An Alternative Kaluza Theory Identifying 5D Momentum and Charge

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. Here an alternative approach is presented that more fully unifies gravity and electromagnetism. Emphasis is placed on admitting important electromagnetic fields not present in Kaluza's original theory without constraints, and on deriving a Lorentz force law. This is done by identifying 5D momentum with a kinetic charge. By doing so the usual assumption of Ricci flatness corresponding to sourceless electromagnetic fields is replaced by the weaker constraint of vanishing 5D momentum outside of charge models. A weak field limit is also used. An electromagnetic limit is imposed by assuming a constant scalar field. A further extended postulate set involving a super-energy divergence law and a conformal factor is also suggested that allows for a varying scalar field, within what then becomes a type of geometrical conformal gauge theory.

1 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [1][2][3][4] using a fifth wrapped-up spatial dimension gives a taste of unification of electromagnetism with gravity in a way that has problems and is often believed to be untenable. However the underlying aim was particularly promising in terms of explanatory power. Modern works hold out hope for higher dimensional theories and non-abelian gauges [1][25], and the consequent hope for unification with quantum mechanics. Here an alternative approach is implemented that goes back to a simpler (and in the author's opinion more practical) root: fully unifying just gravity and electromagnetism. Certain requirements are evident: a Lorentz force law [6] must be explained, Maxwell's laws [6] must be present, the Lorentz transformation [6] must define inertial frames, general relativity [6] must be a limit for gravitational physics. The Lorentz force law is the most conceptually unsatisfying law within classical theory. It may not even be compatible with n-dimensional Noether theorems [26] - all the more reason to construct it, or an approximation, from first principles. Whilst it does come from the Einstein-Maxwell stress-energy tensor [6], where does *that* come from? Kaluza's original theory only partially answers that and, in addition, problems arise in the coupling of the scalar field with the electromagnetic field. On the other hand the Lorentz force law is but the relativistic form of Coulomb's law. Surely it should be as fundamental geometrically as the inverse square law of gravity? It may just as equally be approximate in a fully geometrised theory. That is, the usually presumed stress-energy tensor of electromagnetism need not be exactly the correct one in curved space-time. It is in this straight forward and relatively unambitious vein that search for a variant Kaluza theory is undertaken.

The Lorentz force law here requires a constant scalar field, this places constraints on admissible solutions. The emphasis is then on eliminating the constraint in Kaluza theory that prevents the below defined non-nullish electromagnetic solutions when the scalar field is constant. This was done twice elsewhere by the author using torsion, once allowing symmetric components [24], the other time by imposing complete antisymmetry of torsion [28]. In both cases the results were not as simple as those presented here.

It is sufficient to show that certain constraints that cause the problem for the usual Kaluza and Kaluza-Klein theories have here been weakened. The main constraint is the third field equation in [1], equation (2.0.6) here, and the first field equation. When the scalar field is constant the third field equation becomes one of two equations that characterize the null electromagnetic fields. This equation is as follows, and fields that satisfy this will be called 'nullish':

Definition 1.0.1: 'Nullish' electromagnetic fields satisfy: $F_{ab}F^{ab} = 0$. Null electromagnetic fields have the nullish property plus the following condition, where the star is the Hodge star operator: $F_{ab}(*F^{ab}) = 0$.

Kaluza's original theory [1] prohibits non-nullish solutions (or even near non-nullish solutions) for constant scalar field. Nullishness is too tight to admit important electromagnetic fields, in particular the essential electrostatic fields. That electrostatic or near-electrostatic fields are non-nullish and therefore a problem in any theory that omits them can be seen by comparing definition (1.0.1) with the following well-known fact from special relativity, that is by considering a special relativistic limit: $F_{ab}F^{ab} = 2(B \cdot B - E \cdot E)$.

The leading objective here is to find a more natural way to allow for non-nullish solutions.

In this paper the emphasis is on reimplementing the simplicity of Kaluza's original idea: by replacing the 5D Ricci flat condition for electromagnetic fields without charge sources with the identification of the 5D momentum components in the 5D Einstein tensor as charge sources. That is, by relaxing the Ricci-flat 'background' condition of Kaluza in a way tutored by the explorations undertaken in [24] and [28] (where background is defined as outside of matter models). Here, however, torsion will not be used and a simpler construction made, which then clarifies the physical insight into Kaluza's theory. A weak field limit is however used as an additional constraint.

Kinetic charge will be defined as the 5th-dimensional component of momentum [8] and identified with Maxwellian charge in a non-obvious way. A Lorentz force law will then follow. As momentum the kinetic charge has a divergence law via the 5D Einstein tensor. Maxwellian charge also has a vector potential, see (3.4.1), and thus local conservation, but the kinetic charge and corresponding divergence law, being covariant, is taken to be the fundamental charge.

