Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 4 The transfer principle

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Abstract

Between gossamer numbers and the reals, an extended transfer principle founded on approximation is described, with transference between different number systems in both directions, and within the number systems themselves. As a great variety of transfers are possible, hence a mapping notation is given. In $*G$ we find equivalence with a limit with division and comparison to a transfer $*G \mapsto \mathbb{R}$ with comparison.

1 Introduction

The transfer principle in Non-Standard Analysis(NSA) generally translates between the hyperreals $*\mathbb{R}$ and the reals $\mathbb{R}$. We are similarly interested in a transfer principle between the gossamer number system $*G$ [2, Part 1] and real numbers $\mathbb{R}$; including their variants.

For many reasons, we need to work in the more detailed number system. Any such work requires us to interpret or bring back the results into $\mathbb{R}$ or otherwise. We note that $*G$ has $\mathbb{R}$ embedded within, making the transfer from $\mathbb{R}$ to $*G$ possible. However, the nature of a statement in $*G$ may not be able to be expressed in $\mathbb{R}$. While the transfer is possible, the meaning may change. (See Example 2.2)

We provide another view of the transfer principle that is based on approximation, a process with indeterminacy. This is motivated by the fact that if given a number in $*G$ that is not an infinity, we can truncate successive orders of infinitesimals, and when all the infinitesimals are truncated we have only the real component remaining. Truncating all the infinitesimals is defined as the standard part st() function, which results in a transfer from $*G \mapsto \mathbb{R}$.

However, taking just one truncation can change an inequality. Hence, a more general view of a transfer from one state to another is warranted. We can also see the algebra of comparing functions [3] as transfers.

Then truncating a Taylor polynomial is a transfer; $*G \mapsto *G$, as truncating a Taylor polynomial may involve infinitesimals, which are not in $\mathbb{R}$.

More general questions can be asked. Consider the two number systems, one with infinitesimals and infinites, the other the reals. If $a > b$ in $*G$, is this true in $\mathbb{R}$? Under what conditions is this true?

That is, can we in one number system, transfer to the other number system? So, rather than working in reals, and extending the reals which is implicitly done (for example the evaluation of a limit), you can deliberately work in one or the other number systems, and transfer between them.

Surprisingly we are applying the transfer principle all the time, for example in evaluating limits. The limits themselves, having infinitesimals or infinities do not belong in $\mathbb{R}$. By taking the limit, and truncating the infinitesimals that remain, you are effectively taking the standard part of the expression. That this is not discussed but assumed true, is part of our culture.

2 Transference

The transfer principle itself is a realization of the 'Law of continuity'. This law from the outset was used for considering infinite numbers, and their transition.

For geometric surfaces, we can easily visualise the law of continuity applying to computer generated meshes. As the mesh is refined, a smoother surface appears. For a 2D example, at infinity, a polygon of equal sides inside a circle becomes the circle.

That space with infinitesimals existing was predominant in their minds. Leibniz gives the example of two parallel lines infinitesimally close that never meet [12, p.1552]. Du Bois-Reymond constructs an infinity of curves infinitely close therefore parallel to a straight line [3, Part 3 p.12 Example 2.19]. A transfer could be from these curves to the straight line.

The following example is perhaps a more complicated transfer, as a radical state change occurs, but only at infinity. If we describe a fixed ellipse with one focal point at the origin and send the other focal point to infinity. The ellipse becomes a parabola, but only after the variable of the focal point is sent to infinity before the other variables.

