COUNTING 2-WAY MONOTONIC TERRACE FORMS OVER RECTANGULAR LANDSCAPES

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Abstract. A terrace form assigns an integer altitude to each point of a finite two-dimensional square grid such that the maximum altitude difference between a point and its four neighbors is one. It is 2-way monotonic if the sign of this altitude difference is zero or one for steps to the East or steps to the South. We provide tables for the number of 2-way monotonic terrace forms as a function of grid size and maximum altitude difference, and point at the equivalence to the number of 3-colorings of the grid.

1. A Model of Altitude Maps

A mathematical model of altitudes in a landscape is obtained if the terrain is sliced orthogonally in m length units West-to-East (W-E) and n length units North-to-South (N-S), and some (average) altitude is assigned to each of the unit squares. The topology is represented by $n \times m$ matrices of altitudes. One step into one of the four directions of the compass rose (N, E, S, W) means increasing or decreasing the column or row index of the matrix by one. By further refinement of the units of altitudes we shall assume

(1) that the altitudes can be measured in integer units, so a map is an integer matrix $A_{s,e}$ with S-E coordinate pairs $1 \le s \le n$ and $1 \le e \le m$.
(2) that the altitudes are normalized such that the altitude at the NW corner of the map is zero—at sea level—, $A_{1,1} = 0$,
(3) and that the landscape is rising monotonically with unit altitude steps if one walks either one step E or S, such that all level differences $A_{s+1,e} - A_{s,e}$ and $A_{s,e+1} - A_{s,e}$ of the matrix are either 0 or 1.

Definition 1. *The positive integer number $T_{n\times m}$ is the number of 2-way monotonic altitude maps over a terrain of $n \times m$ squares with steps of height 0 or 1 as described above.*

The unit squares that share a common altitude form terraces of the landscape. The horizontal or vertical edges of the terraces are located where the altitude changes by 1. Consider for example the 3×4 map with maximum altitude $h = 4$ represented by the 3×4 matrix

$$\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{matrix} . \quad (1)$$

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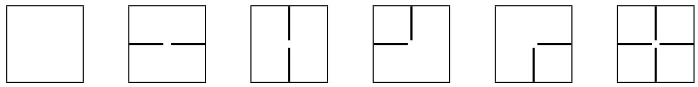

FIGURE 1. The 6 symbols on the cards that cover the $(n-1)\times(m-1)$ two-way terrace forms.

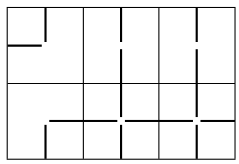

FIGURE 2. The representation of (3) by cards taken from Figure 1.

Insert horizontal or vertical edges where the altitude differs between an entry and any of its four neighbors to the N, E, S or W:

$$
\begin{array}{ccccccc}
0 & | & 1 & | & 2 & | & 3 \\
- & & & & & & \\
1 & & 1 & | & 2 & | & 3 \\
 & & - & & - & & - \\
1 & | & 2 & | & 3 & | & 4
\end{array} . \tag{2}
$$

Erase the integer entries of the matrix—which are redundant information then—and insert crosses where four edges meet and hooks where they switch direction:

$$
\begin{array}{ccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & | & & | & & | & & \cdot \\
\cdot & - & \lrcorner & & | & & | & & \cdot \\
\cdot & & & & | & & | & & \cdot \\
\cdot & & \ulcorner & - & + & - & + & - & \cdot \\
\cdot & & | & & | & & | & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \tag{3}
$$

The hooks $\lrcorner$ and $\ulcorner$ may appear, but not the hooks $\urcorner$ or $\llcorner$, because the latter forms are not supported by the landscapes that rise monotonically to the East and to the South. The constraint also enforces that (i) edges do not terminate inside the rectangle but only at the four sides, and that (ii) there are no points where three edges meet.

A further transcription is obtained by covering the terrain with $(n-1)\times(m-1)$ square cards centered at the 4-way intersections which show faces with the half-edges (fins) intruding from the N or E. There is a set of 6 different cards; one card without half-edge, four different cards with two half-edges, and one card with four half-edges (Figure 1).

This transforms the representation (3) for example into Figure 2.

Remark 1. *There are further interpretations of these objects:*

(1) *The edges may represent light beams that enter an optical switch table at up to $m-1$ equidistant ports from the North and up to $n-1$ equidistant*

ports from the East. The optical table offers $(n-1)(m-1)$ places optionally equipped with flat mirrors with surfaces pointing NW and SE. The mirrors deflect beams running S to W, and deflect beams running W to S. At places marked with crosses beams are reflected by both surfaces of a mirror. The beams leave either at the S or at the W rim of the table. $T_{n\times m}$ counts a number of switch tables of that type.

