = 3\sum_{j=1}^{n} \binom{n-1+j}{2j-1}2^{n-j}$$

This shows that the Mersenne numbers $2^n - 1$ may be represented as the number of n^{th} roots that have 1 in the integer position. The forms of the sums are interesting too. Further investigation will reveal,

$$L_{2n-1} = (2n-1)\sum_{j=1}^{n}\frac{\binom{n-2+j}{2j-2}}{2j-1}$$

$$F_{2n} = \sum_{j=1}^{n}\binom{n-1+j}{2j-1}$$

which are bisections of the Lucas and Fibonacci numbers. If one forms a triangle of the coefficients of the expansion of $(x+1)^n - x^n$, it can be quickly discovered that,

$$(x+1)^n - x^n - 1 \equiv mod\ n\ \ iff\ n\ is\ prime$$

This is the basis of the AKS test. Further investigation reveals that this is not the end all to the story. In fact, this should actually say,

$$(x+1)^n - x^n - 1 \equiv mod\ p\ \ iff\ n = \ p^k \quad\ k \geq 1$$

The AKS test is capitalizing on $k = 1$. But, there are other interesting relationships here as well. If we restrict n to the even numbers $(x+1)^{2n} - x^{2n} - 1$, than one can find that in this form, the smallest number that will be coprime to each and every coefficient is in fact the greatest prime factor of 2n+1, $GPF(2n+1)$. So,

$GPF(2n+1) = min\{k: \forall\in\ \ (x+1)^n - x^n - 1, k \nmid (x+1)^n - x^n - 1\ \}$.

If that notation makes any sense. If one replaces x with i, the imaginary number one can find,

$$\frac{(i+1)^{2n} - i^{2n} - 1}{2n-1} = \mathbb{Z}\ \ if\ \ 2n-1 = p\ \wedge\ p \in 4m-1$$

If you explore this, you can find that,

$$\frac{2^{\frac{n-1}{2}} - (-1)^{\frac{n^2-1}{24}} - 1}{n}\ is\ and\ integer\ if\ n = p \quad n \geq 5$$

In conclusion, there is much more to discover within this area. There may be deeper relationships here. At worst, it may be an interesting area to explore within mathematics.