Example 2.1. *[11, p.1553–1554] With a closed curve the ellipse becomes an open curve the* *parabola, but only with the focus at infinity. Send the focus h to infinity.*

$$(x^2+y^2)^{\frac{1}{2}} + (x^2+(y-h)^2)^{\frac{1}{2}} = h+2|_{h=\infty}$$
$$x^2+y^2+x^2+(y-h)^2+2((x^2+y^2)(x^2+(y-h)^2))^{\frac{1}{2}} = (h+2)^2|_{h=\infty}$$
$$2x^2+2y^2-2yh+2((x^2+y^2)(x^2+(y-h)^2))^{\frac{1}{2}} = 4h+4|_{h=\infty}$$
$$\text{(Apply non-reversible arithmetic [4, Part 5])}$$
$$(2x^2+2y^2-2yh = -2yh|_{h=\infty} \text{ as } -2yh \succ 2x^2+2y^2|_{h=\infty},\ 4h+4=4h|_{h=\infty})$$
$$((x^2+y^2)(x^2+(y-h)^2))^{\frac{1}{2}} = yh+2h|_{h=\infty}$$
$$(x^2+y^2)(x^2+(y-h)^2) = (h(y+2))^2|_{h=\infty}$$
$$(x^2+y^2)h^2 = h^2(y+2)^2|_{h=\infty}$$
$$x^2 = 4y+4$$

That is a closed curve is broken open. The ellipse is broken to form a parabola at infinity. For any finite values the curve is always closed, and is an ellipse.

The example highlights the directional nature of change. After applying non-reversible arithmetic to the equation, a transfer process takes place to the new state.

Definition 2.1. $\overline{\mathbb{R}} = \mathbb{R} \cup \pm\infty$ *the extended real numbers.*

Example 2.2. *From [7] reformed in* $*G$*. Let* $n \in \mathbb{J}_\infty$*,* $w \in \mathbb{J}$ *be finite then* $\sum_{k=1}^{w} 1 < n|_{n=\infty}$ *cannot be transferred to* $\mathbb{R}$ *because it lacks infinity elements in* $\mathbb{R}$ *then* $*G \not\mapsto \mathbb{R}$*. However, since the extended reals* $\overline{\mathbb{R}}$ *have infinity the transfer is possible;* $*G \mapsto \overline{\mathbb{R}}$*:* $\sum_{k=1}^{w} 1 < \infty$*, which is slightly different as the extended reals* $\overline{\mathbb{R}}$ *only have two infinity elements,* $\pm\infty$*.*

Example 2.3. *In* $*G$*,* $2 + \frac{1}{n} > 2|_{n=\infty}$*, but* $*G \not\mapsto \mathbb{R}$*. However, if we replace the strict inequality to include equality, the transfer is possible.* $2 \geq 2$ *in* $\mathbb{R}$*. (See Theorem 2.6)*

> A transfer principle states that all statements of some language that are true for some structure are true for another structure[7].
>
> A sentence in φ in $L(V(S))$ is true in $V(S)$ if and only if its *-transform ${}^*\varphi$ is true in $V({}^*S)$ [9, p.82]

From Example 2.3 we see the transfer definitions given above are not adequate. While it is very important and most useful to take a proposition in one number system, and have the proposition true in another. For example, theorem proving where if true in one system implies the truth in the other. However, the principle as stated is not complete because a transfer can change the relation's meaning.

We put forward a definition of the extended transfer principle, which in part, is based on approximation. Where, by realizing infinitesimals, we can truncate expressions. By seeing the continued truncation of infinitesimals as a sequence of smaller operations, we can transfer within the same number system.

We find such truncation can describe non-reversible processes, which lead to non-reversible arithmetic [4, Part 5].

The second part of the transfer principle generalization is its directional nature. Transfers exist in both directions.

Possibilities arise from non-uniqueness, for example, a single point of discontinuity in $\mathbb{R}$ can be continuous in $*G$; transferring from $*G$ to the point discontinuity in $\mathbb{R}$ can be done in several ways. Perhaps a deeper transfer is the promotion of an infinitesimal to a small value.

Definition 2.2. *Transfer principle: Assume an implementation of the "Law of continuity" between $\mathbb{R}$ or $\overline{\mathbb{R}}$ and $*G$ or $*\overline{G}$. For each number x in the target space, $x \mapsto x'$ in the image space. If true over the domain in the target space, then it is true in the image space.*

Definition 2.3. *Extended transfer principle: Depending on context we can transfer in either direction, and in any combinations of number systems and operations. Further, dependent on the transfer, the relations may change.*

Example 2.2 is an extended transfer. For further examples see Table 1 Mapping examples.

