Some remarks on Cl_3 and Lorentz transformations

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Geometric algebra is a powerful mathematical tool for description of physical phenomena. Algebra Cl_3, based on Euclidean 3D vector space is a possible framework for physical theories, especially the new, broader definition of Lorentz transformations. Here we discuss several consequences from that broader definition. Among other things, we derived some consequences to the special theory of relativity, a change in speed limit for example.

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1. Time

Geometric algebra is a promising platform for mathematical analysis of physical phenomena. The simplicity and naturalness of the initial assumptions and the possibility to formulate all main theories with the same mathematical language imposes the need for a serious study of this beautiful mathematical structure. Many authors have made significant contributions and there is some surprising conclusions. Important one is certainly the possibility of natural defining Minkowski metrics within Euclidean 3D space without the need for the introduction of negative signature, that is, without the need to define time as the fourth dimension ([1, 8]). It is possible to ask many questions about time as a special dimension, but let's focus on Cl_3 now.

2. Brief introduction to geometric algebra

This short introduction has a purpose of motivation. In Appendix one can find some additional material and we also recommend further reading (see references).

If we ask without prejudice how to multiply vectors, then we probably covet real numbers-like multiplication. Clearly, we cannot expect commutativity (think of the cross product), except for parallel vectors. For orthogonal vectors we can expect anticommutativity. It's amazing, but these two simple assumptions, with the additional property that the product of parallel vectors is positive real number, produce an enormous amount of beautiful mathematics. Product of two vectors (*geometric product*, due to Clifford) can be written as a sum of symmetric and antisymmetric parts

$$ab=\frac{1}{2}\left(ab+ba\right)+\frac{1}{2}\left(ab-ba\right). \qquad (1)$$

From assumption that $aa=a^2$ is positive scalar we have

$$\left(a+b\right)^2=a^2+ab+ba+b^2\Rightarrow ab+ba=\left(a+b\right)^2-a^2-b^2,$$

so symmetric part of the product ab is a scalar (*inner product*). It is not difficult (at least for orthogonal unit vectors) to show that antisymmetric part of the product is not a vector (*outer* *product*), because its square is negative. Orthogonal vectors are anticommutative and symmetric part of geometric product is $ab+ba=ab-ab=0$, antisymmetric part is $(ab-ba)/2=(ab+ab)/2=ab \Rightarrow abab=-baab=-a^2b^2=-1$. Geometric product is associative and distributive and there is a possibility to divide by vectors, so, to define inverse of vector, $a^{-1}=a/a^2$. These arguments are often seen in the literature, but their simplicity and motivational strength justify (to some extent) repeating them here. Typically, geometric product is written as $ab=a\cdot b+a\wedge b$, where $a\cdot b$ is symmetric (*inner product*), a $a\wedge b$ antisymmetric (*wedge* or *outer product*) part.

Geometric algebra is a language in which we can make sentences like *abbcdca* …, which is very suitable for programming. An example of the simplicity is the rotation and scaling of vector *a* by operator ba^{-1}: $ba^{-1}a=b$. Or one can take *abba* to find connection between scalar and bivector part of the geometric product.

In Euclidean 3D space we define orthogonal unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ with the property

$$\boldsymbol{e}_i^2=1,\ \boldsymbol{e}_i\boldsymbol{e}_j+\boldsymbol{e}_j\boldsymbol{e}_i=0,$$

so one could recognize the rule for multiplication of Pauli matrices (they are just one of the representations of unit vectors). Each element of algebra (Cl_3) can be expressed as linear combination of elements of 2^3 – dimensional basis (*Clifford basis*)

$$\{1,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3,\ \boldsymbol{e}_1\boldsymbol{e}_2,\ \boldsymbol{e}_3\boldsymbol{e}_1,\ \boldsymbol{e}_2\boldsymbol{e}_3,\ \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\},$$

where we have a scalar, three vectors, three bivectors and pseudoscalar. According to the number of unit vectors in the product we are talking about the odd or even elements. If we define $j=\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ it is easy to show that pseudoscalar *j* has two interesting properties: 1) $j^2=-1$, 2) $jX=Xj$, for any element X of algebra, so behaves like an ordinary imaginary unit. Complex structure of Cl_3 is rich and fascinating. This property we have for *n* = 3, 7, … [3]. Bivectors can be expressed as product of pseudoscalars and vectors, $j\vec{\boldsymbol{v}}$. It is worthwhile to note that unit vectors can be used to create multivectors $a+b\boldsymbol{n},\ a,b\in\mathbb{R}$ (*paravectors*) having multiplication table like spacetime (hypercomplex) numbers (for fixed $\boldsymbol{n}$) ([13]).

We define a general element of algebra (multivector)

$$M=t+\vec{\boldsymbol{x}}+j\vec{\boldsymbol{n}}+jb=z+\boldsymbol{F},\qquad z=t+jb,\qquad \boldsymbol{F}=\vec{\boldsymbol{x}}+j\vec{\boldsymbol{n}}$$

where $\boldsymbol{z}$ is complex scalar (element of center of algebra), while $\boldsymbol{F}$, by analogy, is a *complex vector*. *Complex conjugation* we define as $z^*=z^\dagger=t-jb,\ \boldsymbol{F}^*=\boldsymbol{F}^\dagger=\vec{\boldsymbol{x}}-j\vec{\boldsymbol{n}}$. The complex structure allows different ways to express multivectors, one is

$$M=t+\vec{\boldsymbol{x}}+j\vec{\boldsymbol{n}}+jb=t+j\vec{\boldsymbol{n}}+j\left(b-j\vec{\boldsymbol{x}}\right),$$

where multivector of the form $a+vj\hat{\boldsymbol{v}}$ belongs to even part of algebra and can be associated with rotations, spinors or quaternions. Also we could treat multivector as ([15]) $M=\alpha_0+\sum_{i=1}^{3}\alpha_i\boldsymbol{e}_i,\quad \alpha_k\in\mathbb{C}$ and implement it relying on ordinary complex numbers.

