Solution for Euler Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Euler equations, following the description of the Lagrangian movement of an element of fluid, for spatial dimension n = 3. As we had seen in other previous articles, there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Euler equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Bernouilli's law.

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Essentially the Euler (and Navier-Stokes) equations relate to the velocity u and pressure p suffered for a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Euler equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl book[2] for a more detailed description of the deduction of these equations (including Navier-Stokes equations).

In spatial dimension $n = 3$, the Euler equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1) \qquad \begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y} + u_3 \dfrac{\partial u_1}{\partial z} = f_1 \\ \dfrac{\partial p}{\partial y} + \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y} + u_3 \dfrac{\partial u_2}{\partial z} = f_2 \\ \dfrac{\partial p}{\partial z} + \dfrac{\partial u_3}{\partial t} + u_1 \dfrac{\partial u_3}{\partial x} + u_2 \dfrac{\partial u_3}{\partial y} + u_3 \dfrac{\partial u_3}{\partial z} = f_3 \end{cases}$$

where $u(x, y, z, t) = \big(u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t)\big)$, $u: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$, and $f(x, y, z, t) = \big(f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t)\big)$, $f: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x, y, z) and at the instant of time t, for example, gravity force per mass unity, with $x, y, z, t \in \mathbb{R}$, $t \geq 0$. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ and these x, y, z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

(2) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\frac{dx}{dt} = u_1$, $\frac{dy}{dt} = u_2$, $\frac{dz}{dt} = u_3$, comes

(3) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u_1 + \frac{\partial u}{\partial y}u_2 + \frac{\partial u}{\partial z}u_3,$$

and therefore

(4) $$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1\frac{\partial u_i}{\partial x} + u_2\frac{\partial u_i}{\partial y} + u_3\frac{\partial u_i}{\partial z}, 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluation of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x, y, z) variable in time (Lagrangian formulation) and then consider (x, y, z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, leaving from Euler equations in Eulerian description, next the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\, dy\, dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = cte.$, $u = u(t)$. Done the solution in Lagrangian formulation, the solution in Eulerian formulation will be given.

Following this definition, the system (1) above is transformed into

$$(5) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{Du_1}{Dt} = f_1 \\ \frac{\partial p}{\partial y} + \frac{Du_2}{Dt} = f_2 \\ \frac{\partial p}{\partial z} + \frac{Du_3}{Dt} = f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity.

The system (5) always has a solution if the difference between the external force f and the acceleration $\frac{Du}{Dt}$ is a gradient function[3], for example, dependent only on the time variable, and the components velocity are C^1 class in the domain of u.

Given $u = u(x, y, z, t) \in C^1(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and a vector function f (both when in Eulerian description) such that the difference $f - \frac{Du}{Dt}$ is gradient, the system's solution (5) is

$$(6) \qquad p = \int_L \left(f - \frac{Du}{Dt}\right) \cdot dl + \theta(t),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$.

In Eulerian description and in special case when $f - \frac{Du}{Dt}$ is a constant vector or a dependent function only on the time variable, we come to

$$(7) \qquad \begin{aligned} & p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0), \\ & S_i(t) = f_i - \frac{Du_i}{Dt}, \end{aligned}$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t. When the variables x, y, z in (6) as well as f and u are in Lagrangian description, representing a motion over time, we need eliminate the dependence of the position using in (6)

$$(8) \qquad dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

$$(9) \qquad p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\, u_i(t)\, dt,$$

$$p^0 = p(0) = cte.$$

This expression can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (9) is equal to

(10) $$p(t) = p^0 - \frac{1}{2}\sum_{i=1}^{3}\left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^{3} f_i(t)\, u_i(t)\, dt,$$

i.e.,

(11) $$p(t) = p^0 - \frac{1}{2}(u^2 - u^{0\,2}) + \int_0^t f \cdot u\, dt,$$

$p, p^0 \in \mathbb{R}, u, u^0, f \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t),\ u^0 = (u_1^0, u_2^0, u_3^0) = u(0)$, $f = (f_1, f_2, f_3)$, in Lagrangian description, as well as

(12) $$p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L f \cdot dl,$$

in Eulerian description, $p^0(x, y, z) = p(x, y, z, 0)$, $u^0 = u^0(x, y, z) = u(x, y, z, 0)$, both formulations supposing f a gradient vector function ($\nabla \times f = 0,\ f = \nabla\phi,\ \phi$ potential function of f) when in Eulerian description (because we do not expect there is a contradiction between the Lagrangian and Eulerian descriptions). $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ it is simply

(13) $$p = p^0 - \frac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in a general format, in Lagrangian and Eulerian descriptions, and according Bernoulli's law without external force (gravity, in special).

Again we have seen that the system of Euler equations has no unique solution, only given initial conditions. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (11) that complies with the initial conditions (Lagrangian formulation), if we do not use (12) directly we also can choose appropriate $u(x, y, z, t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking these functions in (12). This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

(14) $$\begin{cases} \frac{dx}{dt} = u_1(x,y,z,t) \\ \frac{dy}{dt} = u_2(x,y,z,t) \\ \frac{dz}{dt} = u_3(x,y,z,t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (14) such that it is always possible to make any point (x,y,z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0,y_0,z_0) conveniently calculated according to (14). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x,y,z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions $(x_0,\ y_0,\ z_0)$.

We can escape the need to solve (14), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

(15) $$\begin{cases} u_1(t) = \frac{dx}{dt} \\ u_2(t) = \frac{dy}{dt} \\ u_3(t) = \frac{dz}{dt} \end{cases}$$

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (9) or (11), conveniently transforming the initial position $(x_0,\ y_0,\ z_0)$ as function of a generic position (x,y,z) and time t, we will have a correct value of the pressure in Eulerian formulation, as it was indicated in (12) based on (11). The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

Apply these methods to the Navier-Stokes equations and to the famous 6th Millennium Problem[4] on existence and smoothness of the Navier-Stokes equations apparently is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ is almost immediate.

It is no longer true that the Euler equations do not have a general solution.

To Leonard Euler, in memorian,
the greatest mathematician of all time.
He was brilliant, great intuitive genius.

Euler, forgive me for my mistakes…
This subject is very difficult!

References

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