The simplicity of the presented model over previous works is a benefit and is very naturally a development of Kaluza's original model. It is an alternative approach also to induced-matter Kaluza theory [1], in that here the cylinder condition is still maintained.

A further extension of the theory is also provided which allows the scalar field to vary, but from which a constant scalar field can be derived by using a conformal transformation. Further, the extension allows for super-energy conditions to be applied, which could then substitute for more usual ways to deal with such issues as causality using the energy conditions of general relativity. This extension is however fundamentally different from the first postulate set presented here, and similar theories, in that the physics is determined up to conformal transformation and represents a set of geometries rather than a single one. This however also makes the extended postulate set more obviously a type of gauge theory.

2 Conventions

The following conventions are adopted unless otherwise specified.

Five dimensional metrics, tensors and pseudo-tensors and operators are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. Lower case indices can either be 4D or generic for definitions depending on context. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope. Space-time is given signature $(-, +, +, +)$, Kaluza space $(-, +, +, +, +)$ in keeping with [6]. Under the Wheeler et al [6] nomenclature the sign conventions used here as a default are $[+, +, +]$. The first dimension (index 0) is time and the 5^{th} dimension (index 4) is the topologically closed Kaluza dimension. Time and distance are geometrized throughout such that c = 1. **G** is the gravitational constant, which may also be set to 1 unless otherwise specified. The scalar field component is labelled ϕ^2. The matrix of g_{cd} can be written as $|g_{cd}|$. The Einstein summation convention may be used without special mention. $\Box$ represents the 4D D'Alembertian [6].

The unique Levi-Civita connection can be specified with: Γ^c_{ab}, and the covariant Levi-Civita derivative operator: ∇_a. Define:

$$F_{ab} = \partial_a A_b - \partial_b A_a = \nabla_a A_b - \nabla_b A_a$$
$$F = dA \tag{2.0.1}$$

Some familiar defining equations consistent with [1] define the Ricci tensor and Einstein tensors in terms of the applied connection coefficients along usual lines:

$$R_{ba} = \partial_c \Gamma^c_{\ ba} - \partial_b \Gamma^c_{\ ca} + \Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc} \tag{2.0.2}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \mathbf{G} T_{ab} \tag{2.0.3}$$

We will define $\alpha = \frac{1}{8\pi\mathbf{G}}$.

We also make reference to Kaluza's original field equations [1] in the text:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4} g_{ab} F_{cd}F^{cd} - F^c_{\ a} F_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b \phi) - g_{ab}\Box\phi\} \tag{2.0.4}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a \phi}{\phi} F_{ab} \tag{2.0.5}$$

$$\Box\phi = \frac{k^2\phi^3}{4} F_{ab}F^{ab} \tag{2.0.6}$$

The following result from [24] and [28] will be rederived (5.3.8) and used as required: that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{\ (bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \to (Q_M/\rho_0) F^a_{\ b}\frac{dx^b}{d\tau'} \quad \text{and} \quad k = 2\sqrt{\mathbf{G}} \tag{2.0.7}$$

That is, when the Gravitational constant is set to unity, we can set by convention $k = 2$ to make the units of this text consistent with other works, in particular Wald [7]. This sheds some light on the tricky problem of units in electromagnetism more generally. However, for calculation purposes setting $k = 1$ is equally good, merely representing a change of how units are interpreted. This will also be allowed here.

In this paper some of the working has been omitted as is usual. Some of workings are however available in [24], for reference, though care must be taken in translating them to the current context.

Orders of magnitude notation are used. Following [28], to indicate when terms are of a certain order of magnitude: $O(X)$ will be used. Further, when rounding has occured by ignoring terms of $O(X)$, ie "$O(X)$ terms discounted", this will be denoted $\backslash O(X)$.

3 Overview Of Alternative Kaluza Theory

3.1 Postulates

The following K1-K4 are the core postulates of the present Kaluza theory.

POSTULATE (K1): **Geometry**. The geometry, the Kaluza space, under consideration is a 5D smooth Lorentzian manifold.

POSTULATE (K2): **Well-behaved**. Kaluza space is assumed globally hyperbolic in the sense that there exists at each point 4D spatial cauchy surface plus time, such that the 4D hypersurface is a simply connected 3D space extended around a 1D loop. And Kaluza space is oriented and time-oriented.

