A difficulty in proof theory

Reasonable assumptions cause the Gödeliar to collapse to the Liar

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Advice for refereeing: this paper should not be refereed by logicians but by mathematicians with some experience with empirical science. Otherwise it is useless to submit this paper. Of course such referees may consult with logicians but all involved should heed the warnings in this paper on dealing with logicians and mathematicians remote from empirical science in general.

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Summary

Adding some reasonable properties to the Gödelian system of Peano Arithmetic creates a new system for which Gödel's completeness theorems collapse and the Gödeliar becomes the Liar paradox again. Rejection of those properties is difficult since they are reasonable. Three-valued logic is a better option to deal with the Liar and its variants.

Introduction

Since January / Februari 2007 when Colignatus (1981, 2007a), "A logic of exceptions" (ALOE), finally made it from typescript in 1981 to print in 2007, logicians in Holland have hardly reacted but now there is a small question about the validity of the method by which some lemmas have been proven. Rather than going into the method of proof itself, it appears a good road to present the proofs in a different form. Since ALOE is intended for first year students and thus uses accessible ways of proof, it is less of an option to change ALOE. Those other ways of proof can best be explained in a separate article like this present one. One will understand that this is reverse didactics: a simple and accessible manner of proof for students is explained in a more complex fashion for teachers who have their doubts about that new simplicity. This approach has the added advantage that the main new result can also be presented in a short fashion to a larger audience. We now apply a double reverse Escher: the objective of this paper becomes the presentation of these new results in undebateable fashion.

The point of this discussion is to clarify that Gödel's two completeness theorems are a proof-theoretic variant of the age-old Liar paradox, where the latter is not to be interpreted semantically but only historically. A decent solution of that paradox requires three-valued logic (and a solution of the paradoxes of three-valued logic). This can be clarified by extending axiomatic system S with some additional nontrivial and rational properties P such that Gödel's theorems collapse. Rather than using the method of posing and retracting hypotheses we will now explicitly state those properties P.

Let the Gödeliar be $g = \neg (S \vdash g)$. Let c be the statement that S is consistent, and recall that Gödel showed that $c \Leftrightarrow g$. First we assume properties P that are perfectly acceptable when jugded on their own. Then we show that $(S \vdash \neg g)$ so that S refutes something that is true. We thus have $(S \vdash \neg c)$ too so that S proves that it is inconsistent. Secondly we show that $(S \vdash c)$. Thus under these P we find that S becomes inconsistent. To maintain a consistent system we would have to reject reasonable properties P or switch to three-valued logic.

1. Reasonable properties

The following properties state something about S and they don't show how it works in S. This manner has great economy since it allows for a wide variety for internal methods in systems while at the same time it allows us to discuss the consequences. An objection might be that this lacks an existence proof that there is at least one internal method to generate those outward properties. However, we will give one small example ("a robot") and this shows how that question can be dealt with more generally. While we lack a robot, a practical solution would be to ask someone to sit in a black box and act as that robot, and we would all be able to tell when that person would cheat.

Property 1 is that if p is proven and the consequence is that q is proven, then the system is smart enough to see itself that $p \Rightarrow q$. The system doesn't have to be real smart, it may only require a robot to put a p in a box "hypothesis", to note that some q falls in the box "consequence", and then pick up both statements and transfer $p \Rightarrow q$ to the box "proven". But when we forget about this robot then we merely have the property itself:

1. $\forall p$: $((S \vdash p) \Rightarrow (S \vdash q)) \Rightarrow (S \vdash (p \Rightarrow q))$

Property 2 is that when someone says p then this person is also willing to say that he or she said p. The property is accepted by logicians for Peano Arithmetic but it is useful to accept it explicitly for an abstract system. Though it appears to be known under other names it is my preference to call it "proof-consequentness" since this expresses that S is consequent for its notion of proof.

2. $\forall p$: $(S \vdash p) \Rightarrow (S \vdash (S \vdash p))$

One will quickly note that we can substitute $q = (S \vdash p)$ in (1) and that modus ponens on (1) and (2) generates:

(1) & (2) $\forall p$: $S \vdash (p \Rightarrow (S \vdash p))$

which is half of the "definition of truth" construction, i.e. where S would declare that proof is its concept of truth or assertion. We are already getting close to the Liar paradox.