We differentiate between infinitesimals and zero. Similarly we differentiate between and an infinity such as $n^2|_{n=\infty}$ and the number ∞.

We will define an operation to convert from "an infinitesimal" to zero, and an operation to convert from "an infinity" to infinity. In other words, zero is a generalization of infinitesimals and its own unique number. Similarly, infinity is a generalization of infinities, and its own unique number.

Definition 2.4. *We say "realizing an infinitesimal" is to set the infinitesimal to* 0*, and "realizing an infinity" is to set an infinity to* ∞*.*

With these definitions an infinitesimal is not 0, but a realization of it. "An infinity" is not infinity, but an instance of it. By the 'realization' operation we convert infinitesimals and infinities respectively to 0 or ∞. The numbers 0 or ∞, while mutual inverses, have no specific inverses. After a realization, you cannot go back.

Example 2.4. $\infty \notin \Phi^{-1}$*, but* $\Phi^{-1} = \infty$ *as a left-to-right generalization is true. Similarly* $0 \notin \Phi$*, but* $\Phi = 0$*.*

Example 2.5. *We may have* $n^2|_{n=\infty} = \infty$*. The left side is a specific instance of the right side generalization. Similarly for zero,* $\frac{1}{n}|_{n=\infty} = 0$*.*

Example 2.6. $(\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots)|_{n=\infty} = (0, 0, 0, \ldots)$ *is a null sequence,* $\frac{1}{n}|_{n=\infty} \in \Phi$

Example 2.7. *If we consider realizing an infinitesimal before dividing,* $1/\delta = 1/0$*,* $\delta \in \Phi$*, then we can interpret* $1/0 = \infty$ *as a generalization of a specific infinity which we do not know about, nor necessarily care.*

Definition 2.5. *Let* $\mathrm{rz}(x)$ *realize infinitesimals and infinities in an expression* x*. Lower order of magnitude terms are deleted by non-reversible addition: if* $a \succ b$ *then* $a + b = a$ *[4, Part 5].*

Example 2.8. $\mathrm{rz}(n^2 + n + 1)|_{n=\infty} = n^2|_{n=\infty}$ *as* $n^2 \succ n + 1|_{n=\infty}$*.*

Definition 2.6. *Let* $\mathrm{rz}(z)$ *realize infinitesimals and infinities in an additive sense about the relation* z*.*

Example 2.9. *Realize relation* $n\ z\ n^2|_{n=\infty}$*,* $n\ \mathrm{rz}(z)\ n^2|_{n=\infty}$*,* $0\ \mathrm{z}\ n^2|_{n=\infty}$ *because* $n\ z\ n^2$*,* $0\ z\ n^2 - n$*,* $0\ z\ n^2|_{n=\infty}$ *as* $n^2 - n = n^2|_{n=\infty}$*.*

Fisher [8, p.115] comments, while following du Bois-Reymond's infinitary calculus:

> For the objects of his infinitary calculus, ..., are functions which do not form a field under the operations he considers(addition and composition), whereas his reference to ordinary mathematical quantities obeying the same rules that hold for finite quantities might be taken to refer to a field.

While $*G$ we believe forms a field [2, Part 1], this is the first step before applying arithmetic that has no inverse. Realizing can simplify the expression, however such an operation is non-reversible. Continued application could lead to 0 or ∞, which could then be captured as a theorem. However in this case 0 and ∞ have no inverses, with respect to their state before realization. While we define ∞ and 0 as mutual inverses, the realization was likely done before this, perhaps through a limit.

The properties which make infinitary calculus not a field are the valuable properties. We approximate and use the field. The transfer(when realizing) is not independent of the number system, but part of it.

If after realizing a number, has the number changed type? Has the meaning of the relation changed?

