Does Geometric Algebra provide a loophole to Bell's Theorem?

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Abstract

Geometric algebra has been proposed as an alternative framework to the quantum mechanics of interacting qubits in a number of pioneering papers by Chris Doran, Anthony Lasenby and others, building on the foundations laid by David Hestenes. This line of work is summarised in two chapters of the book Doran and Lasenby (2003). Since then however, the approach has been pretty much completely neglected, with one exception: in 2007 Joy Christian published the first of a series of works, culminating in a book Christian (2014), which took off in a completely different direction: he claimed to have refuted Bell's theorem and to have obtained a local realistic model of the famous singlet correlations by taking account of the geometry of space, as expressed through geometric algebra. His geometric algebraic model of the singlet correlations is completely different from that of Doran and Lasenby.

One of the aims of the paper is to explore geometric algebra as a tool for quantum information and to explain why it did not live up to its early promise. The short answer is: because the mapping between 3D geometry and the mathematics of one qubit is already thoroughly understood, while the extension to a system of entangled qubits does not bring in new geometric insights but on the contrary merely reproduces the usual complex Hilbert space approach in a slightly clumsy way. We also work through the mathematical core of two of Christian's shortest, least technical, and most accessible works (Christian 2007, 2011), exposing both a conceptual and an algebraic error at their heart.

1 Introduction

In 2007, Joy Christian surprised the world with his announcement that Bell's theorem was incorrect, because, according to Christian, Bell had unnecessarily restricted the co-domain of the measurement outcomes to be the traditional real numbers. Christian's first publication on arXiv spawned a host of rebuttals and these in turn spawned rebuttals of rebuttals by Christian. Moreover, Christian also published many sequels to his first paper, most of which became chapters of his book Christian (2014), second edition; the first edition came out in 2012.

His original paper Christian (2007) took the Clifford algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ as outcome space. This is "the" basic geometric algebra for the geometry of $\mathbb{R}^3$, but geometric algebra goes far beyond this. Its roots are indeed with Clifford in the nineteenth century but the emphasis on geometry and the discovery of new geometric features is due to the pioneering work of David Hestenes who likes to promote geometric algebra as *the* language of physics, due to its seamless combination of algebra and geometry. The reader is referred to the standard texts Doran and Lasenby (2003) and Dorst, Fontijne and Mann (2007); the former focussing on applications in physics, the latter on applications in computer science, especially computer graphics. The wikipedia pages on geometric algebra, and on Clifford algebra, are two splendid mines of information, but the connections between the two sometimes hard to decode.

David Hestenes was interested in marrying quantum theory and geometric algebra. A number of authors, foremost among them Doran and Lazenby, explored the possibility of rewriting quantum information theory in the language of geometric algebra. This research programme culminated in the previously mentioned textbook Doran and Lasenby (2003) which contains two whole chapters carefully elucidating this approach. However, it did not catch on, for reasons which I will attempt to explain at the end of this paper.

It seems that many of Christian's critics were not familiar enough with geometric algebra in order to work through Christian's papers in detail. Thus the criticism of his work was often based on general structural features of the model. One of the few authors who did go to the trouble to work through the mathematical details was Florin Moldoveanu. But this led to new problems: the mathematical details of many of Christian's papers seem to be different: the target is a moving target. So Christian claimed again and again that the critics had simply misunderstood him; he came up with more and more elaborate versions of his theory, whereby he also claimed to have countered all the objections which had been raised.

It seems to this author that it is better to focus on the shortest and most

self-contained presentations by Christian since these are the ones which a newcomer has the best chance of actually "understanding" totally, in the sense of being able to work through the *mathematics*, from beginning to end. In this way he or she can also get a feeling for whether the accompanying *words* (the English text) actually reflect the mathematics. Because of unfamiliarity with technical aspects and the sheer volume of knowledge which is encapsulated under the heading "geometric algebra", readers tend to follow the *words*, and just trust that the (impressive looking) formulas match the words.

In order to work through the mathematics, some knowledge of the basics of geometric algebra is needed. That is difficult for the novice to come by, and almost everyone is a novice, despite the popularising efforts of David Hestenes and others, and the popularity of geometric algebra as a computational tool in computer graphics. Wikipedia has a page on "Clifford algebra" and another page on "geometric algebra" but it is hard to find out what is the connection between the two. There are different notations around and there is a plethora of different products: geometric product, wedge product, dot product, outer product. The standard introductory texts on geometric algebra start with the familiar 3D vector products (dot product, cross product) and only come to an abstract or general definition of geometric product after several chapters of preliminaries. The algebraic literature jumps straight into abstract Clifford algebras, without geometry, and wastes little time on that one particular Clifford algebra which happens to have so many connections with the geometry of three dimensional Euclidean space, *and* with the mathematics of the qubit.