Property 3 is a bit more complex. If S proves p and if for some q the combination of p *and* q causes an inconsistency, then S is smart enough to conclude that $p \Rightarrow \neg q$ (since p has already been accepted). One will quickly note that this also means that $\neg q$ will be accepted but it is better to concentrate on the weaker rule.

3. $\forall p, q$: $((S \vdash p) \land (S \vdash ((p \land q) \Rightarrow \neg c)) \Rightarrow (S \vdash (p \Rightarrow \neg q))$

The relevance of property 3 will become clearer when it is shown that this leads to ($S \vdash c$). Its discussion will also show that property 3 is quite reasonable.

2. $S \vdash \neg g$

When we take $p = g$ in (1) & (2) then we get $S \vdash (g \Rightarrow (S \vdash g))$ or $S \vdash (g \Rightarrow \neg g)$, and hence $(S \vdash \neg g)$. QED. From Gödel's second theorem we also know that g is equivalent to c and thus $(S \vdash \neg c)$. QED.

For readers who prefer the slow track, we may do the following. In the general condition $(S \vdash p) \Rightarrow (S \vdash (S \vdash p))$ we substitute $p = g$ (line 2) and $(S \vdash g) = \neg g$ (line 3). We use the axiom $((S \vdash p) \Rightarrow (S \vdash q)) \Rightarrow (S \vdash (p \Rightarrow q))$ and then substitute $p = g$ and $q = \neg g$. Subsequently, we apply modus ponens on line 5 and 3.

1	$(S \vdash (g \lor \neg g))$
2	$((S \vdash g) \Rightarrow (S \vdash (S \vdash g)))$
3	$((S \vdash g) \Rightarrow (S \vdash \neg g))$
4	$(((S \vdash p) \Rightarrow (S \vdash q)) \Rightarrow (S \vdash (p \Rightarrow q)))$
5	$(((S \vdash g) \Rightarrow (S \vdash \neg g)) \Rightarrow (S \vdash (g \Rightarrow \neg g)))$
6	$(S \vdash (g \Rightarrow \neg g))$
Ergo	————
7	$(S \vdash \neg g)$

Note that $S \vdash \neg g$ means that S refutes g, so that, if the system is consistent, g is not provable, and since it says so, it is true. Hence the system refutes a true statement. That $S \vdash \neg c$ means that the system proves is own inconsistency (while we would tend to assume its consistency). Note also that g is decidable i.e. not undecidable.

3. $S \vdash c$

When the system accepts Tertium non Datur (TND) then it cannot accept ¬TND since this would cause a contradiction. This provides us with a way for proving consistency. Consistency is equivalent to the *nonsequitur* of the *denial* of *tertium non datur* (yes, three times *not*). The key point is that "∃ .. and ..." *in* the system is the same as "∃ ... and ..." *about* the system.

definition $((S \vdash \text{TND}) \Leftrightarrow (S \vdash \forall_p (p \vee \neg p)))$

thus $(S \vdash \neg\, \text{TND}) \Leftrightarrow (S \vdash \exists_p (p \wedge \neg p)) \Leftrightarrow \exists_p ((S \vdash p) \wedge (S \vdash \neg p)) \Leftrightarrow \neg c$

thus $(\neg (S \vdash \neg\, \text{TND})) \Leftrightarrow \forall_p ((\neg (S \vdash p)) \vee (\neg (S \vdash \neg p))) \Leftrightarrow c$

Then we get the following proof. It will be useful to substitute $A = (S \vdash \text{TND})$ and $B = (S \vdash \neg\text{TND})$,