Example 2.10. $x^2 + \frac{1}{x}|_{x=\infty}$ *realized to* x^2 *by discarding the infinitesimal, apply a transfer principle to bring back to* $\mathbb{R}$*,* $y = x^2$*.*

When transferring from a higher dimension number system to a lower dimension number system, in general, we need to consider the law of continuity after truncating.

The purpose of working in one number system is often to transfer the results to another.

For example, we could solve a problem in integers with real numbers, and transfer the result back to working in the integer domain.

Definition 2.7. *Let A and B be number systems, we say $A \mapsto B$ to mean the number system A is projected onto or maps to the number system B.*

Hence when realizing infinitesimals in $*G$ to $\mathbb{R}$ we approximate and simplify expressions. Let $x = a+\delta$, $a \in \mathbb{R}$, $\delta \in \Phi$. Truncating all the infinitesimals (assuming $\Phi^{-1} \notin x$) is the same as Robinson's NSA, "taking the standard part" [10, p.57], $\text{st}(a+\delta) = a$. If we interpret this as converting a number with infinitesimals to a real number, we can see $*G$ as a more detailed space, where the numbers can ultimately be realized as reals. So 0 in reals may be expressed as an infinitesimal in $*\overline{G}$, as 0 is a projection of an infinitesimal to the reals.

Proposition 2.1. *$\delta \in \Phi$; $*G \mapsto \mathbb{R} : \delta \mapsto 0$*

Proof. By approximation, repeated truncation of the infinitesimals leaves 0. □

In the realization of infinitesimals and infinities, information can be lost. $\frac{1}{n} \prec 1|_{n=\infty}$ becomes $0 < 1$. If the relation was reordered differently, the much greater than relation could remain, but the realization would not be to the real number system, but the extended reals. $\frac{1}{n} \prec 1|_{n=\infty}$, $1 \prec n|_{n=\infty}$, $1 \prec \infty$.

Example 2.11. *Let $\delta \in \Phi$, definition $f/g = \delta$ in $*G$ becomes $f/g = 0$ in $\mathbb{R}$.*

A similar process exists for infinities, where the "infinities" are realized and converted to the "infinity".

Example 2.12. *$\frac{1}{n+1} < \frac{1}{n}|_{n=\infty}$ in $*\overline{G}$ is true, but contradicts in $\mathbb{R}$ when we realize the infinitesimals: $0 < 0$. Similarly rearranging to compare infinities, $\frac{1}{n+1} < \frac{1}{n}|_{n=\infty}$, $\frac{n}{n+1} < 1|_{n=\infty}$, $n < n+1|_{n=\infty}$, realizing the infinities contradicts; $\infty < \infty$.*

Example 2.13. *$\frac{1}{n} > \frac{1}{n^2}|_{n=\infty}$, $n^2\frac{1}{n} > n^2\frac{1}{n^2}|_{n=\infty}$, $n > 1|_{n=\infty}$, $\infty > 1$ is true.*

$\frac{1}{n} > \frac{1}{n^2}|_{n=\infty}$, realizing $*G \mapsto \mathbb{R}$, $0 \not> 0$ is false, but $> \Rightarrow \geq$ is true.

Infinitesimals being smaller than any number in $\mathbb{R}$. Within an inequality, they can change to equality when removed.

Example 2.14. *Let $\delta \in \Phi$, $(*G, e^f < e^{f+\delta}) \mapsto (\mathbb{R}, e^f == e^f)$*

The following theorems may, if unfamiliar, seem trivial. If some proposition is true for a range, it is also true for its subrange: why would we make this into a theorem?

The very reduction of range can greatly simplify the complexity of cases involved. Hence, why construct theorems for reals and infinities if we only need to handle the infinite case?

Doing so, we believe leads to a radically different view of convergence and a new way to integrate: [6, Convergence sums ...] and rearrangement theorems with order on the infinite interval [1, Rearrangements of convergence sums at infinity].