POSTULATE (K3) **Cylinder condition**. One spatial dimension is topologically closed and 'small', the Kaluza dimension. This is taken to mean that there are global unit vectors that define this direction, the Kaluza direction. The partial derivatives of all tensors in this Kaluza direction are taken to be zero in some coordinate system.

POSTULATE (K4): **Geodesic Assumption**. That any model of a charged particle (or for that matter uncharged particle) follows 5D geodesics.

LIMIT POSTULATE (B1): There is a Kaluza atlas, see definition (3.2.1), possibly only over a region, such that $\phi^2 = 1$ at every point. The scalar field results from the the decomposition of the Kaluza metric into 4D metric, potential vector and scalar field. It is contained within the metric explicitly in (3.4.1). Thus B1 is a constraint on the 5D metric. This defines the electromagnetic limit.

L1 below constitutes a *weak field limit* that will be applied by way of approximation for the classical limit of behaviour. The deviation from the 5D-Minkowski metric is given by a tensor $\hat{h}_{AB}$. This tensor belongs to a set of small tensors we label $O(h)$. Whilst this uses a notation similar to orders of magnitude, and is indeed analogous, the meaning here is a little different. This is the weak field approximation of general relativity using a more flexible notation. Partial derivatives, to whatever order, of metric terms in a particular set $O(x)$ will be in that same set at the weak field limit. In principle we are doing nothing more than following the weak field limit procedure [6] of general relativity. In the weak field approximation of general relativity, terms that consist of two $O(h)$ terms multiplied together get discounted and are treated as vanishing at the limit. We might use the notation $O(h^2)$ to signify such terms. There is the weak field approximation given by discounting $O(h^2)$ terms. But we might also have a less aggressive limit given by, say, discounting $O(h^3)$ terms, and so on. We can talk about weak field limits (plural) that discount $O(h^n)$ terms for $n > 1$ based on the same underlying construction. The weak field is in some sense a choice of scale that gives meaning to the smallness of the cylinder condition. The weak field condition and the cylinder condition together correspond to the existence of a classical scale in the theory.

LIMIT POSTULATE (L1): The metric can be written as follows in terms of the 5D Minkowski tensor and $\hat{h} \in O(h)$: $\hat{g}_{AB} = \hat{\mu}_{AB} + \hat{h}_{AB}$.

3.2 The Cylinder Condition And Charts

The cylinder condition by construction allows for an atlas of charts wherein the Kaluza dimension is approximately presented by the fourth index. The atlases that are compliant with this may be constructed by restricting them accordingly. This means that the cylinder condition can be represented by a subatlas of the maximal atlas. The set of local coordinate transformations that are compliant with this atlas (called a Kaluza atlas) is non-maximal by construction. A further reduction in how the atlas might be interpreted is also implied by setting c=1, and constant **G**. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension is here equivalent to a (kinetic) charge inversion. However, given orientability and an orientation we can remove even this ambiguity. We can further reduce a Kaluza atlas by removing boosts in the Kaluza dimension. Space-time is taken to be a subframe within a 5D frame within a Kaluza subatlas of a region wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic charge. The Kaluza atlas represents the 4D view that kinetic charge is 4D covariant. Rotations into the Kaluza axis can likewise be omitted. This results in additional constraints on the connection coefficients associated with charts of this subatlas, and enables certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit.

Definition 3.2.1: A*Kaluza atlas* is:

(i) A subatlas (possibly just over a region) of the maximal atlas of Kaluza space where boosts and rotations into the Kaluza dimension (as defined by the cylinder condition K3) are explicitly omitted.

(ii) All partial derivatives in the Kaluza direction are vanishing.

(iii) Inversion in the Kaluza direction and rescalings can also be omitted so as to establish units and orientation.

(iv) For each point on the Kaluza atlas a chart exists with normal coordinates where index 4 is the Kaluza dimension.

3.3 Kinetic Charge

Kinetic charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Prov. Definition 3.3.1: kinetic charge: $Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0$

This provisional definition (refined below) makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. This kinetic charge can be treated in 4D space-time (and the Kaluza atlas) as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector, and it is also conserved as shown. In general relativity at the special relativistic Minkowski limit the conservation of momenergy can be given in terms of the stress-energy tensor as follows [9], $j \neq 0$. This is approximately true at a weak field limit and can be applied equally to Kaluza theory. We have a description of conservation of momentum in the 5th dimension.

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0\ ,\ \frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \text{ and } \frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{3.3.2}$$

We also have i=4 vanishing by the cylinder condition. Thus the conservation of kinetic charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be locally conserved: $\partial_0\hat{T}^{04} + \partial_1\hat{T}^{14} + \partial_2\hat{T}^{24} + \partial_3\hat{T}^{34} = 0$.