(2) *Representing the presence or absence of a light beam at the $n+m-2$ ports of entrance at the optical table with a 1 or a 0, and the presence or absence at the $n+m-2$ ports of exit again with a 1 or 0, $T_{n\times m}$ counts a family of gate-array logic which preserves the number of bits set on entry.*

(3) *The terraces form polyominos that cover the $n\times m$ rectangle. $T_{n\times m}$ counts covers by polyominos of any size which do not need the two absent hooks mentioned above to construct two forms "internal" bending. The example (3) covers the rectangle with a total surface of $3\times 4 = 12$ units by four 1×1 mono-tiles (one at the NW corner, three at the South rim), one 4-omino with a double-L shape, and two dominos.*

2. Refinement According to Hill Height

With these constraints the maximum altitude in the map is in the SE corner, which is the value of $A_{n,m}$, the hill height h of the geography.

Definition 2. *The number $T_{n\times m}(h)$, $n,m\geq 1$, is the number of 2-way monotonic altitude maps with steps of height 0 or 1 as described above with maximum altitude $h = A_{n,m}$.*

Example 1. *If the maximum altitude at the SE corner is zero, the landscape is entirely flat, and there is only a single configuration (without edges) that supports this:*

$$T_{n\times m}(0) = 1. \tag{4}$$

The maps can be counted by summing over all possible heights at the corners opposite to the minimum altitude:

$$T_{n\times m} = \sum_{h=0}^{n+m-2} T_{n\times m}(h). \tag{5}$$

Example 2. *A simple case counts the staircase shapes along uni-directional stairs, $n = 1$. There are $m-1$ steps which may individually be up-steps that change the altitude by +1 or flat stapes that do not change the altitude. The binomial numbers count the number of ways of distributing the up-steps over the $m-1$ steps:*

$$T_{1\times m}(h) = \binom{m-1}{h}; \quad T_{1\times m} = \sum_{h=0}^{m-1}\binom{m-1}{h} = 2^{m-1}. \tag{6}$$

Example 3. *The maximum hill height is $h = m+n-2$ because walking from the NW to the SE corner implies $m-1$ steps to the East and $n-1$ steps to the South. To gain that maximum height, all steps must be up-steps, which is one single configuration:*

$$T_{n\times m}(m+n-2) = 1. \tag{7}$$

3. Symmetries

3.1. **Swapping Rows and Columns.** A 2-way monotonic altitude map stays 2-way monotonic if the associated matrices A are mirrored along the diagonal, because this swaps the meaning of steps to the S and steps to the E. Therefore $T_{n\times m}$ and $T_{n\times m}(h)$ are symmetric with respect to exchange of width and height of the rectangle:

$$T_{n\times m} = T_{m\times n}, \tag{8}$$

$$T_{n\times m}(h) = T_{m\times n}(h). \tag{9}$$

3.2. **Conjugation.** Each 2-way monotonic altitude map has a unique *conjugate* 2-way monotonic map which is constructed by replacing each up-step to the East or to the South (entering another terrace) by a leveled/flat step (staying on the terrace) and replacing each leveled step to the East or South by an up-step.

Conjugation replaces each of the cards of Figure 1 by a conjugate card; the conjugate card has half-edges where the card has not, and vice versa. (There are three pairs of conjugate cards in Figure 1.)

If a walk form the NW to the SE corner in the original map is represented by the binary number of the step heights, the walk through the conjugate map is represented by the binary complement, therefore

$$T_{n\times m}(h) = T_{n\times m}(n+m-2-h). \tag{10}$$

3.3. **Inversion.** A rotation of the terrace form by 180 degrees is again a terrace form; the N rim changes place with the S rim and the E rim changes place with the W rim. A rotation by 180 degrees acts like a permutation on the set of cards of Figure 1 and flips their row and column order in representations like (3).

In the matrix representation this is equivalent to a transformation which (i) converts all up-steps to down-steps and keeps the flat steps, so the former hill becomes a valley and all matrix values switch sign, (ii) swaps the matrix elements by the rotation, so steps to the E or to the S are up-steps or flat steps again, and (iii) adds h to each matrix element: $A_{s,e} \to h - A_{m-s+1,n-e+1}$.

4. Recurrence: Attaching one Column

4.1. **Subclassification: NE value and E rim steps.** Another classification of the terrace forms is by considering the altitude $l \equiv A_{1,m}$ at the NE corner of the rectangle, $0 \le l \le m-1$, and step distribution of the staircase of walking from there along the E rim to the SE corner. These combinations are characterized by the number l and the binary vectors of length $n-1$ of zeros and ones of the altitude differences walking along the eastern rim of the rectangle: Then the values in the last column, m, of the matrix are

$$A_{s,m} = l + \sum_{i=0}^{s-2} d_i, \quad 0 \le l < m, \quad d_i \in \{0,1\}, \tag{11}$$

by accumulating the partial sum of the digits of a number

$$b = \sum_{i\ge 0} d_i 2^i \tag{12}$$

which signifies in its binary representations the locations of up-steps in column m, $0 \le b < 2^{n-1}$.

Definition 3.

$$H(b) \equiv \sum_{i\ge 0} d_i \tag{13}$$

is the non-negative sum of the binary digits d_i *of* $b = \sum_i d_i 2^i$, $0 \le d_i \le 1$.