Interestingly, in recent years David Hestenes also has ardently championed mathematical modelling as the study of stand-alone mathematical realities. I think the real problem with Christian's works is that the mathematics does not match the words; the mismatch is so great that one cannot actually identify a well-defined mathematical world behind the verbal description and corresponding to the mathematical formulas. Since his model assumptions do not correspond to a well-defined mathematical world, it is not possible to hold an objective discussion about properties of the model. As a mathematical statistician, the present author is particularly aware of the distinction between mathematical model and reality – one has to admit that a mathematical model used in some statistical application, e.g., in psychology or economics, is really "just" a model, in the sense of being a tiny and oversimplified representation of the scientific phenomenon of interest. All models are wrong, but some are useful. In physics however, the distinction between model and reality is not so commonly made. Physicists use mathematics as a language, the language of nature, to describe reality. They do not use it as a

tool for creating artificial toy realities. For a mathematician, a mathematical model has its own reality in the world of mathematics.

The present author already published (on arXiv) a mathematical analysis, Gill (2012), of one of Christian's shortest papers: the so-called "one page paper", Christian (2011), which moreover contains the substance of the first chapter of Christian's book. In the present paper he analyses in the same spirit the paper Christian (2007), which was the foundation or starting shot in Christian's project. The present note will lay bare the same fundamental issues which are present in all these works. In fact, the model is not precisely the same in every publication, but constantly changes. We will find out that Christian's (2007) model is not complete: the central feature, a definition of local measurement functions, is omitted. However there is little choice in how to fill the gap, and that is done in the 2011 version (though the author does not state explicitly that he is actually adding something to the earlier version of the model).

What is common to both these short versions of Christian's theory is a basic conceptual error and a simple algebraic (sign) error. I think it is important to identify these with maximal clarity and that is why it is best to focus on the shortest and most self-contained presentations of the theory. One sees that the model is based on a dream which evaporates in the cold light of day. If one only uses mathematics as an intuitive language in which to describe *physical* insights, if a particular mathematical formalism does not match physical intuition, then the mathematics is at fault: not the intuition. However if there is no stand-alone mathematical model behind the dream... then it remains a dream which cannot be shared with other scientists.

I shall return to both projects: Doran and Lazenby's project to geometrise quantum information, and Christian's to disprove Bell; at the end of this paper. First it is necessary to set up some basic theory of geometric algebra. What is the Clifford algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ and what does it have to do with Euclidean (three-dimensional) geometry? What does it have to do with the quantum information theory of a system of qubits? We will actually start very close to a single qubit by discussing an alternative way of looking at the two-by-two complex matrices. Since each of the four complex entries of such a matrix has a real and an imaginary part, the whole matrix is determined by 8 real numbers. Restricting attention to the addition and subtraction of these matrices, and scalar multiplication, we are looking at an eight-dimensional real vector space. We will study the interaction between the real vector space structure and the operation of matrix multiplication, together giving us what the pure mathematicians call a *real algebra*.

2   Geometric Algebra

I will first discuss the real algebra of the two-by-two complex matrices. We can add such matrices and obtain a new one; we can multiply two such matrices and obtain a new one. The zero matrix acts as a zero and the identity matrix acts as a unit. We can multiply a two-by-two complex matrix by a *real*. In this role, we talk about scalar multiplication. We can therefore take *real* linear combinations. Matrix multiplication is associative so we have a *real unital associative algebra*. Unital just means: with a unit; real means that the scalars in scalar multiplication are reals, and therefore, as a vector space (i.e., restricting attention to addition and scalar multiplication), our algebra is a vector space over the real numbers.

The four complex number entries in a two-by-two complex matrix can be separated into their real and imaginary parts. In this way, each two-by-two complex matrix can be expressed as a *real* linear combination of eight "basis" matrices: the matrices having either $+1$ or i in just one position, and zeros in the other three positions. Obviously, if a real linear combination of those 8 matrices is zero, then all eight coefficients are zero. The vector-space dimension of our space is therefore eight.