There is theorem (with some minor exceptions, [8])

$$e^Ae^B=e^{A+B},\ \text{iff}\ AB=BA.$$

In the case of multivectors A and B that do not commute we can solve the equation $e^{A}e^{B}=e^{X}$ using results from [3]. For multivector $M = z + \boldsymbol{F}$ we have

$$e^{M}=e^{z}\left(\cos\left|\boldsymbol{F}\right|+\hat{\boldsymbol{F}}\sin\left|\boldsymbol{F}\right|\right),\qquad \boldsymbol{F}=\left|\boldsymbol{F}\right|\hat{\boldsymbol{F}}, \tag{2}$$

$$e^{\boldsymbol{F}}=1+\boldsymbol{F},\quad \text{for}\quad \boldsymbol{F}^{2}=0, \tag{2a}$$

$$\log M=\log\left|M\right|+\phi\hat{\boldsymbol{F}},\qquad \phi=\arg M=\arctan\frac{\left|\boldsymbol{F}\right|}{z} \tag{2c}$$

(definition of $\left|M\right|$ is given below, see [2]). Frequently we have $\left|M\right|=1$, so principal value of logarithm is zero and there remains complex vector $\phi\hat{\boldsymbol{F}}$. Now

$$C=e^{A}e^{B}=e^{X}\Rightarrow X=\log C.$$

In order to define metrics let us discuss some involutions in geometric algebra. Usually (like [2, 3]) one defines three involutions:

1) grade involution: $\hat{M}=t-\vec{\boldsymbol{x}}+j\vec{\boldsymbol{n}}-jb$
2) reverse (*adjoint*): $M^{\dagger}=t+\vec{\boldsymbol{x}}-j\vec{\boldsymbol{n}}-jb=z^{*}+\boldsymbol{F}^{*}$
3) Clifford conjugation: $\bar{M}=t-\vec{\boldsymbol{x}}-j\vec{\boldsymbol{n}}+jb=\bar{z}+\bar{\boldsymbol{F}}=z-\boldsymbol{F}$,

asterisk stands for a complex conjugate. Grade involution is transformation $\hat{\vec{\boldsymbol{x}}}=-\vec{\boldsymbol{x}}$ (*space inversion*), while reverse in Cl_3 is like complex conjugation, $\vec{\boldsymbol{x}}^{\dagger}=\vec{\boldsymbol{x}},\ j^{\dagger}=-j$. Clifford conjugation is combination $\bar{M}=\hat{M}^{\dagger},\ \bar{\vec{\boldsymbol{x}}}=-\vec{\boldsymbol{x}},\ \bar{j}=j$. Bivectors given as a wedge product could be expressed as $\vec{\boldsymbol{x}}\wedge\vec{\boldsymbol{y}}=j\vec{\boldsymbol{x}}\times\vec{\boldsymbol{y}}$, where $\vec{\boldsymbol{x}}\times\vec{\boldsymbol{y}}$ is a *cross product*.

Defining paravector $p=t+\vec{\boldsymbol{x}}$ we have $p\bar{p}=\left|t+\vec{\boldsymbol{x}}\right|^{2}=\left(t+\vec{\boldsymbol{x}}\right)\left(t-\vec{\boldsymbol{x}}\right)=t^{2}-x^{2}$ and we have usual metric of special relativity.

From $M=\hat{M}\Rightarrow M=t+j\vec{\boldsymbol{n}}$, even part of algebra (spinors).

From $M=M^{\dagger}\Rightarrow M=t+\vec{\boldsymbol{x}}$, paravector; reverse is anti-automorphism $\left(MM^{\dagger}\right)^{\dagger}=MM^{\dagger}$, so $MM^{\dagger}$ (square of *multivector magnitude*, [2]) is a paravector.

From $M=\bar{M}\Rightarrow M=t+jb=z$, complex scalar. Clifford conjugation is anti-automorphism, $\overline{M\bar{M}}=M\bar{M}$, so $M\bar{M}$ (square of *multivector amplitude*, [2]) is a complex scalar and there is no other "amplitude" with such a property. Let's look at all involutions $I\left(M\right)$ such that $I\left(z\right)\in\mathbb{C}$ (center of algebra), $I\left(z\right)M=MI\left(z\right)$.

Theorem 1. *If* $I\left(z\right)\in\mathbb{C}$, *then* $I\left(M\right)M=MI\left(M\right)$ *iff* $I\left(\boldsymbol{F}\right)=\pm\boldsymbol{F}$.

Condition $I\left(M\right)M=MI\left(M\right)$ leads to

$$\left[z+\boldsymbol{F}\right]\left[I\left(z\right)+I\left(\boldsymbol{F}\right)\right]-\left[I\left(z\right)+I\left(\boldsymbol{F}\right)\right]\left[z+\boldsymbol{F}\right]=0\Rightarrow$$

$$\boldsymbol{F}I\left(\boldsymbol{F}\right)-I\left(\boldsymbol{F}\right)\boldsymbol{F}=0\Rightarrow I\left(\boldsymbol{F}\right)=\pm\boldsymbol{F}.$$

This condition is met by Clifford conjugation (up to a sign), but also by, say, $z^* - \boldsymbol{F}$, unfortunately lacking in complex scalar amplitude $MI(M)$.

Theorem 2. *Clifford conjugation is a unique involution that gives* $MI(M) \in \mathbb{C}$.

Proof relays on fact that $\boldsymbol{F}^2$ is a complex scalar, while $\boldsymbol{F}I(\boldsymbol{F})$ generally contains vector $j\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{n}}$ as component, orthogonal to vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{n}}$, except for Clifford conjugation $\boldsymbol{F}I(\boldsymbol{F}) = -\boldsymbol{F}^2 \in \mathbb{C}$. Straightforward proof can be easily obtained by multiplying multivectors ([2], formula (6))

$$\left(t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb\right)\left(s_0 t + s_1 \vec{\boldsymbol{x}} + js_2 \vec{\boldsymbol{n}} + js_3 b\right), \quad s_i = \pm 1,$$

where will appear $j\left(s_1 \vec{\boldsymbol{n}}\vec{\boldsymbol{x}} + s_2 \vec{\boldsymbol{x}}\vec{\boldsymbol{n}}\right) \Rightarrow s_1 = s_2$ (to have double inner product in brackets).

3. Bilinear transformations (BT) and relativity

In [2] is defined *multivector amplitude* $|M|$ (hereinafter MA)

$$M\bar{M} = |M|^2 = t^2 - x^2 + n^2 - b^2 + 2j\left(tb - \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}}\right) \qquad (3)$$

and postulated that all bilinear transformations that preserve $M\bar{M}$ are Lorentz transformations (extended). From $M' = XMY$ follows that the general transformations that preserve $M\bar{M}$ can be written as

$$M' = e^{\vec{\boldsymbol{p}} + j\vec{\boldsymbol{q}}} M e^{\vec{\boldsymbol{r}} + j\vec{\boldsymbol{s}}}, \qquad (4)$$

which is transformation with twelve parameters. It follows from

$$M' = XMY \Rightarrow M'\bar{M}' = |M'|^2 = XMY\bar{Y}\bar{M}\bar{X} = |X|^2 |Y|^2 |M|^2,$$

so, obvious solution is $|X|^2 = |Y|^2 = 1$, but we could include transformations with $|X|^2 = |Y|^2 = -1$ (discussed later). From $e^{z+\boldsymbol{F}} e^{z-\boldsymbol{F}} = e^{2z}$ a question arises about phase transformations with $z = j\vartheta/2$ such that $MI(M)$ is invariant (*multivector magnitude* $\sqrt{MM^\dagger}$ discussed in [2] obviously is).