By partitioning an interval between the infinireals and other numbers, we can separate arguments on finite numbers and infinireals to arguments on infinireals alone, and transfer when we need to go back to real or gossamer numbers.

Particularly important is the implication that partitioning by a finite bound, and including the infinity cases implies the infinity case, be it infinitely small or infinitely large.

We have kept the transfer notation $\mapsto$ as we are losing information in the process.

Definition 2.8. *A 'bounded number' is a number that is not an infinireal. All reals are bounded numbers, and so are all reals except* 0 *with infinitesimals. If x is a bounded number then $x \in *G - \mathbb{R}_\infty$*

Definition 2.9. *We say an 'implicit infinite condition' has a domain that includes both finite numbers (which can include infinitesimals) and infinireals $\mathbb{R}_\infty$. Let x and x_0 be either real or gossamer numbers.*
1. $\forall x > x_0$ where x_0 is finite.
2. $\forall x : |x| < x_0$ where x_0 is finite.

Since the finite numbers are partitioned by the infinite numbers (the infinireals), we remove the finite condition.

Theorem 2.1. *If an implicit infinite condition at infinity determines some proposition P then we can transfer to the infinitely large domain.*

$$(\forall x : x > x_0) \mapsto (x \in +\Phi^{-1})$$

Proof.

$$[x > x_0] = [x > x_0][x < +\Phi^{-1}] + [x > x_0][x \in +\Phi^{-1}]$$

Since choosing $x \in +\Phi^{-1}$ in the domain always satisfies the condition, the transfer is always possible. □

Theorem 2.2. *If an implicit infinite condition at the infinitely small determines some proposition P then we can transfer to the infinitely small domain.*

$$(\forall x : -x_0 < x < x_0) \mapsto (x \in \Phi)$$

Proof.

$$[|x| < x_0] = [|x| < x_0][|x| \in +\Phi] + [x < x_0][x \notin +\Phi]$$

Since choosing $x \in \Phi$ always satisfies $|x| < x_0$ in the above, the infinitely small case is always true and the transfer is always possible. □

Definition 2.10. *In context, a variable x can be described at infinity $|_{x=\infty}$ corresponds with Theorem 2.1*

Definition 2.11. *In context, a variable x can be described at zero $|_{x=0}$ corresponds with Theorem 2.2*

Theorems 2.1 and 2.2 are a common reduction within the transfer. Because of the variations of mapping involving the transfer from one domain to another, with different relations, we have developed a loose and not exact notation to communicate the mapping, and its context.

Definition 2.12. *Let $(K, <f_b>, <x>)$ describe the number system and context, where the angle brackets indicate optional arguments. K is the number type, $f_b \in \mathbb{B}$ a binary relation.*

$$(\textit{number type}, <\textit{relation}>, <\textit{number}>)$$

A mapping between domains can be described by

$$(K, <f_b>, <x>) \mapsto (K', <f'_b>, <x'>)$$

Remark: 2.1. *A number may be input that is not of the same type as its result. For example f/g may be in number system K but neither f nor g need necessarily be in K. Limit calculations happily accept input with infinities and infinitesimals, but the limit can be in $\mathbb{R}$.*

Mapping can occur in different contexts: realization, rearrangement of expression, transfer principle. The mapping can be in many different combinations. We summarize with a flexible notation; it is not at all strict.