I will now specify an alternative vector-space basis of our space. Define the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define

$$\mathbb{1} = I, \ e_1 = \sigma_x, \ e_2 = \sigma_y, \ e_3 = \sigma_z, \ \beta_1 = i\sigma_x, \ \beta_2 = i\sigma_y, \ \beta_3 = i\sigma_z, \ M = iI.$$

It is not difficult to check that this eight-tuple is also a vector space basis. Note the following:

$$e_1^2 = e_2^2 = e_3^2 = \mathbb{1},$$

$$e_1 e_2 = -e_2 e_1 = -\beta_3, e_2 e_3 = -e_3 e_2 = -\beta_2, e_3 e_1 = -e_1 e_3 = -\beta_1,$$

$$e_1 e_2 e_3 = M.$$

These relations show us that the two-by-two complex matrices, thought of as an algebra over the reals, are isomorphic to $\mathcal{C}\ell_{3,0}(\mathbb{R})$, or if you prefer, form a *representation* of this algebra. By definition, $\mathcal{C}\ell_{3,0}(\mathbb{R})$, is the associative

unital algebra over the reals generated from $\mathbb{1}$, e_1, e_2 and e_3, with three of e_1, e_2 and e_3 squaring to $\mathbb{1}$ and none of them squaring to $-\mathbb{1}$, which is the meaning of the "3" and the "0" : three squares are positive, zero are negative. Moreover, the three basis elements e_1, e_2 and e_3 anti-commute. The largest possible algebra over the real numbers which can be created with these rules has vector space dimension $2^{3+0} = 8$ and as a real vector space, a basis can be taken to be $\mathbb{1}$, e_1, e_2, e_3, $e_1e_2 = -\beta_3$, $e_1e_3 = \beta_2$, $e_2e_3 = -\beta_1$, $e_1e_2e_3 = M$. The algebra is associative (like all Clifford algebras), but not commutative.

This description does not quite tell you the "official" definition of Clifford algebra in general, but is sufficient for our purposes. We are interested in just one particular Clifford algebra, $\mathcal{C}\ell_{3,0}(\mathbb{R})$, which is *the* Clifford algebra of three dimensional real geometry. It is called the (or a) *geometric algebra* and the product is called the geometric product, for reasons which will soon become clear.

I wrote $\mathbb{1}$ for the identity matrix and will later also write $\mathbb{0}$ for the zero matrix, as one step towards actually indentifying these with the scalars 1 and 0. However, for the time being, we should remember that we are not (at the outset) talking about numbers, but about elements of a complex matrix algebra over the reals. Complex matrices can be added, multiplied, and multiplied by scalars (reals).

Within this algebra, various famous and familiar structures are embedded.

To start with something famous, we can identify the quaternions. Look at just the four elements $\mathbb{1}, \beta_1, \beta_2, \beta_3$. Consider all real linear combinations of these four. Notice that

$$\beta_1^2 = \beta_2^2 = \beta_3^2 = -\mathbb{1},$$

$$\beta_1\beta_2 = -\beta_2\beta_1 = -\beta_3,$$

$$\beta_2\beta_3 = -\beta_3\beta_2 = -\beta_1,$$

$$\beta_3\beta_1 = -\beta_1\beta_3 = -\beta_2.$$

This is close to the conventional multiplication table of the quaternionic roots of minus one: the only thing that is wrong is the last minus sign in each of the last three rows. However, the sign can be fixed in many ways: for instance, by taking an odd permutation of $(\beta_1, \beta_2, \beta_3)$, or an odd number of sign changes of elements of $(\beta_1, \beta_2, \beta_3)$. In particular, changing the signs of all three does the job. But also the triple $e_1e_2 = -\beta_3, e_1e_3 = \beta_2, e_2e_3 = -\beta_1$ (the order is crucial) does the job too. Notice that we reversed the order of the three β_js, which is an odd permutation, *and* changed two signs, an even number. We ended up with e_1e_2, e_1e_3, e_2e_3 in the conventional *lexicographic* ordering.

Switching some signs or permuting the order does not change the algebra generated by $\mathbb{1}, \beta_1, \beta_2, \beta_3$. These four elements do generate a sub-algebra of $\mathcal{Cl}_{3,0}(\mathbb{R})$ which is isomorphic to the quaternions. The quaternion sub-algebra is called the "even sub-algebra" of our Clifford algebra since it is built from just the products of even numbers of e_1, e_2 and e_3.