Given definition of Lorentz transformations (LT) differs from the usual ([1, 8]), where it is assumed that LT should act on the real part only (real scalars and vectors), which leads to $M' = LML^\dagger$. The key idea here is that starting with point in space we obtain a line by translation, then we obtain area by rotating line, and translating area follows volume. Which is the most important one? Neither. So, let’s try to treat them equally.

Important example of BT is a *boost* $B = e^{\varphi \boldsymbol{n}}$, $\boldsymbol{n}$ is a unit vector, $\tanh\varphi = v$, so

$$e^{\varphi \boldsymbol{n}} = \cosh\varphi + \boldsymbol{n}\sinh\varphi = \gamma\left(1 + v\boldsymbol{n}\right), \quad \gamma = 1/\sqrt{1 - v^2}, \qquad (5)$$

$\varphi = \ln\sqrt{(1+v)/(1-v)} = \ln k$, k is *Bondi factor*.

Note that the speed can be obtained as the ratio of the norm of vector part and scalar part of paravector $e^{\varphi \boldsymbol{n}} = \cosh\varphi + \boldsymbol{n}\sinh\varphi = \gamma\left(1 + v\boldsymbol{n}\right)$. New boost gives ($\boldsymbol{n}_\parallel$ is parallel to $\boldsymbol{m}$, $\boldsymbol{n}_\perp$ is orthogonal to $\boldsymbol{m}$)

$$e^{\varphi_2 \boldsymbol{m}/2} e^{\varphi_1 \boldsymbol{n}} e^{\varphi_2 \boldsymbol{m}/2} = e^{\varphi_2 \boldsymbol{m}/2} \gamma_1 \left(1 + v_1 \boldsymbol{n}\right) e^{\varphi_2 \boldsymbol{m}/2} = \gamma_1 e^{\varphi_2 \boldsymbol{m}} + \gamma_1 v_1 e^{\varphi_2 \boldsymbol{m}/2} \boldsymbol{n} e^{\varphi_2 \boldsymbol{m}/2} \Rightarrow$$
$$\gamma_1 \gamma_2 \left(1 + v_1 \boldsymbol{n}_{\parallel}\right)\left(1 + v_2 \boldsymbol{m}\right) + \gamma_1 v_1 \boldsymbol{n}_{\perp} . \qquad (6)$$

Boosts do not change an orthogonal component of a vector. Two boosts $e^{\varphi_1 \boldsymbol{n}} e^{\varphi_2 \boldsymbol{m}} = e^{\boldsymbol{v} + j\boldsymbol{w}}$ generally do not result in a boost (Thomas rotation, [2]).

For two boosts in $\boldsymbol{n}$ direction we have

$$\gamma_1 \gamma_2 \left(1 + v_1 \boldsymbol{n}\right)\left(1 + v_2 \boldsymbol{n}\right) = \gamma_1 \gamma_2 \left(1 + v_1 v_2 + \boldsymbol{n}\left(v_1 + v_2\right)\right) \qquad (7)$$

and ratio immediately gives overall speed $\left(v_1 + v_2\right)/\left(1 + v_1 v_2\right)$, so we obtained a new boost

$$\gamma_1 \gamma_2 \left(1 + v_1 v_2\right)\left(1 + \boldsymbol{n} \frac{v_1 + v_2}{1 + v_1 v_2}\right) \Rightarrow \gamma = \gamma_1 \gamma_2 \left(1 + v_1 v_2\right). \qquad (8)$$

It is convenient to express a boost (paravector) in new basis, using idempotents

$$\boldsymbol{u}_+ = \left(1 + \boldsymbol{n}\right)/2, \quad \boldsymbol{u}_- = \left(1 - \boldsymbol{n}\right)/2, \ \boldsymbol{u}_{\pm}^2 = \boldsymbol{u}_{\pm}, \quad \boldsymbol{u}_+ - \boldsymbol{u}_- = \boldsymbol{n}, \quad \boldsymbol{u}_+ + \boldsymbol{u}_- = 1, \quad \boldsymbol{u}_+ \boldsymbol{u}_- = 0, \qquad (9)$$
$$\gamma\left(1 + v\boldsymbol{n}\right) = a_+ \boldsymbol{u}_+ + a_- \boldsymbol{u}_- \Rightarrow a_+ = 1/a_- = k = \sqrt{\left(1 + v\right)/\left(1 - v\right)}, \qquad k = k_1 k_2 . \qquad (9a)$$

Mathematics for boosts could be extended to general LT, obtaining generalized *Bondi factor*.

From general expression for MA comparing real and imaginary part we have

$$t^2 - x^2 + n^2 - b^2 = t'^2 - x'^2 + n'^2 - b'^2 \qquad (10)$$
$$tb - \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} = t'b' - \vec{\boldsymbol{x}}' \cdot \vec{\boldsymbol{n}}' . \qquad (10a)$$

Defining differential of multivector $dX = dt + d\vec{\boldsymbol{x}} + jd\vec{\boldsymbol{n}} + jdb$, we have MA of differential

$$\left|dX\right| = dt^2 - dx^2 + dn^2 - db^2 + 2j\left(dbdt - d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{n}}\right), \qquad (11)$$

so we can ask the question which conditions must be met to be defined real proper time τ. This question could be related to Lagrangian [2], because if we can define real proper time $d\tau = \left|dX\right|$ we could also define a real action (up to a scale factor – mass) and Lagrangian independent of multivector variables

$$S = \int \left|dX\right| = \int \frac{\left|dX\right|}{d\tau} d\tau \Rightarrow \mathcal{L} = \frac{\left|dX\right|}{d\tau} = 1 . \qquad (12)$$

A direct consequence is the existence of conserved quantities ([2]).

There is a possibility to define proper time as $d\tau = \left|dX'\right|_{v'=0} \in \mathbb{R}$, but then generally remains dependence of ratio dt'/dt on quantities from different referent frames. In [2] is commented possibility of defining a complex Lagrangian, but that would be inconsistent with reality of proper time. Certainly, the idea that model with invariant MA serves as mathematical basis for the principle of relativity opens up a whole series of questions. The fact that classical relativistic results follow from this model could not be ignored, but there is a question of existence of physical arguments to possibly limit a use of model. Where are the opportunities for new physics, and where is the model too broad?

One can easily obtain proper time assuming that all quantities in $|dX'|$, except dt', are equal to zero. Assuming that this is not the case and still regarding reality of a proper time, imaginary part of MA must be zero for every referent frame:

$$dbdt - d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{n}} = dt^2 \left(d\dot{b} - d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}} \right) = dt^2 \left(h - d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}} \right) = 0 \Rightarrow h = d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}}, \qquad (13)$$

where we defined $d\dot{b} = h$, and $h' = d\dot{\vec{\boldsymbol{x}}}' \cdot d\dot{\vec{\boldsymbol{n}}}'$. Defining $\dot{\vec{\boldsymbol{n}}} = \vec{\boldsymbol{w}}$ we have $h = \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}$. Bivector part of multivector is not transforming like area ([2]), so is reasonable to assume vector $\vec{\boldsymbol{w}}$ to be proportional to some *angular momentum-like quantity* (AMLQ). Now $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}$ may be associated with flow of AMLQ. It turns out that this quantity could be associated with a new law of conservation ([2], see below).