Then the value at the SE corner of the matrix is

$$h = l + H(b). \tag{14}$$

Definition 4. $T_{n\times m}(l,b)$ *is the number of 2-way monotonic terrace forms with an altitude* l *in the NE corner and a step distribution* b *along the E edge (column* m*).*

The $T_{n\times m}(l,b)$ contribute to $T_{n\times m}(h)$ if the number of up-steps matches (14):

$$T_{n\times m}(h) = \sum_{0\le l<m} \sum_{0\le b\le 2^{n-1}} T_{n\times m}(l,b)\delta_{H(b),h-l}. \tag{15}$$

4.2. **Compatibility of step distributions.** Augmenting the size of the matrix by one column, counting $T_{n\times(1+m)}$, is recursively done by constructing $T_{n\times m}$ and counting the different ways of either copying the l at the NE corner to become the NE corner of the augmented matrix, $l' = l$, or increasing it by one, $l' = l+1$. In both cases we will write $l' = l + a$ with $a \in \{0,1\}$ to simplify the notation. In each of the two cases there are distributions b' of the steps in column $m' = 1+m$ which are compatible with the E rim, b, of the $n\times m$ matrix, in the sense that $A(s,m') = l' + \sum_{i=0}^{s-2} d'_i$ is either $A(s,m)$ or $A(s,m)+1$ for all $1\le s\le n$ as required by the monotonicity:

$$A(s,m) \le A(s,m') \le 1 + A(s,m); \tag{16}$$

$$\Leftrightarrow l + \sum_{i=0}^{s-2} d_i \le l + a + \sum_{i=0}^{s-2} d'_i \le 1 + l + \sum_{i=1}^{s-2} d_i. \tag{17}$$

$$\Leftrightarrow \sum_{i=0}^{s-2} d_i \le a + \sum_{i=0}^{s-2} d'_i \le 1 + \sum_{i=1}^{s-2} d_i. \tag{18}$$

We construct two $2^{n-1}\times 2^{n-1}$ compatibility matrices $C^{(a)}_{b,b'}$ which have a value of 1 if the vectors of binary digits d_i and d'_i satisfy this pair of inequalities, and a value of 0 if they do not.

Remark 2. $C^{(0)}_{b,b'}$ *and* $C^{(1)}_{b,b'}$ *are actually transposed of each other because the pair of inequalities demands that the partial sum of* d'_i *runs at most 1 ahead of the partial sum of* d_i *if* $a = 0$ *and lags at most 1 behind* d_i *if* $a = 1$.

Example 4. *Keeping the step distribution for the new column results in a valid step distribution independent which of the two* a *is applied:*

$$C^{(0)}_{b,b} = C^{(1)}_{b,b} = 1. \tag{19}$$

Example 5. *If the* $n\times m$ *E edge had no rises the new edge may have at most one rise. If* $a = 1$ *the* b' *must also be flat:*

$$C^{(1)}_{0,b'} = \delta_{b',0}, \tag{20}$$

and if $a = 0$ *the* b' *may have at most a single rise:*

$$C^{(0)}_{0,b'} = \begin{cases} 1, & b' = 0; \\ 1, & b' > 0 \wedge H(b') = 1; \\ 0, & b' > 0 \wedge H(b') > 1. \end{cases} \quad (21)$$

$T_{n\times m}(l,b)$ is a matrix with rows enumerated by $0 \le l < m$ and columns enumerated by $0 \le b < 2^{n-1}$. The two matrix multiplications

$$T^{(0)}_{n\times(m+1)}(l,b') \equiv \sum_{b=0}^{2^{n-1}} T_{n\times m}(l,b) C^{(0)}_{b,b'}, \quad (22)$$

$$T^{(1)}_{n\times(m+1)}(l+1,b') \equiv \sum_{b}^{2^{n-1}} T_{n\times m}(l,b) C^{(1)}_{b,b'}, \quad (23)$$

count the 2-way monotonic terrace forms derived from the two possible values of a. Shifting and adding the two matrices represents the result with $1+m$ columns:

$$T_{n\times(m+1)}(l,b') = T^{(0)}_{n\times(m+1)}(l,b') + T^{(1)}_{n\times(m+1)}(l,b'). \quad (24)$$

The recurrence is started at $m = 1$ as described in Example 2 with a $1 \times 2^{n-1}$ matrix:

$$T_{n\times 1}(l,b) = 1; \quad l = 0; \quad 0 \le b < 2^{n-1}. \quad (25)$$

This describes the numerical implementation via the program in Section A.

4.3. **Transfer Matrix Method.** One can imagine a different implementation where the representation of the E (rightmost) column of the terrace form is not the step distribution, b, but a vector of $n-1$ symbols taken from Figure 1. This is not favorable numerically because the compatibility table would grow in both dimensions as 6^{n-1} and not 2^{n-1}, as there are 6 candidates at each place. (It also has the disadvantage that one must keep track of stacking only symbols downwards the new column where the lines that enter at the top and leave at the bottom stay connected.)