Obviously, M commutes with everything, and obviously $M^2 = -\mathbb{1}$. Also $Me_1 = \beta_1$, $Me_2 = \beta_2$, $Me_3 = \beta_3$ and, in duality with this, $e_1 = -M\beta_1$, $e_2 = -M\beta_2$, $e_3 = -M\beta_3$. Thus any element of $\mathcal{Cl}_{3,0}(\mathbb{R})$ can be expressed in the form $p + Mq$ where p and q are quaternions, M commutes with the quaternions, and M is itself another square root of minus one. However one must be careful with parametrisation and be aware of "left-handed" and "right-handed" ways to define and work with the quaternions.

Notice that $(\mathbb{1} - e_i)(\mathbb{1} + e_i) = \mathbb{0}$ (the zero two-by-two matrix). So the algebra $\mathcal{Cl}_{3,0}(\mathbb{R})$ possesses zero divisors (in fact, very many!), and hence not all of its elements have inverses.

The real reason for calling this algebra a *geometric algebra* comes from locating real 3D space within it, and also recognising geometric operations and further geometric structures with the algebra. To start with, look at the linear span of e_1, e_2 and e_3. This can be identified with $\mathbb{R}^3$: from now on, real 3D vectors *are* real linear combinations of e_1, e_2 and e_3. So let us look at two elements a, b of $\mathbb{R}^3$, thus simultaneously elements of the linear span of e_1, e_2 and e_3. In the latter incarnation (i.e., as elements of $\mathcal{Cl}_{3,0}(\mathbb{R})$), we can multiply them; what do we find? The answer is easily seen to be the following:

$$ab = (a \cdot b)\,\mathbb{1} + M\,(a \times b).$$

Here, $a \cdot b$ stands for the (ordinary) inner product of two real (three dimensional) vectors, hence a real number, or a scalar; $a \times b$ stands for the (ordinary) cross product of two real three dimensional vectors, hence a real vector. As such, it is a real linear combination of e_1, e_2 and e_3. Multiplying by M gives us the same real linear combination of β_1, β_2, β_3.

Thus the geometric algebra $\mathcal{Cl}_{3,0}(\mathbb{R})$ contains as a linear subspace the real vectors of $\mathbb{R}^3$; the *geometric product* of two such elements encodes both their vector dot product and their vector cross product. The dot product and the cross product of real vectors can both be recovered from the geometric product, since these parts of the geometric product live in parts of the eight dimensional real linear space $\mathcal{Cl}_{3,0}(\mathbb{R})$ spanned by disjoint sets of basis elements. The real linear subspace generated by β_1, β_2, β_3 is called the set of *bivectors*, just as the real linear subspace generated by e_1, e_2, e_3 is identified with the set of *vectors*. The real linear subspace generated by the single element $\mathbb{1}$ is identified with the real numbers or scalars and in particular, we

identify the scalar 1 and the element $\mathbb{1}$, the scalar 0 and the element $\mathbb{0}$. The real linear subspace generated by the single element M is called the set of *trivectors*, also known as *pseudo-scalars*.

So we have seen that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ is a beautiful object, containing within itself the quaternions, the complex numbers, the two-by-two complex matrices, three dimensional real vectors; and its product, sometimes called "geometric product", contains within it both the inner and outer product of real vectors. Moreover, everything comes in dual pairs: the vectors and the bivectors; the quaternions and the product of the quaternions with M, the scalars and the pseudo-scalars. Every element of the algebra can be split into four components: scalar, vector, bivector and pseudo-scalar. The algebra is called *graded*. The parts we just named constitute the zeroth, first, second and third grades of the algebra. The product of elements of grades r and s belong to the grade $r + s$ modulo 4.

The fact that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ has as a representation the algebra of complex two-by-two matrices over the reals is, on the whole, not much more than a side-note in the literature on geometric algebra. In the theory of Clifford algebra, it is just one tiny piece in the theory of classification of Clifford algebras in general. It seems to this author that this representation should help very much to "anchor" the theory for those coming from quantum theory and quantum foundations. However, it must be pointed out that the standard textbook on geometric algebra for physicists, Doran and Lasenby (2003), has two whole chapters on quantum mechanics, the first containing a geometric algebraic representation of the quantum theory of one spin half particle, the second doing the same for two entangled spin half particles. This even contains a carefully worked out analysis of the singlet state, including derivation of the famous EPR-B correlation $-\cos(a \cdot b)$. The one-particle analysis uses the even sub-algebra of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ and the two-particle analysis uses an 8-dimensional subspace of the product of the 16-dimensional algebra (but not a Clifford algebra) $\mathcal{C}\ell_{3,0}(\mathbb{R})$ with itself.