One could regard conditions for real proper time to be:

i) $d\tau \in \mathbb{R}$

ii) $dt'/dt \equiv \gamma(M, M') = \gamma(M) = dt'/d\tau.$

Condition ii) is natural, relativistic factor now depends on quantities from single reference frame only. From i) and ii) follows

$$|dX| = |dX'| = d\tau^2 = dt^2 - dx^2 + dn^2 - db^2,$$

$$1 = \frac{dt^2}{d\tau^2}\left(1 - \frac{dx^2}{dt^2} + \frac{dn^2}{dt^2} - \frac{db^2}{dt^2}\right) = \gamma^2\left(1 - v^2 + w^2 - h^2\right),$$

$$\gamma = 1/\sqrt{1 - v^2 + w^2 - \left(\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}\right)^2} = 1/\sqrt{1 - v^2 + w^2 - w^2 v^2 \cos^2\alpha}. \qquad (14)$$

Recalling that factor γ is real (ratio of two reals) we have condition

$$1 - v^2 + w^2 - w^2 v^2 \cos^2\alpha > 0 \Rightarrow v_{\max} < \sqrt{\frac{1 + w^2}{1 + w^2 \cos^2\alpha}}. \qquad (15)$$

For $\cos\alpha = \pm 1$ is $v_{\max} = 1$, but $v_{\max} > 1$ otherwise.

So, for vector $\vec{\boldsymbol{w}}$ given a physical meaning it follows that the maximum speed varies. Natural assumption is that we do not require $w' = 0$ generally because it could be an internal characteristic of a system (like spin) and could not be reduced to zero by the selection of a suitable reference frame, i.e., there is no reference frame for an electron ceased to be a fermion.

We have real $|dX'| = dt'^2\left(1 - v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha'\right)$, so it would be easiest to conclude that the $w' = 0$, $v' = 0$, as discussed. Regarding $v' = 0$ relativistic factor γ becomes dependent on w', so, remains possibility $-v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha' = 0$, which means

$$v'_\tau = \frac{w'}{\sqrt{1 + w'^2 \cos^2(\alpha')}}, \qquad (16)$$

and one has a proper time in a referent frame of moving particle. What could be a physical meaning of that? In relativistic physics one usually relays on real scalars and real vectors and defines a proper time regarding $\vec{\boldsymbol{p}} = 0$. But under bilinear transformations that preserve MA one could regard $-v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha' = 0$. This could be possible to justify physically, because extending Lorentz transformations and including all other motions and their symmetries there is no preferable momentum-zero condition but „center of energy-momentum-AMLQ-flow-zero“ condition, whatever that means. Conclusion on limiting speed 1 is based on preferring momentum as the main form of motion in space-time. Also, important motivation for the use of geometry contained in Cl_3 is just equal treatment of all kind of movements (for author surely). It is interesting that speed v'_τ generally could be greater than 1, having upper limit $1/\cos\alpha'$ (but there is a question of limiting AMLQ somehow). Finally, is there any evidence of rest electrons?

Having a (really) real proper time we could define derivative of multivector by proper time

$$V=\frac{dX}{d\tau}=\frac{dt}{d\tau}+\frac{d\vec{\boldsymbol{x}}}{dt}\frac{dt}{d\tau}+j\frac{d\vec{\boldsymbol{n}}}{dt}\frac{dt}{d\tau}+j\frac{db}{dt}\frac{dt}{d\tau}=\gamma\left(1+\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}}+jh\right), \tag{17}$$

$$\left|V\right|=1\Rightarrow\frac{d\left|V\right|^2}{d\tau}=\frac{dV}{d\tau}\bar{V}+V\frac{d\bar{V}}{d\tau}=0, \tag{18}$$

which we could understand as kind of orthogonality of multivectors (velocity and acceleration). Defining $A=dV/d\tau$ we have $A\bar{V}+V\bar{A}=A\bar{V}+\overline{A\bar{V}}=0$ which means that multivector $A\bar{V}$ (or $V\bar{A}$) is a complex vector, so, using $V=z_V+\boldsymbol{F}_V$, $A=z_A+\boldsymbol{F}_A$ we have from (A.2) condition $z_V z_A-\boldsymbol{F}_V\cdot\boldsymbol{F}_A=0,\quad \boldsymbol{F}_V\cdot\boldsymbol{F}_A=\left(\boldsymbol{F}_V\boldsymbol{F}_A+\boldsymbol{F}_A\boldsymbol{F}_V\right)/2$. (19)

Multiplying $V=dX/d\tau=\gamma\left(1+\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}}+jh\right)$ by mass m (rest mass) one obtains ([2])

$$P=mV=E+\vec{\boldsymbol{p}}+j\vec{\boldsymbol{l}}+jH,\quad \left|P\right|=E^2-p^2+l^2-H^2=m^2, \tag{20}$$

but according to i) and ii) we have $H=\gamma mh=\vec{l}\cdot\vec{v}$ (flow) and so AMLQ related to conserved quantity. Geometry of model is in multivector *V*, mass is just a scaling factor.

Let summarize a little:

1) Pseudoscalar part of multivector is related to flow, wherein part of area is replaced by AMLQ (it does not transforms like area)
2) Limiting speed varies (depending on AMLQ and his flow)
3) for $\gamma=E/m$ with non-zero AMLQ speed is greater than with zero AMLQ
4) flow of AMLQ is conserved

Statement 3) is easy to show:

$$\gamma=E/m=1/\sqrt{1-v^2+w^2-\left(\vec{\boldsymbol{w}}\cdot\vec{\boldsymbol{v}}\right)^2}\Rightarrow$$

$$v=\sqrt{\frac{1+w^2-m^2/E^2}{1+w^2\cos^2\alpha}}=\sqrt{\frac{1+\left(l/E\right)^2-m^2/E^2}{1+\left(l/E\right)^2\cos^2\alpha}}\geq\sqrt{1-m^2/E^2}. \tag{21}$$

Relation (20) shows that quantities *E*, ***p***, *l* and *H* are in a way equal, especially if we connect them to their origin in multivector. Scalars, vectors, bivectors and pseudoscalars are defining subspaces in the linear space Cl_3, conserved quantities are result of symmetries in that space, and so, there is no reason to treat them unequally. It suggests that we should really rethink hard on old chap Euclidean 3D space in regard to branching physics tree.