1)	$S \vdash ((\text{S} \vdash \text{TND}) \wedge (\text{S} \vdash \neg\text{TND})) \Rightarrow \neg c$	S uses the definition of c
2)	$S \vdash ((A \wedge B) \Rightarrow \neg c)$	line 1 with $A = (\text{S} \vdash \text{TND})$ and $B = (\text{S} \vdash \neg\text{TND})$
3)	$S \vdash \text{TND}$	axiom
4)	$S \vdash (S \vdash \text{TND})$	proof-consequent
5)	$S \vdash A$	line 4 with $A = (\text{S} \vdash \text{TND})$
6)	$((S \vdash p) \wedge (S \vdash ((p \wedge q) \Rightarrow \neg c)) \Rightarrow (S \vdash (p \Rightarrow \neg q))$	property 4 for some p and q
7)	$S \vdash (A \Rightarrow \neg B)$	from 2, 5, and 6 for $p = A$ and $q = B$
8)	$S \vdash ((S \vdash \text{TND}) \Rightarrow \neg(S \vdash \neg\text{TND}))$	translate 7 back
9)	$S \vdash \neg(\text{S} \vdash \neg\text{TND})$	modus ponens on 3 and 8
10)	$S \vdash c$	translate 9 again

Which gives an inconsistency with the earlier $S \vdash \neg c$. Thus when reasonable properties P are added to the normal ones then mathematics can only create inconsistent systems, not only those that say that they are inconsistent but also ones that really are inconsistent (which makes them true on the points where they are true, but false at the same time, since they also assert that those truths are false).

As said, this discussion and the explicit listing of the proof also shows that property 3 is reasonable indeed.

4. A historical analogy

The current situation reminds of the foundation crisis in voting theory. Mathematician Kenneth Arrow (1950, 1951, 1963) posed some reasonable properties that caused an impossibility in the theory of voting and democracy. Amongst the avalanche of responses, one could consider DeLong (1991) and Colignatus (1990c, 2007b) for more perspective. In an inverted manner, Kurt Gödel (1931) suggested that the Liar Paradox could be solved by switching from truth to provability, i.e. not using the Theory of Types of Russell & Whitehead's Principia *Mathematica*, but allowing selfreference and exploiting the weaker properties of just provability. Of the avalanche that he caused one could consider DeLong (1971) and Smorynski (1977). This present paper follows Kenneth Arrow's method and poses some reasonable properties that cause an impossibility to do mathematics and logic in two-valued logic. Colignatus (1981, 2007a) comes to the conclusion that a particular system of three-valued logic would be the proper approach to do math and logic with selfreference and consistency.

5. Metamathematically speaking

The term "metamathematical" is best used with respect to a former mathematical exercise, and not in an absolute sense, since otherwise we would not be able to do mathematics now. Thus, with respect to the former exercise, it must be noted that mathematics in general has the structure $\alpha \Rightarrow \beta$ or that we derive some consequences from some assumptions. We can never deduce more than we assumed, hence essentially mathematics is the begging of the question $\alpha \Rightarrow \alpha$, only less obvious, for we seek a β that may follow from α but originally seemed to differ from it. Also, the logician pur sang cannot say much about the selection of α and concentrates on $\alpha \Rightarrow \beta$. A pure logical refutation of Gödel's theorems thus is rather a non-affair since we can choose $\alpha = \gamma$ sufficiently weak so that they hold and sufficiently strong $\alpha = \gamma \wedge P$ so that they don't hold (or hold anyway but to help to create the inconsistency). The only relevant conclusion is that for consistency in two-valued logic one is forced to the weak assumptions. This is the mathematical position. On the other hand there is the metamathematical position of the empirical scientist. The empirical description of the world contains a mathematical substratum worth considering, that causes the empirical scientist to employ his or her mathematical faculty of mind. It would be an empirical scientist (e.g. the author of ALOE when not speaking purely logically) who would select α on some empirical grounds. The choice of $\alpha = \gamma$ has a high price. For empirical science we rather would use $\alpha = \epsilon \wedge P$, in fact $\gamma = \neg(\epsilon \wedge P)$, where the difference between γ and ϵ for example concerns the difference between two-valued and three-valued logic. From an empirical point of view much of the mathematical study of two-valued logic and the consequences of Gödel's theorems is nonsensical. Consistent but nonsensical. Modern mathematics adheres to some dogma of two-valued logic and thus chooses to loop out of science. These metamathematical considerations thus emphasize the difference between math and empirical science and highlight that modern mathematics is insulated against criticism since it uses only consistency as its criterion and the choice of weak $\alpha = \gamma = \neg(\epsilon \wedge P)$ indeed is consistent for some $\neg(\epsilon \wedge P) \Rightarrow \beta$.