The formal advantage of that method is that it clearly becomes a Transfer Matrix Method because each vector of the $n-1$ symbols at the E rim of the $n\times m$ matrix has a (finite) number of compatible vectors of the $n-1$ symbols of the next column. Following standard arguments of an associated state diagram method [2] this proves that the generating function

$$G_n(z) \equiv \sum_{m\ge 1} T_{n\times m} z^m \quad (26)$$

is a rational polynomial function of z. The bivariate generating function is symmetric as a consequence of (8):

$$G(z,t) \equiv \sum_{n\ge 1}\sum_{m\ge 1} T_{n\times m} t^n z^m = G(t,z). \quad (27)$$

5. Results

5.1. **Tabulation.** The results are represented by a table which has two types of lines.

- The first line type displays 4 integers separated by blanks which show n, m, h and $T_{n\times m}(h)$. The values for $2h > n+m-2$ are redundant according to (10) and not listed.

- The second line type displays 3 integers separated by blanks which show n, m, and $T_{n\times m}$. The second type is redundant according to (5).

```
1 1 0 1
1 1 1

2 1 0 1
2 1 2

2 2 0 1
2 2 1 4
2 2 6

3 1 0 1
3 1 1 2
3 1 4

3 2 0 1
3 2 1 8
3 2 18

3 3 0 1
3 3 1 18
3 3 2 44
3 3 82

4 1 0 1
4 1 1 3
4 1 8

4 2 0 1
4 2 1 13
4 2 2 26
4 2 54

4 3 0 1
4 3 1 33
4 3 2 153
4 3 374

4 4 0 1
4 4 1 68
4 4 2 615
4 4 3 1236
4 4 2604

5 1 0 1
5 1 1 4
5 1 2 6
5 1 16

5 2 0 1
5 2 1 19
```

```
5 2 2 61
5 2 162

5 3 0 1
5 3 1 54
5 3 2 413
5 3 3 770
5 3 1706

5 4 0 1
5 4 1 124
5 4 2 1953
5 4 3 6997
5 4 18150

5 5 0 1
5 5 1 250
5 5 2 7313
5 5 3 46812
5 5 4 84910
5 5 193662

6 1 0 1
6 1 1 5
6 1 2 10
6 1 32

6 2 0 1
6 2 1 26
6 2 2 120
6 2 3 192
6 2 486

6 3 0 1
6 3 1 82
6 3 2 949
6 3 3 2859
6 3 7782

6 4 0 1
6 4 1 208
6 4 2 5281
6 4 3 30802
6 4 4 53950
6 4 126534

6 5 0 1
6 5 1 460
6 5 2 23203
6 5 3 248182
6 5 4 762227
6 5 2068146
```

6 6 0 1
6 6 1 922
6 6 2 85801
6 6 3 1592348
6 6 4 8241540
6 6 5 14024408
6 6 33865632

7 1 0 1
7 1 1 6
7 1 2 15
7 1 3 20
7 1 64

7 2 0 1
7 2 1 34
7 2 2 211
7 2 3 483
7 2 1458

7 3 0 1
7 3 1 118
7 3 2 1948
7 3 3 8694
7 3 4 13976
7 3 35498

7 4 0 1
7 4 1 328
7 4 2 12686
7 4 3 112877
7 4 4 315198
7 4 882180

7 5 0 1
7 5 1 790
7 5 2 64920
7 5 3 1100210
7 5 4 5385305
7 5 5 8989062
7 5 22091514

7 6 0 1
7 6 1 1714
7 6 2 277585
7 6 3 8528422
7 6 4 71297441
7 6 5 197352882
7 6 554916090

7 7 0 1
7 7 1 3430
7 7 2 1030330

```
7 7 3 54926890
7 7 4 759337545
7 7 5 3397542544
7 7 6 5530983756
7 7 13956665236

8 1 0 1
8 1 1 7
8 1 2 21
8 1 3 35
8 1 128

8 2 0 1
8 2 1 43
8 2 2 343
8 2 3 1050
8 2 4 1500
8 2 4374

8 3 0 1
8 3 1 163
8 3 2 3676
8 3 3 22924
8 3 4 54199
8 3 161926

8 4 0 1
8 4 1 493
8 4 2 27805
8 4 3 359550
8 4 4 1499394
8 4 5 2376024
8 4 6150510

8 5 0 1
8 5 1 1285
8 5 2 164399
8 5 3 4230324
8 5 4 31454256
8 5 5 82146778
8 5 235994086

8 6 0 1
8 6 1 3001
8 6 2 806347
8 6 3 39423196
8 6 4 512868867
8 6 5 2213802873
8 6 6 3561146170
8 6 9094954740

8 7 0 1
8 7 1 6433
```

8 7 2 3407823
8 7 3 303382053
8 7 4 6725497344
8 7 5 47269152002
8 7 6 121303742075
8 7 351210375462

8 8 0 1
8 8 1 12868
8 8 2 12742873
8 8 3 1988261908
8 8 4 73117894428
8 8 5 819944490812
8 8 6 3296290368486
8 8 7 5192169001644
8 8 13574876544396