As example of non-existence of real proper time let look at the formally constructed multivector

(22)

$$W = t + \vec{x} + j\vec{k} + j\omega \Rightarrow |W| = t^2 - x^2 + k^2 - \omega^2 + 2j\left(\omega t - \vec{k}\cdot\vec{x}\right).$$

Imaginary part is a phase of wave. For electromagnetic wave in vacuum real part could be zero, so, there is no proper time, there is no rest frame. For electromagnetic field in vacuum we have

$$E = B,\ \vec{E} \perp \vec{B} \Rightarrow \boldsymbol{F} = \vec{E} + j\vec{B} \Rightarrow |\boldsymbol{F}| = \sqrt{-\boldsymbol{F}^2} = 0. \tag{23}$$

4. Transformations, transformations

Besides boosts and rotations we can now define other transformations.

Generally is valid $\exp(\boldsymbol{F}) = \exp(\boldsymbol{F}_1)\exp(\boldsymbol{F}_2) \neq \exp(\boldsymbol{F}_1 + \boldsymbol{F}_2)$. From $\exp(\boldsymbol{F}) = z + \boldsymbol{f}$ we have

$$\boldsymbol{F} = \ln(z + \boldsymbol{f}) = \ln|z + \boldsymbol{f}| + \phi\hat{\boldsymbol{f}} = \phi\hat{\boldsymbol{f}}, \qquad \phi = \arctan\left(|\boldsymbol{f}|/z\right), \tag{24}$$

because $|z + \boldsymbol{f}|^2 = \exp(\boldsymbol{F}_1)\exp(\boldsymbol{F}_2)\exp(-\boldsymbol{F}_2)\exp(-\boldsymbol{F}_1) = 1$. Usually there is a vector component in $\boldsymbol{F}$ orthogonal to defining vectors for $\boldsymbol{F}_i$.

1) Another example of bilinear transformation is ([14, 16])

$$e^{-\frac{\varphi}{2}\boldsymbol{n}} M e^{\frac{\varphi}{2}\boldsymbol{n}}, \tag{25}$$

which leaves complex scalar unchanged, so let see effect on vectors. Using components of vector parallel and orthogonal to $\boldsymbol{n}$ we have

$$e^{-\frac{\varphi}{2}\boldsymbol{n}}\left(\vec{x}_{\parallel} + \vec{x}_{\perp}\right)e^{\frac{\varphi}{2}\boldsymbol{n}} = \vec{x}_{\parallel} + \vec{x}_{\perp}e^{\varphi\boldsymbol{n}} = \vec{x}_{\parallel} + \vec{x}_{\perp}\cosh\varphi + \vec{x}_{\perp}\wedge\boldsymbol{n}\sinh\varphi, \tag{26}$$

$$e^{-\frac{\varphi}{2}\boldsymbol{e}_2}\boldsymbol{e}_1 e^{\frac{\varphi}{2}\boldsymbol{e}_2} = \boldsymbol{e}_1 e^{\varphi\boldsymbol{e}_2} = \boldsymbol{e}_1\cosh\varphi + \boldsymbol{e}_1\wedge\boldsymbol{e}_2\sinh\varphi,$$

so, there is a bivector as a part of result. Physical interpretation is needed.

2) Next interesting example is

$$\left(e^M\right)^2 = -1 \tag{27}$$

and ignoring solutions from center of algebra we have $\exp(M) = \boldsymbol{F},\quad \boldsymbol{F}^2 = -1$, so

$$M = \frac{\pi}{2}\hat{\boldsymbol{F}} = \frac{\pi}{2}\left(\boldsymbol{n}\sinh\varphi + j\boldsymbol{m}\cosh\varphi\right),\quad n = m = 1,\quad \boldsymbol{n}\perp\boldsymbol{m}. \tag{28}$$

Now we have $\boldsymbol{F}t\boldsymbol{F} = -t$.

3) For $\boldsymbol{F}^2 = 1$ is $\boldsymbol{F} = \exp\left(j\pi(1-\boldsymbol{F})/2\right)$, $\boldsymbol{F}\bar{\boldsymbol{F}} = -\boldsymbol{F}^2 = -1$, so $\boldsymbol{F}M\boldsymbol{F}$ could be regarded as transformation keeping MA.

4) Reflection of multivector M is defined as $-\boldsymbol{n}M\boldsymbol{n}$, $\boldsymbol{n}$ is unit vector with MA equal to $-\boldsymbol{n}\boldsymbol{n} = \boldsymbol{n}(-\boldsymbol{n}) = -1$, so, reflection meets condition $|X|^2 = -1$. Logarithm of unit vector is $\log\boldsymbol{n} = j\pi(1-\boldsymbol{n})/2$, so $\pm\boldsymbol{n} = \exp\left(j\pi(1\pm\boldsymbol{n})/2\right)$.

5) From $\boldsymbol{N}^2 = 0$ follows $\left|\exp(\boldsymbol{N})\right|^2 = (1+\boldsymbol{N})(1-\boldsymbol{N}) = 1$ and we have another bilinear transformation preserving MA.

There are many other possibilities, recall that bilinear transformations that preserve MA have twelve parameters (from four defining vectors).

Conclusion

Starting from the articles [2, 3] is shown a few consequences of introduction of bilinear transformations of multivectors that preserve multivector amplitude, defined using Clifford conjugation in Cl_3. As a result of regarding that proper time is a real positive scalar dependent on quantities from single reference frame follows several conclusions. The first is that there is a new conserved quantity associated with the flow of angular momentum-like quantities (AMLQ). The second is that the maximum speed of the particles with AMLQ may be greater than 1, and that the speed at fixed energy is higher with than without AMLQ. Limiting speed varies depending on angle between momentum vector and AMLQ. Finally, from request that reference frame exist such that dt/dt' is independent of components of multivector M' follows existence of minimal speed of particles with AMLQ in such reference frame, meaning that real proper time is not defined using $\boldsymbol{p}' = 0$, but rather including other possible motions in Euclidean 3D space.

Appendix

As the Lorentz transformation here are generally defined by expression $\exp(\vec{\boldsymbol{v}} + j\vec{\boldsymbol{w}})$, let's look at some properties of the complex vector. For $\boldsymbol{F} = \vec{\boldsymbol{v}} + j\vec{\boldsymbol{w}}$ we have

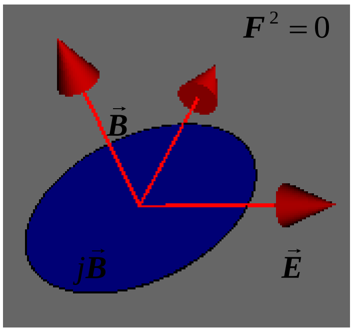
$$\left|F\right| = -\boldsymbol{F}^2 = v^2 - w^2 + 2j\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}, \quad \hat{\boldsymbol{F}} = \boldsymbol{F}/\sqrt{-\boldsymbol{F}^2}\,. \tag{A.1}$$