Mapping	Comment
$(*G, /) \mapsto \mathbb{R}/\overline{\mathbb{R}}$	Limit $\frac{a_n}{b_n}\|_{n=\infty}$ evaluation
$(\mathbb{R}, /) \mapsto (*G, /)$	Undoing an implicit limit
$(\forall x > x_0) \mapsto (\Phi^{-1}, \|_{x=\infty})$ $(\forall \|x\| < x_0) \mapsto (\Phi, \|_{x=0})$	Law of continuity from $\mathbb{R}$ to $*G$ Theorems 2.1, 2.2
$(*G, \Phi^{-1}) \mapsto (\overline{\mathbb{R}}, \infty)$	Realize infinities
$(*G, \Phi) \mapsto (\mathbb{R}, 0)$	Realize infinitesimals, apply st() the standard part
If $(*G, \not<)$ then $(*G, \text{rz}(<)) \mapsto (\text{R}/\overline{\mathbb{R}}, <)$	Theorem 2.4
If $(*G, \not\leq)$ then $(*G\backslash\{\Phi^{-1}\}, <) \mapsto (\mathbb{R}, <)$	Corollary 2.2
$(*G, <) \mapsto (\mathbb{R}/\overline{\mathbb{R}}, \leq)$	Loses information, Theorem 2.6
$(*G, (\Phi < \Phi)) \mapsto (\mathbb{R}, (0 < 0))$	See Example 2.12
$(*\overline{G}, \infty) \not\mapsto (\mathbb{R}, \infty)$	Infinity is not in $\mathbb{R}$
$(\mathbb{R}, f \notin C^0) \mapsto (*G, f_2 \in C^0)$	Adding information [5]
$(J_\infty, n) \mapsto (*G, n)$	Discrete to continuous domain
$(\Phi, \delta_n) \mapsto (\mathbb{R}, \delta_n)$	Algorithm example [4, Example 2.4]
$\sum a_n\|_{n=\infty} \mapsto \sum_{k=k_0}^{\infty} a_k$	Convergences sums to sums [6, Theorem 11.1]

Table 1: Mapping examples

Consider a limit. While the image space may by $\mathbb{R}$, the solution space is $*G$ as it holds infinitesimals and infinities. Hence, given $*G \mapsto \mathbb{R}$, we can consider $*G$ and postpone or avoid the transfer. The implicit nature of the limit can be undone.

Example 2.15. *$\frac{n^2+1}{n^2}|_{n=\infty} = 1 \in \mathbb{R}$, but $n^2+1|_{n=\infty} \in *G$ and $n^2|_{n=\infty} \in *G$. Hence $\frac{n^2+1}{n^2}|_{n=\infty} \in *G$. However, the limit calculation can be described by $*G \mapsto \mathbb{R}$.*

If we consider the more general question of function evaluation, we have numbers that may be transfered between the number systems $\mathbb{R}$ and $*G$. While a function returns a value, by a transfer process it may not actually be calculated in that type. A transfer can occur between the calculation and the function's returned value.

Consider now, the function return value location as holding a local variable. If the function type does not match the location type, a transfer is made.

In the evaluation, we can show the implicit transfer. $(*G, =) \mapsto (\mathbb{R}, =)$. Then $\frac{a_n}{b_n}|_{n=\infty} = 1$ can be multiplied through to $a_n = b_n|_{n=\infty}$ in $*G$.

We introduce a notation to explicitly describe a relation, to help describe the transference rather than of practical use. With transference, the relation argument types are likely to be in the less detailed number system, but the evaluation in the more detailed number system $*G$.

Definition 2.13. *Let two arguments of a binary relation be described by their type $T1$, $T2$ where z is the binary relation.*

$$(T1 \; z \; T2)$$

Example 2.16. *To undo an infinite operation, we need to promote the numbers to $*G$. The comments indicate the left and right types on either side of the equality relation.*

$$\frac{n^2+1}{n^2}|_{n=\infty} = 1 \qquad (*G = \mathbb{R} \text{ or } *G)$$

$$*G \mapsto \mathbb{R}$$

$$\frac{n^2+1}{n^2}|_{n=\infty} = 1 \qquad (*G = *G)$$

$$n^2+1 = n^2|_{n=\infty} \qquad (*G = *G)$$

Example 2.17. *Promote a limit to a limit in $*G$. $(\mathbb{R}, 0) \mapsto (*G, \Phi)$*

$$\frac{\sin\frac{1}{n}}{n}|_{n=\infty} = 0 \qquad (*G = \mathbb{R})$$

$$\frac{\sin\frac{1}{n}}{n}|_{n=\infty} = \delta; \quad \delta \in \Phi \qquad (*G = *G)$$

$\mathbb{R} \mapsto *G$ is one-one as $\mathbb{R}$ is embedded in $*G$. However $*G \mapsto \mathbb{R}$ is different, as information about the infinitesimals is lost. Because $*G$ is more dense than $\mathbb{R}$, the transfer principle applied to the strict inequalities $\{<, >\}$ for variables/functions which are infinitesimally close, fail. Examples 2.12 and 2.13 implicitly worked in $*G$ and the relations changed when projected onto $\mathbb{R}$.

