

# Nine steps transforming incompressible Navier-Stokes equations into parabolic partial differential equations

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## Abstract

It was shown that using spatial transform obtained by applying the difference to spatial global coordinates and time integral of velocities non linear Navier-Stokes equation transforms into parabolic equations.

## Nine steps of linearisation

Let's start from Navier-Stokes equations [1, 2, 3]

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}$$

We know that in the left side is full time derivative

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \quad (1)$$

Now we could integrate both sides by time and obtain for  $\mathbf{v}(\mathbf{r}, t)$

$$\mathbf{v}(\mathbf{r}, t) = -\frac{1}{\rho} \int_0^t \nabla_{\mathbf{r}} p(\mathbf{r}, \tau) d\tau + \nu \int_0^t \nabla_{\mathbf{r}}^2 \mathbf{v}(\mathbf{r}, \tau) d\tau \quad (2)$$

We obtained formal solution for velocities as follow

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{F}(\mathbf{r}, t) \quad (3)$$

Eq. (3) is valid for any  $\mathbf{r}'$  in the boundary where we try to solve NS equation. We could choose  $\mathbf{r}'$  so that

$$\mathbf{r}' = \mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau \quad (4)$$

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and

$$\mathbf{v}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t) = \mathbf{F}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t) \quad (5)$$

or by taking full derivative of both sides we obtain

$$\frac{D\mathbf{v}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t)}{Dt} = \frac{D\mathbf{F}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t)}{Dt} = \quad (6)$$

$$-\nabla p(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t) + \nu \nabla_{(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t)}^2 \mathbf{v}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t) \quad (7)$$

It is easy to prove that

$$\frac{D\mathbf{v}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t)}{Dt} = \frac{\partial \mathbf{v}(\mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau, t)}{\partial t}. \quad (8)$$

So, we obtain parabolic equations

$$\frac{\partial \mathbf{v}(\mathbf{r}', t)}{\partial t} = -\frac{1}{\rho} \nabla_{\mathbf{r}'} p(\mathbf{r}', t) + \nu \nabla_{\mathbf{r}'}^2 \mathbf{v}(\mathbf{r}', t) \quad (9)$$

which are analytically solvable for defined boundaries using for example Green function method. When we obtain velocities  $\mathbf{v}(\mathbf{r}', t)$  in the local coordinates we could go back to the global coordinates by inserting coordinates  $\mathbf{r}'$  (4) into obtained analytical solution.

## References

- [1] Navier, C. L. M. H. Mem acad. R. sci. paris, Vol. 6, 389-416, 1823.
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- [3] Stokes. G. G. trans. Camb. Phil. Soc., vol 8, 287-305, 1845.