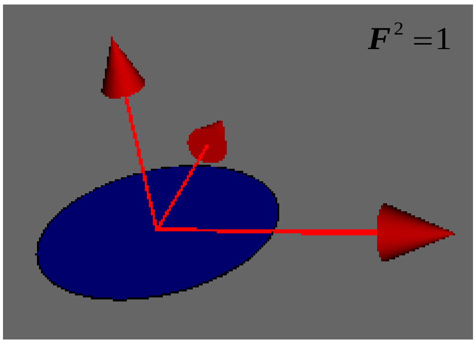
Commutator and anticommutator of two complex vectors satisfy

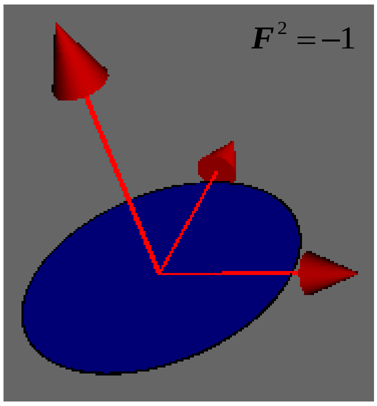
$$\overline{\boldsymbol{F}_1\boldsymbol{F}_2 - \boldsymbol{F}_2\boldsymbol{F}_1} = -\left(\boldsymbol{F}_1\boldsymbol{F}_2 - \boldsymbol{F}_2\boldsymbol{F}\right), \ \overline{\boldsymbol{F}_1\boldsymbol{F}_2 + \boldsymbol{F}_2\boldsymbol{F}_1} = \boldsymbol{F}_1\boldsymbol{F}_2 + \boldsymbol{F}_2\boldsymbol{F}\ , \tag{A.2}$$

so, commutator is a complex vector and anticommutator is a complex scalar.

(fig. 1., figures are created in GAViewer)







In [3] was shown that for $M = z + \boldsymbol{F}$ demanding $M^2 = c + jd$ we have or $\boldsymbol{F} = 0$ or $\boldsymbol{F} \neq 0$ and $z = 0$, $c = v^2 - w^2$, $d = 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$. Multivectors that square to complex scalars and have some vector component could be also found like this

$$(z+\boldsymbol{F})^2 = c + jd = z^2 + 2z\boldsymbol{F} + \boldsymbol{F}^2 = z^2 + 2z\boldsymbol{F} - |\boldsymbol{F}|^2 \Rightarrow z = 0 \Rightarrow \boldsymbol{F}^2 = c + jd = v^2 - w^2 + 2j\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$$
$$\Rightarrow c = v^2 - w^2,\ d = 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}. \quad \text{(A.3)}$$

Recalling that $\boldsymbol{F}^2 \in \mathbb{C}$, one could expect $\sqrt{z} = \boldsymbol{F} \Rightarrow z = \boldsymbol{F}^2$. Of course, there is a usual solution for $\sqrt{z}$, just use ordinary imaginary unit instead of j.

From $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{w}}$ follows that vector $\vec{\boldsymbol{v}}$ belongs to plane defined by bivector $j\vec{\boldsymbol{w}}$ (we could name such multivector *whirl*, for example, for EMV in vacuum $\boldsymbol{F} = \vec{\boldsymbol{E}} + j\vec{\boldsymbol{B}}$, vector $\vec{\boldsymbol{E}}$ belongs to plane defined by $j\vec{\boldsymbol{B}}$). Let's look at three interesting cases (we introduce a label for *whirl* $\boldsymbol{F}_c \,|\, \boldsymbol{F}_c^2 = c \in \mathbb{R}$).

1) $\boldsymbol{F}_0^2 = \boldsymbol{N}^2 = 0 \Rightarrow v = w$

We have $\exp(\boldsymbol{N}) = 1 + \boldsymbol{N}$ and such multivector gives general form of nilpotent $\boldsymbol{F}_0 = \boldsymbol{N}$, up to some factor ([12]). We also have $|\boldsymbol{N}| = \sqrt{-\boldsymbol{N}^2} = 0$, so there is no inverse. Defining $\hat{\boldsymbol{k}} = -j\hat{\boldsymbol{v}} \wedge \hat{\boldsymbol{w}} = \hat{\boldsymbol{v}} \times \hat{\boldsymbol{w}}$ (*direction of nilpotent*) it is easy to show that

$$(z\boldsymbol{N})^2 = z\boldsymbol{N}z\boldsymbol{N} = z^2\boldsymbol{N}^2 = 0, \quad \text{(A.4)}$$

$$\hat{\boldsymbol{k}}\boldsymbol{N} = -\boldsymbol{N}\hat{\boldsymbol{k}} = \boldsymbol{N}, \qquad (1+\hat{\boldsymbol{k}})\boldsymbol{N} = 2\boldsymbol{N}. \quad \text{(A.4a)}$$

In ([12]) is shown that multiplication of nilpotent by complex phase is equivalent to rotation of nilpotent about vector $\hat{\boldsymbol{k}}$, so that rotation around direction of nilpotent is thus reduced to the transformation U(1). Here is a simple proof

$$\hat{\boldsymbol{k}}\boldsymbol{N} = -\boldsymbol{N}\hat{\boldsymbol{k}} = \boldsymbol{N} \Rightarrow \mathrm{e}^{-j\frac{\theta}{2}\hat{\boldsymbol{k}}}\boldsymbol{N}\,\mathrm{e}^{j\frac{\theta}{2}\hat{\boldsymbol{k}}} = \mathrm{e}^{-j\theta\hat{\boldsymbol{k}}}\boldsymbol{N} = \boldsymbol{N}\mathrm{e}^{-j\theta}. \quad \text{(A.5)}$$

EMV in vacuum is nilpotent. In [8] was derived solution of Maxwell's equations in vacuum

$$\boldsymbol{F} = \boldsymbol{F}_0 e^{j(\omega t \pm \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}})} = \boldsymbol{F}_0 e^{\pm j\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}}} e^{j\omega t}, \quad \text{(A.6)}$$

where $\boldsymbol{F}_0 = \vec{\boldsymbol{E}}(0,0) + j\vec{\boldsymbol{B}}(0,0)$, $E(0,0) = B(0,0)$, $\vec{\boldsymbol{E}}(0,0) \perp \vec{\boldsymbol{B}}(0,0)$ is nilpotent. One could regard $\boldsymbol{F}_0 e^{\pm j\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}}}$ as nilpotent field (nilpotent $\boldsymbol{F}_0$ rotated about his direction vector by angle dependent on space coordinate), so, that field is multiplied by time depending complex phase. Nilpotent field $\boldsymbol{F}_0 e^{\pm j\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}}}$ is rotating around $\vec{\boldsymbol{k}}$ with frequency ω.