9 1 0 1
9 1 1 8
9 1 2 28
9 1 3 56
9 1 4 70
9 1 256

9 2 0 1
9 2 1 53
9 2 2 526
9 2 3 2058
9 2 4 3923
9 2 13122

9 3 0 1
9 3 1 218
9 3 2 6497
9 3 3 54272
9 3 4 177848
9 3 5 260962
9 3 738634

9 4 0 1
9 4 1 713
9 4 2 56624
9 4 3 1024773
9 4 4 6083808
9 4 5 14274628
9 4 42881094

9 5 0 1
9 5 1 2000
9 5 2 383735
9 5 3 14477724
9 5 4 157376166
9 5 5 612222006

9 5 6 952152558
9 5 2521075822

9 6 0 1
9 6 1 5003
9 6 2 2142634
9 6 3 161160206
9 6 4 3160111147
9 6 5 20525389173
9 6 6 50689903445
9 6 149077423218

9 7 0 1
9 7 1 11438
9 7 2 10237249
9 7 3 1471499970
9 7 4 50869309436
9 7 5 546793506964
9 7 6 2144980290812
9 7 7 3351708981962
9 7 8839958693702

9 8 0 1
9 8 1 24308
9 8 2 42993671
9 8 3 11360377192
9 8 4 675539536773
9 8 5 11837958868428
9 8 6 73048480647796
9 8 7 176885984094690
9 8 524918733085718

9 9 0 1
9 9 1 48618
9 9 2 161937617
9 9 3 75922639116
9 9 4 7578889491370
9 9 5 212879678428784
9 9 6 2038576101280635
9 9 7 7545165464129370
9 9 8 11583105980431652
9 9 31191658416342674

10 1 0 1
10 1 1 9
10 1 2 36
10 1 3 84
10 1 4 126
10 1 512

10 2 0 1
10 2 1 64
10 2 2 771

10 2 3 3732
10 2 4 9069
10 2 5 12092
10 2 39366

10 3 0 1
10 3 1 284
10 3 2 10894
10 3 3 118057
10 3 4 513905
10 3 5 1041518
10 3 3369318

10 4 0 1
10 4 1 999
10 4 2 108549
10 4 3 2667554
10 4 4 21733215
10 4 5 71899369
10 4 6 106145902
10 4 298965276

10 5 0 1
10 5 1 3001
10 5 2 836797
10 5 3 44951694
10 5 4 692347393
10 5 5 3861354450
10 5 6 8866654233
10 5 26932295138

10 6 0 1
10 6 1 8006
10 6 2 5281314
10 6 3 593478797
10 6 4 17063990547
10 6 5 161858075052
10 6 6 591078807754
10 6 7 902439668964
10 6 2443638951906

10 7 0 1
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10 7 3 6383377435
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10 7 6 31432924990292
10 7 7 74101002378750
10 7 222522561716048

10 8 0 1
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10 8 2 132872804
10 8 3 57644900961
10 8 4 5411459549576
10 8 5 145256401778490
10 8 6 1349062860032829
10 8 7 4906423896412527
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10 10 3 3044977814280
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11 1 2 45
11 1 3 120
11 1 4 210
11 1 5 252
11 1 1024

11 2 0 1
11 2 1 76
11 2 2 1090
11 2 3 6369
11 2 4 19095
11 2 5 32418
11 2 118098

11 3 0 1
11 3 1 362
11 3 2 17492
11 3 3 239798
11 3 4 1342933

11 3 5 3600608
11 3 6 4966934
11 3 15369322

11 4 0 1
11 4 1 1363
11 4 2 197804
11 4 3 6438457
11 4 4 69921908
11 4 5 314303269
11 4 6 651324265
11 4 2084374134

11 5 0 1
11 5 1 4366
11 5 2 1722394
11 5 3 128484004
11 5 4 2731553014
11 5 5 21180858856
11 5 6 69023191877
11 5 7 101582773162
11 5 287714402186

11 6 0 1
11 6 1 12374
11 6 2 12207123
11 6 3 1997447386
11 6 4 82211677028
11 6 5 1110247570840
11 6 6 5815403108223
11 6 7 13018111367891
11 6 40055966781732

11 7 0 1
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11 7 2 73128376
11 7 3 25130419118
11 7 4 1963313662947
11 7 5 46046994751248
11 7 6 390637403100019
11 7 7 1349331496630738
11 7 8 2025629900536780
11 7 5601638723985318

11 8 0 1
11 8 1 75580
11 8 2 380660536
11 8 3 263660741716
11 8 4 38210014068388
11 8 5 1541741349322761
11 8 6 21162752529241540
11 8 7 113212128532755874
11 8 8 256682233387533018

11 8 785274659708798828

11 9 0 1
11 9 1 167958
11 9 2 1758369389
11 9 3 2362795756242
11 9 4 620079831016189
11 9 5 42492215048872994
11 9 6 938336407734666461
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12 1 2 55
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12 1 2048