The reader is referred to Chapters 8 and 9 of Doran and Lasenby (2003), whose endnotes contain many, many further references. An important point to notice is that prior to Christian's work, there already was a large literature devoted to expressing quantum theory in terms of geometric algebra. Though it seems that this literature did not gain a place in mainstream quantum theory, one of Christian's aims had already been fulfilled: geometric algebra had already been shown to be a suitable vehicle for re-writing quantum theory with geometry instead of Hilbert space at the forefront. I will later try to explain why, I believe, Doran and Lazenby's project to geometrise quantum information theory never caught on.

2.1 Remarks on computation

Before continuing, I would like to make some remarks on computation. In order to test formulas, or in order to simulate models, it is convenient to have access to computer languages which can "do" computer algebra.

First of all, the fact that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ can be identified with the two-by-two complex matrices means that one can implement geometric algebra as soon as one can "do" complex matrices. One needs to figure out how to get the eight coordinates out of the matrix. There are easy formulas but the result would be clumsy and involve a lot of programming at a low level. The relation with the quaternions is more promising: we can represent any element of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ as a pair of quaternions, and the real and the imaginary parts of the two component quaternions immediately give us access to the scalar and bivector, respectively trivector (also called "pseudo scalar") and vector parts of the element of the algebra. For those who like to programme in the statistical language `R` it is good to know that there is an `R` package called "`onion`" which implements both the quaternions and the octonions.

Here a small digression starts. There exists a huge menagerie of somewhat related mathematical objects, and sometimes notations and nomenclature are confused and confusing. For instance, the octonions are another eight dimensional algebra which however is not associative. There are weird and wonderful and deep connections between these various objects and in particular, concerning the sequence 1, 2, 4, 8, which is where a particular construction taking us from the reals to the complex numbers to the quaternions and finally to the octonions stops. But this is not part of our story. Another Clifford algebra $\mathcal{C}\ell_{0,3}(\mathbb{R})$ is also related to pairs of quaternions, but also not part of our story. End of digression.

Nicest of all is to use a computer system for doing real geometric algebra, and for this purpose I highly recommend the programme `GAViewer` which accompanies the book Dorst, Fontijne and Mann (2007). It can be obtained from the authors' own book website `http://www.geometricalgebra.net`. It not only does geometric algebra computations, it also visualises them, i.e., connects to the associated geometry. Moreover the book and the programme are a nice starting point for two higher dimensional geometric algebras, of dimension 16 and 32 respectively, which encode more geometric objects (for instance, circles and affine subspaces), and more geometric operations, with the help of the further dimensions of the algebra.

3 Christian's first model

After several pages of general introduction, Christian (2007) gives a very brief specification of his (allegedly) local realist model for the singlet (or EPR-B) correlations, obtained through the device of taking the co-domain of Bell's measurement functions to be elements of the geometric algebra $\mathcal{Cl}_{3,0}(\mathbb{R})$ rather than the conventional (one dimensional) real line.

Conventionally, a local hidden variables model for the singlet correlations consists of the following ingredients. First of all, there is a hidden variable λ which is an element of some arbitrary space over which there is a probability distribution referred to in physicist's language sometimes as $\rho(\lambda)\mathrm{d}\lambda$, sometimes as $\mathrm{d}\rho(\lambda)$. This hidden variable is often thought to reside in the two particles sent to the two measurement devices in the two wings of the experiment, and therefore to come from the source; but one can also think of λ as an assemblage of classical variables in the source and in both particles and in both measurement devices which together determine the outcomes of measurement at the two locations. Any "local stochastic" hidden variables model can also be re-written as a deterministic local hidden variables model. This rewriting (thinking of random variables as simply deterministic functions of some lower level random elements) might not correspond to physical intuition but as a mathematical device it is a legitimate and powerful simplifying agent.

Secondly, we have two functions $A(a, \lambda)$ and $B(b, \lambda)$ which take the values ± 1 only, and which denote the measurement outcomes at Alice's and Bob's locations, when Alice uses measurement setting a and Bob uses measurement setting b. Here, a and b are 3D spatial directions conventionally represented by unit vectors in $\mathbb{R}^3$. The set of unit vectors is of course also known as the unit sphere S^2.