2) $\boldsymbol{F}_1^2 = 1 \Rightarrow v^2 - w^2 = 1 \Rightarrow \boldsymbol{F}_1 = \hat{\boldsymbol{v}} cosh\varphi + j\hat{\boldsymbol{v}}_\perp sinh\varphi$ (A.7)

We have $\boldsymbol{F}_1 = \exp(j\pi(1-\boldsymbol{F}_1)/2)$, $\boldsymbol{F}_1^{-1} = \boldsymbol{F}_1$ and

$$[(1+\boldsymbol{F}_1)/2]^2 = (1+\boldsymbol{F}_1)(1+\boldsymbol{F}_1)/4 = (1+\boldsymbol{F}_1)/2,$$

So we have all idempotents of algebra [12, 16].

Theorem 3. *All idempotents in Cl*$_3$ *have form* $(1+\boldsymbol{F}_1)/2$.

Simple proof is

$$(z+\boldsymbol{F})^2 = z^2 + 2z\boldsymbol{F} + \boldsymbol{F}^2 = z + \boldsymbol{F} \Rightarrow z = 1/2,\ \ z^2 + \boldsymbol{F}^2 = z \Rightarrow \boldsymbol{F} = \boldsymbol{F}_1/2.$$

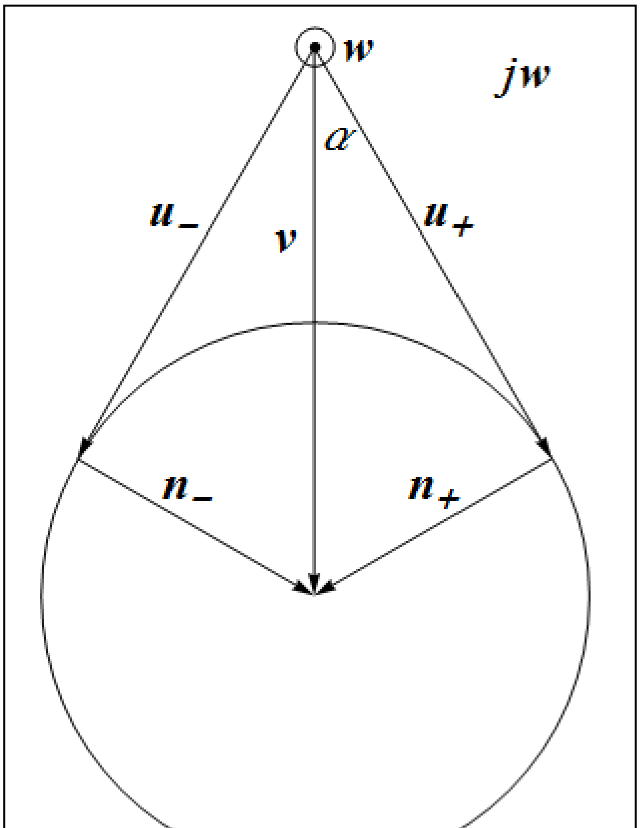


Fig. 2

Every such idempotent could be expressed as a sum of simple idempotent (like $(1+\boldsymbol{e}_1)/2$) and nilpotent ([12]). Simple nilpotent is paravector of form $(1+\hat{\boldsymbol{v}})/2$.

Theorem 4. *Every idempotent in Cl*$_3$ *can be expressed as a sum of simple idempotent and nilpotent.*

Let $\mathrm{I} = (1+\boldsymbol{n})/2$ is simple idempotent, then

$$(\mathrm{I}+\boldsymbol{N})^2 = \mathrm{I} + \mathrm{I}\boldsymbol{N} + \boldsymbol{N}\mathrm{I} =$$
$$\mathrm{I} + \boldsymbol{N} + \boldsymbol{nN} + \boldsymbol{Nn} \Rightarrow \boldsymbol{nN} + \boldsymbol{Nn} = 0,$$

so vector $\boldsymbol{n}$ must be orthogonal on vectors from nilpotent. It means that is not every sum of simple idempotent and nilpotent idempotent.

For non-simple idempotent we can define

$$(1+\boldsymbol{F}_1)/2 = (1+\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}})/2 =$$
$$(1+\boldsymbol{n})/2 + \boldsymbol{N},\quad \boldsymbol{N} = (\vec{\boldsymbol{u}}+j\vec{\boldsymbol{w}})/2,$$

where bivector part of $\boldsymbol{N}$ must be unchanged because of linear independency. Now we have $u = w$, $v^2 - w^2 = 1 = v^2 - u^2$, $\vec{\boldsymbol{v}} = \boldsymbol{n} + \vec{\boldsymbol{u}}$. Vectors $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{u}}$ belong to plane defined by bivector $j\vec{\boldsymbol{w}}$ so must unit vector $\boldsymbol{n}$, too. From $\vec{\boldsymbol{v}} = \boldsymbol{n} + \vec{\boldsymbol{u}} \Rightarrow v^2 = 1 + u^2 + 2\boldsymbol{n}\cdot\vec{\boldsymbol{u}}$ and so must be $\boldsymbol{n} \perp \vec{\boldsymbol{u}}$. We can find vectors $\boldsymbol{n}$, $\vec{\boldsymbol{u}}$ explicitly (fig. 2). Defining a unit vector $\boldsymbol{m} = \hat{\boldsymbol{w}} \times \hat{\boldsymbol{v}}$ (*cross product*) we have $(v = \cosh\varphi,\quad w = \sinh\varphi)$

$$\vec{\boldsymbol{u}}_{\pm} = \hat{\boldsymbol{v}} w\cos\alpha \pm \boldsymbol{m} w\sin\alpha = wv^{-1}(\hat{\boldsymbol{v}}w \pm \boldsymbol{m}) = (\hat{\boldsymbol{v}}\sinh\varphi \pm \hat{\boldsymbol{w}}\times\hat{\boldsymbol{v}})\tanh\varphi, \tag{A.8}$$

$$\boldsymbol{n}_{\mp} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}_{\pm} = (\hat{\boldsymbol{v}} \pm \hat{\boldsymbol{w}}\times\hat{\boldsymbol{v}}\sinh\varphi)\cosh^{-1}\varphi. \tag{A.8a}$$

Scalar 1 is trivial idempotent of algebra, but has decomposition

$$1 = (1+\boldsymbol{F}_1)/2 + (1-\boldsymbol{F}_1)/2,\quad (1+\boldsymbol{F}_1)(1-\boldsymbol{F}_1) = 0. \tag{A.9}$$

Using whirl $\boldsymbol{F}_1$ we could define involutions $I_R(M) = M\boldsymbol{F}_1, I_L(M) = \boldsymbol{F}_1 M$, $I(M) = \pm\boldsymbol{F}_1 M\boldsymbol{F}_1$, where right and left involutions are somewhat permutations of components of multivector in Clifford basis, while last one in special form $\boldsymbol{F}_1 = \boldsymbol{n} + j\boldsymbol{m},\ \boldsymbol{n}\perp\boldsymbol{m},\ n = m = 1$ represents interesting reflections.