How to break the stranglehold that the age-old Liar paradox has on modern mathematics ? This paper follows the approach of showing some properties P that are reasonable by themselves and that, when added to the Gödelian system cause it to collapse. Of course, the mathematician pur sang will say that it is easy to include some axioms, such as $S \vdash (\neg c \Rightarrow c)$ or $S \vdash (p \Rightarrow (S \vdash p))$. That mathematician pur sang will only consider the $\alpha \Rightarrow \beta$ structure and will have no strong opinion on the assumptions. If the addition causes an inconsistency he or she may well conclude that this is a begging of the question so that some of those additions must be perverse. Such reactions can indeed be observed. Namely, $S \vdash (\neg c \Rightarrow c)$ would be rejected since it is directly equivalent to an internal proof of consistency $S \vdash c$, and $S \vdash (p \Rightarrow (S \vdash p))$ would be rejected since it directly introduces the Liar paradox in S. For the empirical scientist it is strange that S would not be allowed to know itself what its concept of proof means. One would rather solve the Liar paradox than forbid its creation. One strategy to break the deadlock is to stay away from those obvious axioms and introduce the weaker P. These properties run up against the same kind of criticism from the mathematician pur sang who is insulated against any consideration that an empirical scientist might have on the choice of assumptions. Nevertheless, rejection becomes harder, and by using the weaker P it can become clearer that reasonable properties are rejected and that this carries the high price of turning mathematics irrelevant. Eventually, something would have to give. One is reminded of the Chinese proverb "The situation was unbearable and could no longer continue, it lasted 300 years".

The most likely outcome is that the mathematician pur sang will adopt three-valued logic and proceed as before, since little will have changed in attitude, only some other $\alpha \Rightarrow \beta$. Nevertheless, for the world, and students sensitive to logical paradoxes, it would be a beneficial change. And for the mathematical community it might become a topic of consideration whether it is so wise to aspire to become a mathematician pur sang. Mathematics is rather a faculty of the mind than a way of life or a mold to shape one's person. Aristotle, Archimede, Newton, Leibniz did some decent math while not getting lost in it. Modern math did get lost. Young minds aspiring at mathematics would better be wise in selecting their role models. Another important notion would be that a Department of Mathematics rather would be central to all other departments of Physics, Economics, Psychology etcetera rather than Physics only. Including a higher regard for engineering. Colignatus (2007c) gives another example, where mathematicians adopt the Penrose square root rule for voting weights, and do this even in an open letter to the governments in the EU, while their assumptions cannot be maintained since they neglect an empirical and statistical argument. In one part of his or her mind, the mathematician insulates him or herself from the world by adopting the $\alpha \Rightarrow \beta$ attitude, but in the other part of the mind is appears hard to do this consequently. Rather than maintain the fiction that the mathematical faculty of mind can also become the person himself or herself, it would be better to immerse mathematicians in the real world so that they are better aware of the pitfalls and uncertainties lurking there. One condition could be that one writes one paper on a real problem once in two years. Mathematicians who cannot deal with the world should not be the core of the Department but rather the fringe so that their fellows can protect the world from their lunacies, instead that the patients take over the asylum and abuse the great practical history of mathematics to lure the world along in those.

Conclusion

Adding some reasonable properties *P* to the Gödelian system of Peano Arithmetic creates a new and stronger system for which Gödel's completeness theorems collapse. Rejection of those properties is difficult since they are reasonable. Though this paper has not discussed the system of three-valued logic as presented in ALOE, it would be one of the options to consider. It isn't so that any system of three-valued logic would suffice. And the properties discussed here are not a definition for three-valued logic. But the conclusion definitely holds that we have discussed properties such that the Gödeliar collapses to the Liar and conditions for a three-valued logic to become necessary and sufficient.

Literature

Colignatus is the name of Thomas Cool in science.

Thomas Cool graduated in 1982 at Groningen University as an econometrician with the licence to teach both economics and mathematics. He has been employed as an econometrician and teacher of mathematics. Momentarily he is a teacher of mathematics at Wolfert van Borselen Highschool in Rotterdam, The Netherlands, and is a consultant in econometrics.

Statement of interest: Thomas Cool sells The Economics Pack that contains software referred to in this article.

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