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12 2 3 10351
12 2 4 37356
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12 2 6 99442
12 2 354294

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12 3 6 20227001
12 3 70107974

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12 4 7 4797897182
12 4 14532174630

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```
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13 1 4096

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13 3 0 1
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13 3 319801226

13 4 0 1
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14 1 4 715
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14 1 8192

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14 2 3 24388
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14 2 6 673699
14 2 7 829168
14 2 3188646

14 3 0 1
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14 3 4 15451358
14 3 5 81130329
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14 3 1458790182

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14 4 4 1444285973
14 4 5 13963135688
14 4 6 65772003524
14 4 7 162521024666
14 4 8 218853011926
14 4 706383387198

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14 5 2 11427851
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14 5 4 100275435009
14 5 5 1822789071353
14 5 6 14387152855881

14 5 7 54462323883180
14 5 8 104612321338519
14 5 350773799961746

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14 6 9 58630276427146132
14 6 176426890096852632

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14 7 89364987835152404174

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14 12 9 37874379873257035544447660311780
14 12 10 245242696258812327996900700173152
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14 12 3094414026884947108271747841121212

14 13 0 1
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14 13 2 18421452918813
14 13 3 1590238459349270018
14 13 4 19639394369841159785004
14 13 5 46772970975347073258631796
14 13 6 27066300640090690577068822371
14 13 7 4585081169707941556795908009930
14 13 8 264749288756650882820648742703643

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14 13 9 5892459630972278966942625805764391
14 13 10 55648424630454980592574874721027094
14 13 11 239288484041873307333738371268471547
14 13 12 490593281165662062018547294744927295
14 13 1583384021903964479573689967000780402

14 14 0 1
14 14 1 40116598
14 14 2 70961632168027
14 14 3 11357451257590359840
14 14 4 250232365919081938956865
14 14 5 1022857577249059021514748742
14 14 6 977702068950765256879289304431
14 14 7 263612023260531856801214095233768
14 14 8 23404197624934775760703541434286424
14 14 9 777053429918809607908553532496030304
14 14 10 10684975270153684612120404751235035986
14 14 11 65789460797825068518108569300698730840
14 14 12 191604921933626366955120969400733752168
14 14 13 272649922373702777601949547955204462396
14 14 810410082813497381147177065840601910384
```

Many of these values are not new because $T_{n\times m}(h)$ where $h = n+m-4$, $h = n+m-5$, or $h = n+m-6$, have been published by Hardin in the Online Encyclopedia of Integer Sequence in Sequences A252876, A252976 and A252930 [3]. Sequences where $h = 1$ are A166810 ($n = 6$), A166812 ($n = 7$) and A166813 ($n = 8$).

5.2. **3-colorings.** The $T_{n\times m}$ are also the number of 3-colorings of the $n \times m$ grid [3, A078099], where the rules are that the color of the NW point is fixed and that none of the four points which is an immediate neighbor to the N, E, S or W from a point has the same color as the point [1]. The reason for this match is that the tiling with 3 colors (enumerated 0 to 2) allows the 6 color combinations

$$\boxed{\begin{matrix}0&2\\2&1\end{matrix}},\boxed{\begin{matrix}0&2\\1&0\end{matrix}},\boxed{\begin{matrix}0&1\\2&0\end{matrix}},\boxed{\begin{matrix}0&1\\1&0\end{matrix}},\boxed{\begin{matrix}0&2\\2&0\end{matrix}},\boxed{\begin{matrix}0&1\\1&2\end{matrix}}, \tag{28}$$

(and what follows by adding 1 or 2 modulo 3) at each of the 4-way crossings. If these are mapped in that order onto the 6 cards of Figure 1, the compatibility rules of attaching a symbol at any of the four sides of another symbol are actually the same. [This bijective map is established if an up-step in Figure 1 is replaced by increasing the color number by 1 (mod 3) and if a flat step in Figure 1 is replaced by increasing the color number by 2 (mod 3).] The first symbol of Figure 1 can be stacked on the top of itself, on top of the second symbol and on top of the fifth symbol, for example; in the same manner the first symbol of (28) can be stacked on top of itself, on top of the second and on top of the fifth symbol of (28).

5.3. **Generating Functions.** Generating functions (26) are easily extracted from the tables:

$$G_{1,z} = \frac{x}{1-2x} \tag{29}$$

is equivalent to (6).

$$G_{2,z} = \frac{2x}{1-3x} \tag{30}$$

has a simple interpretation of counting single lines of cards where the factor 3 means that each card has three candidates (possibly including itself) with matching half-edge for attachment at its E rim.

$$G_{3,z} = \frac{2x(2-x)}{1-5x+2x^2}. \tag{31}$$

[3, A078100] gives

$$G_{4,z} = \frac{2x(4-9x+4x^2)}{1-9x+15x^2-6x^3}. \tag{32}$$

Apart from a factor two there are [3, A207994-A207996]

$$G_{5,z} = \frac{2x(8-47x+77x^2-44x^3+8x^4)}{1-16x+65x^2-92x^3+48x^4-8x^5}. \tag{33}$$

$$G_{6,z} = \frac{2x(16-237x+1257x^2-3198x^3+4206x^4-2736x^5+688x^6}{1-30x+291x^2-1278x^3+2901x^4-3519x^5+2152x^6-516x^7}. \tag{34}$$

(35)

$$G_{7,z} = \frac{2x(32-1031x+13142x^2-89204x^3+360470x^4-909704x^5+1454814x^6-1461492x^7+896144x^8-320568x}{1-55x+1109x^2-11330x^3+67206x^4-247404x^5+582440x^6-881876x^7+846764x^8-499200x^9+172400x}$$

Appendix A. Java Program

The source code of the JAVA program that generated the table of the results is a single file `TWayMono.java` that is reproduced here:

```
import java.util.* ;
import java.math.* ;
public class TWayMono {
    /** Number of rows
    */
    int n ;

    /** Number of columns
    */
    int m ;

    /** row and column dimension in the compatibility table, 2^(n-1)
    */
    BigInteger binn ;

    /** The number of terrasses T[l][b] given l in the range 0 to m-1
    * and b in the range 0 to 2^(n-1)-1.
    */
    BigInteger[][] T ;

    /** The two compatibility tables C[a][b][bprime] for a=0 or a=1.
    * b and b prime in the range 0 to 2^(n-1)-1. a denotes whether
    * the step in the upper right NE corner is an up-step of levelled step.
    */
    byte[][][] compat ;
```

```

    /** Ctor with a specified number of rows and a single column.
    * @param nrows The number of rows. A value larger than 0.
    */
    TWayMono(int nrows)
    {
        n = nrows;
        m = 1;
        binn = BigInteger.ONE.shiftLeft(n-1) ;
        initCompat() ;
        T = new BigInteger[1][binn.intValue()] ;
        for(int b=0 ; b < T[0].length; b++)
            T[0][b] = BigInteger.ONE ;
    } /* ctor */

    /** Ctor by recurrence from m to m+1.
    * @param colless The distribution with one column less.
    * @param ncols The number of rows in colless as well as the one to be constructed.
    */
    TWayMono(TWayMono colless)
    {
        n = colless.n ;
        /* this matrix has one column more than the old one */
        m = colless.m+1;
        binn = colless.binn ;
        /* compatibility table does not change because n does not change
        */
        compat = colless.compat ;

        /* T0 and T1 are the two matrices obtained by multiplying
        * the old T matrix with one of the two compatibility matrices
        * of a=0 or a=1.
        */
        BigInteger[][] T0 = new BigInteger[colless.m][binn.intValue()] ;
        for(int l=0 ; l < T0.length ; l++)
        {
            for(int bprime=0 ; bprime < T0[l].length ; bprime++)
            {
                T0[l][bprime] = BigInteger.ZERO ;
                for(int b=0 ; b < T0[l].length ; b++)
                {
                    if ( compat[0][b][bprime] == 1)
                        T0[l][bprime] = T0[l][bprime].add(colless.T[l][b]) ;
                }
            }
        }

        BigInteger[][] T1 = new BigInteger[colless.m][binn.intValue()] ;
        for(int l=0 ; l < T1.length ; l++)
        {
            for(int bprime=0 ; bprime < T1[l].length ; bprime++)
            {
                T1[l][bprime] = BigInteger.ZERO ;
```

```
                for(int b=0 ; b < T1[l].length ; b++)
                {
                    if ( compat[1][b][bprime] == 1)
                        T1[l][bprime] = T1[l][bprime].add(colless.T[l][b]) ;
                }
            }
        }

        /* construct the table T[l,b] by the shift-and-add
        * superposition of T0 and T1
        */
        T = new BigInteger[m][binn.intValue()] ;
        for(int l=0 ; l < T.length ; l++)
        {
            for(int b=0 ; b <T[l].length ; b++)
            {
                if ( l == 0 )
                    T[l][b] = T0[l][b] ;
                else if ( l == m-1 )
                    T[l][b] = T1[l-1][b] ;
                else
                    T[l][b] = T0[l][b].add(T1[l-1][b]) ;
            }
        }
    } /* ctor */

    /** Total number of all 2-way monotonic terrace forms.
    * @return T_{n x m}
    */
    public BigInteger count()
    {
        BigInteger val= BigInteger.ZERO ;
        /* sum over all 0<=h<=m+n-2.
        * Note that it would be faster to add simply all elements
        * of T without filtering for distinct values of h. But
        * here it's more convenient to print the table of T(h) on the
        * fly.
        */
        for(int h=0 ; h < m+n-1 ; h++)
        {
            final BigInteger Tofh = count(h) ;
            /* print results if not redundant according
            * to the conjugacy. Consider h<=n+m-2-h.
            */
            if ( 2*h+2 <= n+m)
                System.out.println(n + " " + m + " " + h + " " +Tofh) ;
            val = val.add(Tofh) ;
        }
        return val;
    } /* count */