To make sense of this in 5D we need to change the provisional definition above and make it density-based (imagine a ring rather than a particle). The alternative definition can be made in terms of the mass density ρ_0, coupled with the Kaluza dimension's size or Kaluza length λ. In this way we do not presuppose that the rest mass we observe in space-time is necessarily the m_0 above: what happens for example to the apparent rest mass in 4D if the Kaluza distance changes and the density compressed or rarefacted? m_0 makes most sense as a definition of rest mass in 4D when this does not happen. Generalization demands the following definition, replacing m_0 with a density:

Definition 3.3.3: 5D kinetic charge: $Q^* = \lambda\rho_{k0}dx_4/d\tau^* = \lambda\rho_0 dx_4/dt_0$

This leads to a density-slice definition of 4D density-based kinetic charge as follows (noting that it is not 4D-divergence free in the event that λ changes):

Definition 3.3.4: 4D kinetic charge density: $Q^{**} = \rho_{k0}dx_4/d\tau^* = \rho_0 dx_4/dt_0$

Kinetic charge current density is the 4-vector, induced from 5D Kaluza space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{**a} = -\alpha\hat{\mathcal{G}}^{a4} \tag{3.3.5}$$

And a measure of the total current can be give as:

$$J^{*a} = -\alpha\lambda\hat{\mathcal{G}}^{a4} \tag{3.3.6}$$

Using Wheeler et al [6] p.131, and the appropriate space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a(1,0,0,0)^a \tag{3.3.7}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with conserved mass-energy via the Einstein tensor. It follows that the vanishing of the divergence of kinetic charge in 4D is only approximate, in 5D it is exact.

Definition 3.3.8: *Kinetic charge current* is defined to be the 4-vector $J^{*a} = -\alpha\lambda\hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas that represents this total charge current in 4D. Note the divergence of the Einstein tensor:

$$\hat{\nabla}_A\hat{\mathcal{G}}^{AB} = 0 \text{ and } \hat{\nabla}_A\hat{\mathcal{G}}^{A4} = 0 \approx \hat{\nabla}_a\hat{\mathcal{G}}^{a4}$$

3.4 Two Types Of Geometrized Charge

Components used in [1] will be used here as the Kaluza metric. The vector potential and electromagnetic fields formed via the metric are sourced in Maxwell charge Q_M. Maxwell's law are automatically satisfied, using (2.0.1) to define F with respect to the potential: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0. A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time:

Definition 3.4.1: The 5D Kaluza metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \text{ and } \hat{g}^{AB} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{3.4.1}$$

This gives nullish solutions under the original Kaluza theory (cylinder condition, $\mathcal{R}_{ab} = 0$) and constant scalar field, such that $G_{ab} = -\frac{k^2}{2}\, F_{ac}F^c_b$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units for ostensibly the same fields. The units need to be agreed between the two schemes. We would need to set either $k = 2$ or $k = -2$ for compatibility of results and formulas. And this is particularly important as we wish to derive the Lorentz force law with the same units as [7]. N.B. the sign change introduced by [1], which is confusing. This makes no fundamental difference, but must be noted.

The geometrized units, Wald [7] p470-471, define units of mass in terms of fundamental units. This leads to an expression for kinetic charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$. $\mathbf{G}$ and k are not independent however. If we fix one, the other is fixed too: A consequence of requiring the Lorentz force law written in familiar form and compatibility with the units used in [7]. The relation between $\mathbf{G}$ and k is given in equation (5.3.8) via the derivation of the Lorentz force law. Simple compatibility with Wald [7] results where $k = 2$ and $\mathbf{G} = 1$. The sign of k is also fixed by (5.1.4). The result of dimensional analysis gives kinetic charge, Q^*, in terms of a total 5D momentum component P_4 and its corresponding density P_4^*:

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}}\lambda P_4^* = \frac{c}{\sqrt{\mathbf{G}}}P_4 \tag{3.4.2}$$

4 The Field Equations

It is necessary to show that the three field equations that derive from Kaluza compaction do not unduly restrict the possible range of electromagnetic solutions possible. Here the effort is less than in [24] and [28], the results following simply by looking at each field equation in turn. It would of course be expected that such loosening of Kaluza's theory would also involve the loss of the Lorentz force law as derived in the torsion theories of [24] and [28]. This concern will be shown to be unfounded. A Lorentz force law derivation is also provided here. L1 is thus shown to be a sufficient constraint to replace the Ricci flat condition.