Bell's theorem states that there do not exist functions A and B and a probability distribution ρ, on any space of possible λ whatever, such that

$$\int A(a, \lambda)B(b, \lambda)\mathrm{d}\rho(\lambda) = -a \cdot b$$

for all a and b in S^2.

Christian claims to have a counter-example and the first step in his claim is that Bell "unphysically" restricted the co-domain of the functions A and B to be the real line. Now this is a curious line to take: we are supposed to assume that A and B take values in the two-point set $\{-1, +1\}$. In fact, the correlation between A and B in such a context is merely the probability of getting equal (binary) outcomes minus the probability of getting different (binary) outcomes. In other words: Bell's theorem is about measurements

which can only take on two different values, and it is merely by convention that we associate those values with the *numbers* -1 and $+1$. We could just as well have called them "spin up" and "spin down". In the language of probability theory, we can identify λ with the element ω of an underlying probability space, and we have two families of random variables A_a and B_b, taking values in a two point set, *without loss of generality* the set $\{-1, +1\}$, and Bell's claim is: it is impossible to have $\mathrm{Prob}(A_a = B_b) - \mathrm{Prob}(A_a \neq B_b) = a \cdot b$ for all a and b.

However, let us bear with Christian, and allow that the functions A and B might just as well be thought of as taking values in a geometric algebra ... as long as we insist that they each only take on two different values.

Christian used the symbol $\mathbf{n}$ to denote an arbitrary unit vector (element of S^2) and in formulas (15) and (16) makes the following bold suggestion:

$$A_{\mathbf{n}} = B_{\mathbf{n}} = \boldsymbol{\mu} \cdot \mathbf{n} \cong \pm 1 \in S^2$$

where $\boldsymbol{\mu} \cdot \mathbf{n}$ has been previously defined to be $\pm M\mathbf{n}$ (I use the symbol M instead of Christian's I). Christian talks about the dot here standing for a "projection" referring, presumably, to the duality between vectors and bivectors. Christian sees $\boldsymbol{\mu}$ as his local hidden variable, giving us the story that space itself picks at random a "handedness" for the trivector $M = e_1 e_2 e_3$, thought of as a *directed volume element*. This story is odd; after all, the "handedness" of $e_1 e_2 e_3$ is merely the expression of the sign of the evenness of the permutation 123. Of course, the multiplication rule of geometric algebra, bringing up the *cross product* does again involve a handedness convention: but this is nothing to do with physics, it is only to do with mathematical convention, i.e., with book-keeping.

But anyway, within Christian's "model" as we have it so far, we can just as well define λ to be a completely random element of $\{-1, +1\}$, and then define $\boldsymbol{\mu} = \lambda M$. The resulting probability distribution of $\boldsymbol{\mu}$ is the same; we have merely changed some names, without changing the model.

So now we have the model

$$A(\mathbf{a}, \lambda) = \lambda M \mathbf{a}, \quad B(\mathbf{b}, \lambda) = \lambda M \mathbf{b}$$

which says that the two measurement functions have outcomes in the set of pure (unit length) bivectors. Now, those two sets are both isomorphic to S^2, and that is presumably the meaning intended by Christian when using the congruency symbol $\cong$: our measurements can be thought of as taking values in S^2. At the same time, each measurement takes on only one of two different values (given the measurement direction) hence we can also claim congruency with the two point set $\{-1, +1\} = \{\pm 1\}$. But of course, these

are two different congruencies! And they still need to be sorted out. What is mapped to what, exactly ...

This is where things go badly wrong. On the one hand, the model is not yet specified, if Christian does not tell us how, exactly, he means to map the set of two possible values $\pm M\mathbf{a}$ onto $\{\pm 1\}$ and how he means to map the set of two possible values $\pm M\mathbf{b}$ onto $\{\pm 1\}$. On the other hand, Christian proceeds to compute a correlation (in the physicist's sense: the expectation value of a product) between bivector valued outcomes instead of between $\{\pm 1\}$ valued outcomes. No model, wrong correlation.