    /** Number of bits set in the binary representation of b.
    * @return The number of +1 digits. The Hamming weight of b.
```

```
    */
    public static int bitcount(int b)
    {
        int H=0 ;
        /* shift b by one bit to the right and accumulate the LSB */
        while ( b > 0 )
        {
            H += b& 1;
            b >>= 1 ;
        }
        return H;
    } /* bitcount */

    /** Count number of matrices with fixed SE value
    * @param h The maximum value in the matrix. The hill height in the SE.
    * @return T_{n X m)(h)
    */
    public BigInteger count(final int h)
    {
        BigInteger val= BigInteger.ZERO ;
        /* sum over all rows and columns of T obeying H(b)=h-l
        */
        for(int l =0 ; l < T.length ; l++)
        {
            for(int b=0 ; b < T[l].length ; b++)
                if ( l+bitcount(b) == h)
                    val = val.add(T[l][b]) ;
        }
        return val;
    } /* count */

    /** Generate the two compatibility tables for a=0 and a=1.
    */
    private void initCompat()
    {
        compat = new byte[2][][] ;
        for(int a=0 ; a <= 1 ; a++)
        {
            compat[a] = new byte[binn.intValue()][binn.intValue()] ;
            for(int b= 0 ; b < binn.intValue() ; b++)
            for(int bprime=0 ; bprime < binn.intValue() ; bprime++)
            {
                /* obtain the partial sums of the digits
                * by the remainder of b mod 2^(s), 1<=s<=n
                * rather than keeping track of the bit positions
                */
                compat[a][b][bprime] = 1 ;
                int disum =0 ;
                int diprimesum =0 ;
                for(int s=1 ; s <= n ; s++)
                {
                    if ( (b & (1 << (s-1))) != 0 )
                        disum++ ;
```

```
                    if ( (bprime & (1 << (s-1))) != 0 )
                        diprimesum++ ;
                    if ( a+diprimesum < disum || a+diprimesum > 1+disum)
                    {
                        compat[a][b][bprime] =0 ;
                        break;
                    }
                }
            }
        }
    } /* initCompat */

    /** Build a table of the results for n,m>=1
    *  Usage: java -cp . TWayMono
    * @param args
    *
    */
    static public void main(String[] args)
    {
        TWayMono t2 =null;
        /* strict limit of n is internally given by our internal
        * representation of 2^(n-1) as integers, so n<=32. This constraint
        * is not enforced here, because the byte table of the
        * compatibility matrix has 2^(2n-2) entries and that would
        * likely set a lower RAM limit to what can be computed here.
        */
        for(int n=1 ; n < 15 ; n++)
        {
            for(int m=1 ; m <= n ; m++)
            {
                if ( m == 1 )
                    /* all T[0][b]=1 */
                    t2 = new TWayMono(n) ;
                else
                    /* recurrence m->m+1 */
                    t2 = new TWayMono(t2) ;

                System.out.println(n + " " + m + " " + t2.count()) ;
                System.out.println() ;
            }
        }
    } /* main */
} /* TWayMono */
```

Appendix B. Recurrence by Stitching

The subclassification (15) establishes a recurrence for T by stitching 2-way monotonic terrace forms. Let a 2-way monotonic terrace form over a $n \times m$ rectangle be given with some fixed step distribution at its E edge represented by the (binary) number b. Any 2-way monotonic terrace form over a $n \times m'$ rectangle with the same fixed step distribution b at its W edge defines a 2-way monotonic terrace form over a $n \times (m+m'-1)$ rectangle if we let the unit tiles at the E edge of the first rectangle cover the same ground as the unit tiles of the Western edge of the second rectangle, and if we lift the altitudes of the second rectangle by the altitude l at the NE corner of the first rectangle.

Reversing the argument, a $n \times m$ rectangle with a 2-way monotonic terrace cut vertically along any of the $m-1$ separators between adjacent columns of the A-matrices constructs two separate 2-way monotonic terrace forms with smaller edge lengths m–which actually may be the same form if the two residual m are the same and if the sub-matrices are the same after the formally Eastern portion is normalized to zero altitude at its NW corner.

This algorithm of morphing is perhaps of interest if augmenting the forms by attaching columns one at a time appears too slow.

The obstacle starting from the algorithm of Section 4.1 that provides counts $T_{n\times m}(l,b)$ is that are classified by the step distribution b on their E rim. To attach these forms to other forms necessitates to rotate them as described in Section 3.3, which is a matter of reversing the order of the binary digits in b. Unfortunately this also rotates the corner that fixes l; l is not preserved by the rotation. The remedy is to tabulate counts not as a function of l but as a function of h, which is preserved by rotation. (This is at a larger cost of storage because h may grow up to $m+n-2$, whereas l grows only up to $m-1$.) This type of book-keeping is not nice either, because the values of h of the two rectangles merged in that manner are not simply additive; one needs to subtract $H(b)$ of the common row to find the effective total of the morphed form.

In summary, this generalized way of recurrences has not been pursued here.

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