4.1 The First Field Equation, $k = 1$

Looking at the Ricci tensor gives:

$$\hat{\mathcal{R}}_{ab} = \partial_c\hat{\Gamma}^c_{ba} - \partial_b\hat{\Gamma}^c_{ca} + \frac{1}{2}\partial_b(A^d F_{ad}) + \hat{\Gamma}^c_{ba}\hat{\Gamma}^D_{Dc} - \hat{\Gamma}^C_{Da}\hat{\Gamma}^D_{bC} \tag{4.1.1}$$

In the original Kaluza theory, where the electromagnetic fields are identified with a Ricci flat Kaluza vacuum (ie $\hat{\mathcal{R}}_{ab} = 0$), the Ricci flatness leads to a constraint helping to impose nullish solutions when there is no scalar field. This is the analagous equation to (2.0.4). Without sources the remaining significant term is a nullish solution:

$$\begin{aligned}\mathcal{R}_{ab} &= \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab}\\ &= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c + \frac{1}{2}F_{ac}F_b{}^c\\ &-\frac{1}{2}(A_bF_a{}^c + A_aF_b{}^c)\Gamma^d_{dc} + \frac{1}{2}\Gamma^c_{da}A_bF_c{}^d + \frac{1}{2}A_aF_b{}^c\Gamma^d_{bc}\\ &+\frac{1}{4}(A_dF_a{}^c + A_aF_d{}^c)(A_bF_c{}^d + A_cF_b{}^d) + \frac{1}{4}A^dF_{ad}A^cF_{bc}\end{aligned} \tag{4.1.2}$$

In [24] and [28] redefining the theory by using torsion in the definition sufficiently weakened this constraint. Here we take an alternative (and arguably far simpler approach) of instead imposing only sourceless kinetic charge. This has little bearing on $\hat{\mathcal{R}}_{ab}$ and so comparable field equations to either the original Kaluza theory or other variants is simply not possible. There just isn't a constraint to elaborate.

4.2 The Second Field Equation

Derivation of the second field equation gives:

$$\hat{\mathcal{R}}_{a4} = \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a\Gamma^d_{dc} + \frac{1}{4}F^c_a A^d F_{cd} - \frac{1}{2}(\Gamma^c_{da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c$$

Looking at this at an \O(h^2) L1 weak field limit (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{4.2.1}$$

This couldn't be a clearer conception of Maxwell charge. This coincides with the Einstein tensor at the same limit, thus providing an alternative conception of the conservation of Maxwell charge locally (cf 5.1.1 and 5.1.2):

$$\hat{\mathcal{G}}_{a4} \to \hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{4.2.2}$$

Unlike the previous works of [24] and [28] there are no further considerations.

This suggests the identification of Maxwell and kinetic charges at the L1 limit. A completed identification (with lowered indices) will be provided in the sequel.

4.3 The Third Field Equation, $k = 1$

This section shows how the present theory releases the constraint of the third field equation (2.0.6), thus allowing non-nullish solutions. As already mentioned the constraint that the Ricci tensor be zero leads to no non-nullish solutions in the original Kaluza theory if the scalar field is also constant (ie under postulate B1). This is caused by setting $\hat{\mathcal{R}}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The traditional theory sets the following to vanishing by postulate:

$$\hat{\mathcal{R}}_{44} = \partial_C\hat{\Gamma}^C_{44} - \partial_4\hat{\Gamma}^C_{C4} + \hat{\Gamma}^C_{44}\hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{D4}\hat{\Gamma}^D_{4C} = -\hat{\Gamma}^c_{d4}\hat{\Gamma}^d_{4c} = -\frac{1}{4}F^c_d F^d_c \tag{4.3.1}$$

This theory however loosens this constraint so that the Ricci tensor need not be vanishing. As a result the theory admits non-nullish solutions analogously to [24] and [28].

5 The Lorentz Force Law

5.1 Kinetic And Maxwell Charge

Toth [8] outlines a derivation (perhaps not uniquely) of a Lorentz-like force for static scalar field in the original Kaluza theory for a charge that is the momentum term in the fifth dimension. Here we make use of K4 to investigate this further and set the result on a firmer footing. To investigate the relationship between kinetic charge and Maxwell charge we need the \$O(h^2)$ weak field limit defined by L1 (cf equation 4.2.2). Discounting $O(h^2)$ terms:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C\hat{\Gamma}^{C4a} - \partial^4\hat{\Gamma}^{C\ a}_{\ C} + \hat{\Gamma}^{Cba}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C\ a}_{\ D}\hat{\Gamma}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} = \partial_c\hat{\Gamma}^{c4a} \tag{5.1.1}$$