Proposition 2.2. *$\delta \in \Phi$; $*G \mapsto \mathbb{R}/\overline{\mathbb{R}}$; If $h \not\simeq 0$ in $*G$ then $h \neq 0$ in $\mathbb{R}/\overline{\mathbb{R}}$.*

Proof. Either $\mathbb{R} \in h$ or $\Phi^{-1} \in h$, both components map to non-zero elements in $\mathbb{R}/\overline{\mathbb{R}}$. □

Proposition 2.3. *$*G \mapsto \mathbb{R}/\overline{\mathbb{R}}$: If $h > \Phi$ in $*G$ then $h > 0$ in $\mathbb{R}/\overline{\mathbb{R}}$.*

Proof. If $h > \Phi$ in $*G$ then either $h \in +\Phi^{-1} \mapsto \infty$ or h has $\mathbb{R}^+ \mapsto \mathbb{R}^+$. Neither result is 0 in $\mathbb{R}$. □

Corollary 2.1. *If $f > 0$ in $*G$ and $\mathbb{R}$ then $f \not\simeq 0$.*

Proof. Assume true and show a contradiction. Let $f = \delta$ in $*G$, $\delta \in \Phi^+$ then $f > 0$ in $*G$. $(*G,\ \ \delta > 0) \mapsto (\mathbb{R},\ \ 0 > 0)$ is contradictory. □

Example 2.18. $*G \mapsto \mathbb{R}$, $\frac{1}{n+1} < \frac{1}{n} \mapsto \frac{1}{n+1} == \frac{1}{n}|_{n=\infty}$

Theorem 2.3. *If $(*G, \sim)$ then $(*G, <) \mapsto (\mathrm{R}/\overline{\mathbb{R}}, =)$*

Proof. $f < g$, $\frac{f}{g} < 1$, since $f \sim g$ then let $\frac{f}{g} = 1 - \delta$ to preserve the less than relation, $\delta \in \Phi^+$. Apply transfer, $\delta \to 0$, $\frac{f}{g}|_{n=\infty} = 1$, $f = g$. □

Example 2.19. $(*G, n\ \mathrm{rz}(<)\ n^2)|_{n=\infty} \mapsto (\overline{\mathbb{R}}, 0 < \infty)$. $n\ z\ n^2|_{n=\infty}$, $0\ z\ n^2|_{n=\infty}$, $0 < \infty$

Theorem 2.4. *If $(*G, \not\sim)$ then $(*G, \mathrm{rz}(<)) \mapsto (\mathrm{R}/\overline{\mathbb{R}}, <)$*

Proof. Let $f\ z\ g$. If either $\{f, g\} \in \Phi^{-1}$, after realization, for the negative infinity case: $-\Phi^{-1} < 0 \mapsto -\infty < 0$, positive infinity case: $0 < \Phi^{-1} \mapsto 0 < \infty$. Without infinities, let $g = \alpha + f$, $\alpha > 0$ to maintain the inequality, $\mathbb{R} \in \alpha$ as $f \not\simeq g$ then $\alpha \in \mathbb{R}^+$. In $*G$, $f < \alpha + g \mapsto \mathrm{st}(f)\ z_2\ \alpha' + \mathrm{st}(f)$ where $\alpha' \in \mathbb{R}^+$, $0\ z_2\ \alpha'$, $z_2 = <$. □

Corollary 2.2. *If $(*G, \not\simeq)$ then $(*G\backslash\{\Phi^{-1}\}, <) \mapsto (\mathbb{R}, <)$*