Let us take a look at each mis-step separately. Regarding the first (incomplete specification), there are actually only two options (without introducing completely new elements into the story), since Christian already essentially told us that the two values ± 1 of (my) λ are equally likely. Without loss of generality, his model (just for these two measurement directions) becomes *either*

$$A(\mathbf{a}, \lambda) = B(\mathbf{b}, \lambda) = \lambda = \pm 1$$

or

$$-A(\mathbf{a}, \lambda) = B(\mathbf{b}, \lambda) = \lambda = \pm 1$$

and hence his correlation is either $+1$ or -1, respectively.

Regarding the second mis-step, Christian proceeds to compute the geometric product $(\boldsymbol{\mu} \cdot \mathbf{a})(\boldsymbol{\mu} \cdot \mathbf{b})$ (later, he averages this over the possible values of μ). Now as we have seen this is equal to $(\lambda M \mathbf{a})(\lambda M \mathbf{b}) = \lambda^2 M^2 \mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} - M(\mathbf{a} \times \mathbf{b})$ and therefore certainly *not* equal to $-\mathbf{a} \cdot \mathbf{b} - \lambda M(\mathbf{a} \times \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} - \boldsymbol{\mu}(\mathbf{a} \times \mathbf{b})$.

4   Christian's second model

I next would like to take the reader to Christian's one page paper, Christian (2011), simultaneously the main material of the first chapter of his book Christian (2014).

It seems clear to me that by 2011, Christian himself has realised that his "model" of 2007 was incomplete: there was no explicit definition of the measurement functions. So now he does come up with a model, and the model is astoundingly simple ... it is identical to my second model:

$$A(\mathbf{a}, \lambda) = -B(\mathbf{b}, \lambda) = \lambda = \pm 1.$$

However, he still needs to get the singlet correlation from this, and for that purpose, he daringly *redefines* correlation, by noting the following: associated with the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are the unit bivectors (in my notation) $M\mathbf{a}$ and

$M\mathbf{b}$. As purely imaginary elements of the bivector algebra or quaternions, these are square roots of minus one, and we write

$$A(\mathbf{a}, \lambda) = (M\mathbf{a})(-\lambda M\mathbf{a}) = \lambda$$

$$A(\mathbf{b}, \lambda) = (\lambda M\mathbf{b})(M\mathbf{b}) = -\lambda$$

where λ is a "fair coin toss", i.e., equal to ± 1 with equal probabilities $\frac{1}{2}$.

Now the cunning device of representing these two random variables as products of *fixed* bivectors and *random* bivectors allows Christian to propose a generalised Pearson bivector correlation between A and B by dividing the mean value of the product, by the non random "scale quantities" $M\mathbf{a}$ and $M\mathbf{b}$. In other words, $M\mathbf{a}$ and $M\mathbf{b}$ are thought of as standard deviations. The Clifford algebra valued measurement outcome ± 1 is thought of as having a standard deviation which is a bivectorial root of -1.

Since the geometric product is non commutative, one must be careful about the order of these operations (and whether we divide on the left or the right), but I will follow Christian in taking what seems natural choices: "a" always on the left, "b" on the right.

Unfortunately, since $AB = -1$ with probability one, the Christian-Pearson correlation should be $-(M\mathbf{a})^{-1}(M\mathbf{b})^{-1} = -(M\mathbf{a})(M\mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} - M(\mathbf{a} \times \mathbf{b})$, just as before. However, just as in the 2007 paper, Christian succeeds again in 2011 in getting a sign wrong, so as to "erase" the unwanted bivector cross-product term from the "correlation". In Gill (2012) I have analysed where he went wrong and put forward an explanation of how this mistake could occur (ambiguous notation and inaccurate terminology, driven of course by powerful wishful thinking). It should be noted that he also hides the sign error somewhat deeper in complex computations, looking at the average of a large number of realisations and using the law of large numbers, rather than just computing the expectation "theoretically".

I will not discuss the other versions of Christian's theory. He has elaborated more or less the same "theory" with the same repertoire of conceptual and algebraic errors in various papers with increasing levels of complexity. This work shows a remarkable degree of dedication and persistence, and erudition too, as more and more complex mathematical constructions are brought into play. However the flaws in the foundation are unchanged and one of them (as has been said by many before) is unredeemable: Bell's theorem is about probabilities of different combinations of pairs of binary outcomes. A local hidden variable theory has to explain the correlations between binary outcomes which are predicted by quantum mechanics, following standard interpretations: quantum mechanics predicts the probability distribution of measurement outcomes. Experimenters (in this field) do indeed count discrete events.