Putting k back in, and then using (3.3.8), we get:

$$\hat{\mathcal{R}}^{a4} \to \frac{1}{2}\partial_c k F^{ac} \tag{5.1.2}$$

$$J^*_a \to -\frac{\alpha k}{2}\lambda\partial_c F_a^{\ c} \tag{5.1.3}$$

So kinetic and Maxwell charges are related by a simple formula. The right hand side being the Maxwell charge current (see p.81 of [6]), and has the correct sign to identify a positive kinetic charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and Kaluza atlas frame, using (3.3.7), and approaching the \$O(h^2)$ limit given by L1:

$$4\pi Q_M \to +\frac{2}{\alpha k\lambda}Q^* \tag{5.1.4}$$

This correlates the two definitions of charge at the required limit and differs from [24] only due to the use of densities in the definition - allowing for the possibility of varying Kaluza length. Nevertheless we use throughout the same notation as [24], noting that $m_X \equiv p_X\lambda$.

5.2 A Lorentz-Like Force Law

Christoffel symbols will now be used to investigate the geodesic equation. We will here initially use $k = 1$, a general k can be added in later. The working can apply also when there is torsion, though such a generalisation is not needed here.

$$\hat{\Gamma}^c_{(4b)} = \tfrac{1}{2}\phi^2 F^c_b \quad -\tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 \tag{5.2.1}$$

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = \text{-}\ \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 \tag{5.2.2}$$

$\hat{\Gamma}^c_{(ab)} = \Gamma^c_{(ab)} + \frac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) - A^c(\delta_a \phi^2 A_b + \delta_b \phi^2 A_a)$ (5.2.3)

So, for any coordinate system within the maximal atlas:

$$0 = \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$
$$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + (\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \quad (5.2.4)$$

Taking the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \quad (5.2.5)$$

We derive a Lorentz-like force law, putting k back in:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \quad (5.2.6)$$

$$= -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \quad (5.2.7)$$

5.3 Constant Kinetic Charge And The Lorentz Force Law

Having derived a Lorentz-like force law we look also at the momentum of the charge in the Kaluza dimension. We look at this acceleration as with the Lorentz force law. We have ($k = 1$):

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$
$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \frac{1}{2}A^d\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \quad (5.3.1)$$

The two equations (5.3.1),(5.2.7) under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \quad (5.3.2)$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k^2(Q'/m_{k0})A_c F^c_b\frac{dx^b}{d\tau} \quad (5.3.3)$$

(5.3.3) shows that deviations from geodesic behaviour around the closed Kaluza loops are small, which seems consistent with the conservation of charge and the integrity of a charged particle.

Multiplying both sides of (5.3.2) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \quad (5.3.4)$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{5.3.5}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{5.3.6}$$

Using (5.1.4) - only here does the calculation vary from [24] - as its L1 weak field limit is approached, this can be rewritten again in terms of the Maxwell charge:

$$\rightarrow k(\frac{\alpha k}{2}(4\pi Q_M\lambda)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{5.3.7}$$

The result is that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \rightarrow (Q_M/\rho_0)F^a_b\frac{dx^b}{d\tau'} \quad \text{and} \quad k = 2\sqrt{\mathbf{G}} \tag{5.3.8}$$

This shows that the Lorentz force law proper can be derived given (5.1.4) and the required limit ([B1] and [L1]). This of course suggests that in this variant of the theory the universality of Lorentz force law is dependent on the constancy, or approximate constancy, or local constancy, of the Kaluza length. This is in contrast to the analysis in [24] which did not make this apparent.

6 Limits and Extensions

A simple observation suffices to demonstrate the relativistic limit for geodesics:

$$\hat{\Gamma}^c_{ab} - \Gamma^c_{ab} \text{ is order } \mathrm{O(h^2)}$$

At the Newtonian classical limit we also need the approximate constancy of the Kaluza length to ensure an experimentally valid Lorentz Force law. This follows from the geodesic deviation equation and the fact that the Riemannian tensor is $O(h)$, which is defined to be small, meaning $<< 1$, at the classical limit. The vector u at the classical limit has, by construction, components of order $O(1)$. Thus the changes in vector $\hat{n} = (0, 0, 0, 0, 1)$ that might represent a change in the Kaluza length, are proportionately $O(h)$ small relative to when $\hat{n}$ has unit length. This scales down for arbitrarily small Kaluza lengths: deviations at the classical scale always being $O(h)$ times smaller in significance than the reference vector. That is, there is no significant change in Kaluza length until a much larger scale is reached - meaning when net metric variations cease to be $O(h)$ small or equally when or if L1 ceases to apply. The geodesic deviation equation [6] is as follows:

$$\hat{\nabla}_u \hat{\nabla}_u \hat{n} + \hat{Riemann}(..., u, \hat{n}, u) = 0 \tag{6.0.1}$$