Proof. $(*G, \not\sim)$ without infinity becomes $(*G, \not\simeq)$ at infinitely close test only. □

Theorem 2.5. $(\mathbb{R}/\overline{\mathbb{R}}, <) \mapsto (*G, \not\sim)$ *and* $(*G, <)$

Proof. Since $\mathbb{R}$ embedded in $*G$, the $<$ relation follows. Since elements in $\mathbb{R}/\overline{\mathbb{R}}$ are separated by an infinity or real number the elements cannot be asymptotic in $*G$. ☐

Proposition 2.4. $\text{rz}(\sim) = \sim$

Proof. The limit itself considers magnitudes, usually by dividing the infinities and realizing the infinitesimals. ☐

Theorem 2.6. $(*G, <) \mapsto (\mathbb{R}/\overline{\mathbb{R}}, \leq)$

Proof. Since $\sim$ and $\not\sim$ cover all cases, combining Theorem 2.3 and Theorem 2.4 covers all cases, the union of the two images. ☐

The limit calculation can be described as an evaluation in a more detailed number system with infinitesimals and infinities which is projected back to the real numbers.

$\frac{a_n}{b_n}|_{n=\infty} = 1$ can be expressed as $(*G = \mathbb{R})$ or $(\mathbb{R}, =)$, even though the fraction may not be in $\mathbb{R}$. However the result of the ratio is in $\mathbb{R}$, hence we state it this way.

We found the limit to be a transfer as it realizes infinitesimals and infinities. Hence, we provided a number system $*G$, which contains infinireals and better describes the limit calculation.

In what follows, we are able to decouple a fraction about 1, multiplying the numerator and denominator out, while still able to simplify as a fraction, through transfers.

Theorem 2.7. *$z_1, z_2 \in \mathbb{B}$; $a_n > 0$, $b_n > 0$.*

$$\frac{a_n}{b_n}|_{n=\infty} \; z_1 \; 1 \quad \Leftrightarrow \quad a_n \; z_2 \; b_n|_{n=\infty}$$

Condition 1		Condition 2
z_1	z_2	
$<$	$<$	$(*G, \not\sim)$ then $(*G, <) \mapsto (\mathbb{R}/\overline{\mathbb{R}}, <)$
$>$	$>$	$(*G, \not\sim)$ then $(*G, >) \mapsto (\mathbb{R}/\overline{\mathbb{R}}, >)$
$=$	$\sim$	not one of the other cases

Proof. While we can bring the fraction into $*G$ and multiply out the denominator, to show equivalence we need to show that the transfer back from $*G$ to $\mathbb{R}/\overline{\mathbb{R}}$ is equivalent to the limit.

Condition 2 can map $*G \to \mathbb{R}/\overline{\mathbb{R}}$ as the cases are disjoint and cover $*G$. If we consider the fractions equality case, by definition is $\sim$. However, this case is considered by excluding the inequality cases, partitioning $*G$ into the three disjoint cases. For the inequality, condition 2 leads to condition 1 by Theorem 2.3. All cases of $*G \to \mathbb{R}/\overline{\mathbb{R}}$ have been considered. $\square$

Theorem 2.8. *Extend Theorem 2.7: $a_n \neq 0$, $b_n \neq 0$,*

$$\frac{a_n}{b_n}|_{n=\infty} \; z_1 \; 1 \quad \Leftrightarrow \quad a_n \; (\text{sgn}(b_n) z_2) \; b_n|_{n=\infty}$$

Proof. Since multiplying by a negative number inverts the inequality which is included within the theorem, the proof is identical to the proof given in Theorem 2.7. $\square$

While Theorem 2.7 loses information as $*G$ is more dense, so is not reversible, we can map back to $*G$. Consider the table relations $z_1 \to z_2$, if we consider $\mathbb{R}/\overline{\mathbb{R}} \to *G$, as $\mathbb{R}$ is embedded, the inequalities hold.

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