5 Geometrizing quantum information theory

I assume that the reader is already familiar with the standard "Bloch sphere" picture of quantum states, quantum operations, and quantum measurements for a single qubit. This means that the first of Doran and Lazenby's two chapters on quantum information hold few surprises. Quantum states can be represented as points in the Bloch ball, thus as real 3D vectors of length less than or equal to one. Pure states lie on the surface of the ball, mixed states in the interior. Unitary evolutions correspond to rotations of the Bloch sphere and these are represented by right and left multiplication with a unitary matrix and its Hermitian conjugate. In geometric algebra, rotation of a 3D real vector is expressed using an object called a "rotor" and the geometric idea is that we do not rotate *around* a vector, but *in* a plane. Oriented planes are represented by *bivectors*.

One curious point is that Doran and Lasenby also attempt to map the vector (wave function) representation of a pure state into the geometric algebra framework. Since the density matrix is a completely adequate representation of any state, pure or mixed, this seems to me to be superfluous. Why not agree that a pure state is represented by its density matrix, hence a point on the unit sphere, hence a real vector of unit length? They do achieve their aim by thinking of state vectors as being the result of a 3D rotation from a fixed reference direction, hence pure states are represented by rotors which involve a choice a reference direction.

All in all, one could argue that rephrasing the mathematics of a single qubit in terms of real geometric algebra demystifies the usual approach: no more complex numbers, just real 3D geometry. Doran and Lasenby hint several times in their book and other publications that their approach also has the promise of delivering an unconventional *interpretation* of quantum mechanics, but they never appear to follow this up.

Having all the mathematical machinery of the complex two-by-two matrices at their disposal, it is a trivial matter to now also extend the discussion to several (entangled) qubits. Indeed, their book contains a reworking of the computation of the singlet correlations for the special case of a maximally entangled pair of qubits.

The approach is simply to take the (outer) product of as many copies of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ as we have qubits. Thus no new geometry is involved and in fact we move out of the realm of Clifford algebras: a product of several Clifford algebras is not a Clifford algebra itself. Certainly it is the case that the outer product of two copies of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ contains, as an algebra, everything that we need relative to the usual Hilbert space approach. However it contains more than we need. The problem is that in the Hilbert space approach, the tensor

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product of the identity with i times the identity is equal to the tensor product of i times the identity with the identity. But taking the outer product of two Clifford algebras defines two distinct elements: $\mathbb{1} \otimes M$ and $M \otimes \mathbb{1}$.

Thus we should form the basic space for a composite system not just by taking a product of spaces for the subsystems but also by adding a further equality $\mathbb{1} \otimes M = M \otimes \mathbb{1}$.

Notice that the two-by-two complex matrices have real dimension eight, while the four-by-four complex matrices have real dimension 32. However, $8 \times 8 = 64$. Thus if we believe the usual Hilbert space story for the mathematics of two qubits, then states, operators, measurements, . . . can all be represented in a real algebra of dimension 32. Adding an extra equality $\mathbb{1} \otimes M = M \otimes \mathbb{1}$ reduces the dimension of $\mathcal{C}\ell_{3,0}(\mathbb{R})^{\otimes 2}$ from 64 to 32 and now we do have exactly the algebra which we need in order to carry the "usual" Hilbert space based theory.

In fact, Doran and Lazenby, taking their cue from their representation of pure states as rotors, representing a rotation from a reference direction to the direction of the state, construct a different reduction by imposing a different equality. Their point is that geometrically, we need to align the reference directions of the two subsystems. This leads to an algebraically more complicated "base space", with perhaps better geometric interpretability, than the obvious algebraic reduction, which comes down to identifying the complex root of minus one, i, as being "the same" in the two Hilbert spaces of the two subsystems: mathematically that is a natural step to take, but what does it mean physically?

The proof of the pudding should be in the eating. On the one hand, everything one can do with the Clifford algebra approach can also be done in the Hilbert space approach: after fixing the dimension issue, one has two mathematically isomorphic systems, and the choice between the two is merely a matter of "picking a coordinate system". So far, it seems that the Clifford algebra approach did not pay off in terms of easier computations or geometric insight, when we look at systems of several qubits. On the other hand, the geometric insight for a single qubit was already available and has become completely familiar. The geometric algebra approach did not lead to new *interpretational* insights in quantum theory. Finally, it does not extend in an easy way to higher-dimensional systems: where is the geometric algebra of the qutrit?

However I hope that this paper will encourage others to take a fresh look for themselves.

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