Further, in general relativity, there are energy conditions and/or positivity requirements on mass/energy. There is also the issue of causality. Few of these issues are unambiguously dealt with in general relativity in that there is no correct or definitive energy condition, but rather a number of choices. And sometimes negative mass/energy is even considered, so even positivity of mass/energy is not quite as secure as it might usually be assumed. Further, quantum mechanics shows that causality can only be a macroscopic result (thus causality in physics may be subtly different even macroscopically than often assumed). And then there are such adjustments as are made from time to time, such as inflation or the cosmological constant. Stability of Kaluza solutions has also historically been a problem for Kaluza-type theories. These issues are all bound up one with the other to some extent. For example we could change the energy condition used so as to allow for a cosmological constant, or under certain conditions we might derive positivity from dominant energy.

In [24] the idea that the super-energy tensor of the Riemann tensor (ie the generalised Bel tensor) [17][18] could be conserved (or at least of vanishing divergence) in a Kaluza-type theory with torsion was posed. But the arbitrariness of the choice of connection (ie whether with or without torsion) made any single formulation somewhat arbitrary. This became a recurrent issue in [28]. The same approach can be taken here without that ambiguity due to their being only one connection. The super-energy tensor of the Riemannian tensor could be hypothesized to be conserved with respect to the only covariant derivative in sight - the usual Levi-Civita connection. This local conservation or divergence law would suggest that the Riemannian curvature developed causally, as described in [16]. This alone gives some sort of partial causality also for the metric in that the metric must develop consistently with the curvature. But in addition a super-energy condition provides a large number of constraints that constrain the metric locally. Further, the proofs used elsewhere [10] to argue for instability of Kaluza theories fail to apply due to the use of super-energy instead of energy conditions. The use of super-energy conditions is therefore promising. The essential insight is that the squared terms featured in Bel and Bel-Robinson super-energy tensors, if locally conserved by a divergence law, should prohibit the collapse of the Kaluza dimension due to the fact that the squared terms in these tensors would get expensive to conserve with higher curvatures. This is quite different from other Kaluza theories that rely on more linear functions of curvature like the mass-energy of the Einstein tensor. The Bel tensor is provisionally chosen here as more suitable than the Bel-Robinson tensor in the presence of mass, though this is an issue, possibly empirical, demanding further investigation. Both are natural as an extension of Ricci flatness, though further alternatives or variants may also exist.

The general idea of trying to use a super-energy tensor as an alternative in some sense to stress-energy in general relativity has a long history and need not be summarized here. See [17] for extensive references. It is natural to try to apply it to Kaluza theories.

The covariant divergence of the (generalised) Bel tensor (see [17] for definition and uniqueness) is not in general vanishing although it is always vanishing in the case of Einstein manifolds, and a similar result holds for the Bel-Robinson tensor. But Einstein manifolds are not in themselves rich enough to provide matter models. In [17] a formula for the vanishing of the divergence of the Bel tensor is given in terms of the Riemann and Ricci curvatures. A sufficient condition is that the Riemann tensor is harmonic, which is equivalent to the Ricci tensor being a Codazzi tensor [30][31][32]. This is particularly interesting as it makes the Riemann curvature a Yang-Mills potential. This condition also provides for a wider range of solutions that are not Einstein [29]. Further degrees of freedom come from the interpretation in 4 dimensions (in that the Ricci curvatures of the 4D space-time can be richer than the associated Kaluza space), and still further degrees of freedom for the construction of 'proper' matter models (meaning integral to the metric rather than merely super-imposed) come from the extended postulate set (given below) via the scalar field and the conformal transformation thereof. Nevertheless proper matter models are essentially exact solutions and may be difficult to construct.

As mentioned already the weak field limit is in a sense a choice of scale related also to the cylinder condition. Here we go further with the concept of scale, and suggest some additional postulates that lead to an explanation for the constant scalar field. The idea is that the scalar field is constant due to a conformal transformation of the metric. That is, if we start with an arbitrary Kaluza space obeying all the postulates previously presented except B1 then such a geometry has only one member of its conformal class where the scalar field can be reduced to identity. The identification is not 1-1 as re-extracting the original does not follow from knowledge of the conformally derived copy. But up to conformal invariance we may do this. 'Up to conformal invariance' means that any member of the conformal class may equally be usable as a model, just that the one which sets the scalar field constant is the simplest.