3) $\boldsymbol{F}_{-1}^2 = -1 \Rightarrow v^2 - w^2 = -1 \Rightarrow \boldsymbol{F}_{-1} = \hat{\boldsymbol{v}} sinh\varphi + j\hat{\boldsymbol{v}}_{\perp} cosh\varphi$. (A.10)

This is ([3]) non-trivial solution for $\sqrt{-1}$. We have $\exp(\pi\boldsymbol{F}_{-1}/2) = \boldsymbol{F}_{-1}$, $\boldsymbol{F}_{-1}^{-1} = -\boldsymbol{F}_{-1}$. It is interesting to look at periodicity of integer powers of multivector $(1 \pm \boldsymbol{F}_{-1})/\sqrt{2}$.

Everything is "boost"

For complex vector $\boldsymbol{F} = \vec{\boldsymbol{v}} + j\vec{\boldsymbol{w}}$ we have $\sqrt{\boldsymbol{F}^2} \in \mathbb{C}$ or $\sqrt{\boldsymbol{F}^2} = \boldsymbol{F}$, so for $\boldsymbol{F} \neq \boldsymbol{N}$, $\boldsymbol{N}^2 = 0$ we define $\boldsymbol{F}/\sqrt{\boldsymbol{F}^2} = \boldsymbol{F}_1$, $\boldsymbol{F}_1^2 = 1$ and $\boldsymbol{F}/\sqrt{-\boldsymbol{F}^2} = \hat{\boldsymbol{F}} = -j\boldsymbol{F}_1$. Suppose we have exponential form $\exp(\varphi\boldsymbol{F}_1)$, defining ([2, 3]) $W = \sqrt{\boldsymbol{F}^2}$, $\boldsymbol{F}_1 = \boldsymbol{F}/W$, $\tanh\varphi = W$, $\Gamma = 1/\sqrt{1-W^2}$, $\kappa = \sqrt{(1+W)/(1-W)} = \Gamma(1+W)$ (generalized *Bondi factor*, $\varphi = \log\kappa$) and idempotents $\boldsymbol{f}_{\pm} = (1 \pm \boldsymbol{F}_1)/2$, $\boldsymbol{f}_+\boldsymbol{f}_- = 0$ we have

$$e^{\varphi\boldsymbol{F}_1} = \cosh\varphi + \boldsymbol{F}_1 \sinh\varphi = \Gamma(1 + W\boldsymbol{F}_1) = \kappa\boldsymbol{f}_+ + \kappa^{-1}\boldsymbol{f}_-. \quad \text{(A.11)}$$

Now we can read "speed" as ratio *W*/1 and it is easy to find successive "boosts" as

$$e^{\varphi_1\boldsymbol{F}_1} e^{\varphi_2\boldsymbol{F}_1} = e^{(\varphi_1+\varphi_2)\boldsymbol{F}_1} = \Gamma_1\Gamma_2(1+W_1\boldsymbol{F}_1)(1+W_2\boldsymbol{F}_1) = \Gamma_1\Gamma_2(1+W_1W_2)\left(1+\frac{W_1+W_2}{1+W_1W_2}\boldsymbol{F}_1\right), \quad \text{(A.12)}$$

$$\Gamma = \Gamma_1\Gamma_2(1+W_1W_2), \quad W = \frac{W_1+W_2}{1+W_1W_2}, \quad \text{(A.12a)}$$

or

$$e^{\varphi_1\boldsymbol{F}_1} e^{\varphi_2\boldsymbol{F}_1} = (\kappa_1\boldsymbol{f}_+ + \kappa_1^{-1}\boldsymbol{f}_-)(\kappa_2\boldsymbol{f}_+ + \kappa_2^{-1}\boldsymbol{f}_-) = \kappa_1\kappa_2\boldsymbol{f}_+ + \kappa_1^{-1}\kappa_2^{-1}\boldsymbol{f}_- \Rightarrow \kappa = \kappa_1\kappa_2. \quad \text{(A.13)}$$

Generally we have a complex scalar $\varphi = \log\kappa = \varphi_{\mathrm{R}} + j\varphi_{\mathrm{I}}$ (explicit formulae are rather cumbersome, one can use *Mathematica* and $j \to i$, *i* is ordinary imaginary unit) which leads to $\exp(\varphi\boldsymbol{F}_1) = \exp(\varphi_{\mathrm{R}}\boldsymbol{F}_1)\exp(\varphi_{\mathrm{I}}j\boldsymbol{F}_1)$.

From $\boldsymbol{F} = \vec{\boldsymbol{v}} + j\vec{\boldsymbol{w}}$, $W = \sqrt{(\vec{\boldsymbol{v}} + j\vec{\boldsymbol{w}})^2} = \sqrt{v^2w^2 + 2j\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}$, for $\vec{\boldsymbol{w}} = 0$ we have well known relations for boosts.

From $\boldsymbol{F} = j\vec{\boldsymbol{w}}$ we have $W = \sqrt{(j\vec{\boldsymbol{w}})^2} = \sqrt{-w^2} = jw$, $\boldsymbol{F}_1 = j\vec{\boldsymbol{w}}/jw = \hat{\boldsymbol{w}}$, $\Gamma = 1/\sqrt{1+w^2}$, $\kappa = \sqrt{(1+jw)/(1-jw)}$, $\varphi = \log\kappa = j\arctan w$, $\exp(\varphi\hat{\boldsymbol{w}}) = \Gamma(1 + jw\hat{\boldsymbol{w}})$ and for successive transformations we have

$$\Gamma = \Gamma_1\Gamma_2(1 - w_1w_2), \quad w = (w_1 + w_2)/(1 - w_1w_2). \quad \text{(A.14)}$$

It is interesting possibility to interpret such transformations like "boosts", defining new rotating frame of reference with time $t = \Gamma\tau$, introducing thus rotating frames as "inertial". Regarding invariance of MA instigates to reexamine paradigm "inertial frame of reference".

For a well known pure rotations $\exp(\theta j\hat{\boldsymbol{n}})$ we have $\theta j\hat{\boldsymbol{n}} = j\theta(j\theta\hat{\boldsymbol{n}})/j\theta$, $\varphi = j\theta$, $W = \tanh(j\theta) = j\tan\theta$, $\Gamma = 1/\sqrt{1+\tan^2\theta}$, $\kappa = \sqrt{(1+j\tan\theta)/(1-j\tan\theta)}$, $\boldsymbol{f}_\pm = (1\pm\hat{\boldsymbol{n}})/2$ and so

$$\mathrm{e}^{\theta j\hat{\boldsymbol{n}}} = \Gamma(1+j\hat{\boldsymbol{n}}\tan\theta), \qquad \text{(A.15a)}$$

$$\Gamma = \Gamma_1\Gamma_2(1-\tan\theta_1\tan\theta_2), \qquad \text{(A.15b)}$$

$$\tan\theta = (\tan\theta_1+\tan\theta_2)/(1-\tan\theta_1\tan\theta_2) = \tan(\theta_1+\theta_2). \qquad \text{(A.15c)}$$

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