The postulates are then a fairly simple extension of the previous ones:

K1, K2, K3, K4, L1, M1 (vanishing covariant divergence of the chosen super-energy tensor, for the present purposes the generalised Bel tensor [17]), M2 (that the physics is defined by a subset of a conformal class such that the subset satisfies all the other postulates). Call such a set of manifolds 'the super-energy class'.

The super-energy class can be reconstructed in full from its corresponding Kaluza space of constant scalar field. And this Kaluza space is in turn uniquely defined by the super-energy class, though generally the constant scalar field Kaluza space will not be in the super-energy class. Thus the super-energy class can define the physics which we would more normally associate with a

Kaluza space of the previous sections. The super-energy class defines a gauge redundancy.

And that's all that's needed to create a further theory with explanatory power over the scalar field issue of the first postulate set. Its preliminary presentation here gives better context to the first postulate set, though of course it needs further development.

7 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. Here an alternative Kaluza theory is presented - or rather a set of postulates K1, K2, K3, K4, L1, B1 that define that theory, and also an extension of that theory is presented that is more speculative but goes further in explanatory power with postulates: K1, K2, K3, K4, L1, M1, M2. That is, two related but different sets of postulates are considered.

The Kaluza dimension (in the direction of which partial derivatives are treated as vanishing) is identified with a cylinder condition as in Kaluza's original theory. The difference here is that electromagnetic fields outside of charge sources are not identified with Ricci flat 5D Kaluza space, but a more general condition that identifies 5D momentum with kinetic charge sources. When the scalar field is set constant (and well-behaved assumptions are made about the paths of charged particles), and a weak field limit defined, then an improved unification of gravity and electromagnetism results. Improved because the Lorentz force law is derived from first principles, and because a more complete range of electromagnetic fields (i.e. the non-nullish solutions) become possible without making arbitrary assumptions or making too many demands on the scalar field, and without the coupling of the scalar field with the electromagnetic field. Vanishing background Ricci curvature was also not needed as the Lorentz force law was derived independently of any stress-energy tensor. The constancy of the scalar field was here assumed in order to obtain the electromagnetic limit. The theory was in effect derived from the need to allow more naturally for certain solutions (including electrostatic fields) and to derive the Lorentz force law simultaneously; and by searching for the simplest way to do so.

The physicality of the theory is that Kaluza's original 1921 theory is too restrictive and that the fault is not in the cylinder condition but the Ricci flat assumption. What is here shown to be effective is the identification of Maxwellian charge and 5D momentum. In effect replacing the Ricci flat assumption with vanishing 5D momentum - referred to here as kinetic charge - leads to the same results but with a larger set of solutions, provided a weak field limit is also imposed. Since 5D momentum is identified with Maxwell charge, at a suitable limit, the physicality of defining sourceless solutions by its vanishing is intuitive. Previous attempts to do the same thing by the current author had used torsion and are more complex. Torsion, so it turns out, is not necessary if the Ricci flat assumption can be weakened in other ways. The simple expedient of replacing the Ricci flatness assumption with a sourceless assumption was undertaken here.

Stability is a recurring objection to Kaluza theories, and causality is a further issue. However devices such as variant energy conditions, or super-energy conditions, could be used. Here a second set of postulates is suggested based on the generalised Bel super-energy tensor and a conformal transformation. The first postulate set and second are related via conformal transformations, thus the extended set is built firmly on the first as a foundation. This potentially explains how the scalar field can be treated as constant, and sheds some light on causality, and potentially avoids problems of stability. The result is a set of geometries related by what is essentially a gauge transformation in what could be described as a geometrised conformal gauge theory. Exactly how this might be taken further is the object of further study. This second postulate set is presented as an example of how Kaluza theory can be developed further - the ultimate arbiter necessarily being empirical evidence.

Interestingly the super-energy condition is related to harmonic curvature, and therefore Yang-Mills theory.

Why go to the effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences could be detectable given sufficient technology on the one hand, and, on the other, simply because such an attempt at unification might be right or lead in the right direction [22][23]. It may widen the search. This is not so much a controversial idea as merely a laborious one, and the current research has been rather circuitous. This new attempt is however so inspired, and some merit lies in the fact that the results are promising. But why do it this way when other methods may also be available? As with induced-matter theory [1] the idea is to be as geometrical and explanatory as possible, but the approach here is arguably more natural by virtue of requiring the Kaluza dimension to be necessarily small and compact, as in the old Kaluza-Klein theory. This remains a natural constraint, the most natural way so far found to unify electromagnetism and gravity.

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