The Proof for Non-existence of Magic Square of Squares in Order Three

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Abstract

This paper shows the non-existence of magic square of squares in order three by investigating two new tools, the first is representing three perfect squares in arithmetic progression by two numbers and the second is realizing the impossibility of two similar equations for the same problem at the same time in different ways and the variables of one is relatively less than the other.

Proof: we can proof non-existence of magic square of squares in order three, proof by contradition.
Let there is magic square of square in order three.

3X3 Magic square of squares

e_1^2	e_2^2	e_3^2
e_4^2	e_5^2	e_6^2
e_7^2	e_8^2	e_9^2

we can get either of arrangements as follows by proper tranformation of any magic square.

$e_8^2 < e_3^2 < e_4^2 < e_1^2 < e_5^2 < e_9^2 < e_6^2 < e_7^2 < e_2^2$ or

$e_8^2 < e_3^2 < e_1^2 < e_4^2 < e_5^2 < e_6^2 < e_9^2 < e_7^2 < e_2^2$

The second one is inclusive for our case of the proof.

Magic square of Squares of order three has all possible equation as follows

$R_1 = C_1 \to e_2^2 + e_3^2 = e_4^2 + e_7^2$

$R_1 = C_2 \to e_1^2 + e_3^2 = e_5^2 + e_8^2$

$R_1 = C_3 \to e_1^2 + e_2^2 = e_6^2 + e_9^2$

$R_2 = C_1 \to e_1^2 + e_7^2 = e_5^2 + e_6^2$

$R_2 = C_2 = D_1 = D_2 \to e_1^2 + e_9^2 = e_2^2 + e_8^2 = e_3^2 + e_7^2 = e_4^2 + e_6^2$

$R_2 = C_3 \to e_3^2 + e_9^2 = e_4^2 + e_5^2$

$R_3 = C_1 \to e_1^2 + e_4^2 = e_8^2 + e_9^2$

$R_3 = C_2 \to e_2^2 + e_5^2 = e_7^2 + e_9^2$

$R_3 = C_3 \to e_3^2 + e_6^2 = e_7^2 + e_8^2$

We can drive now the following Equations from all the above equations by rearranging them.

$e_3^2 - e_8^2 = e_4^2 - e_3^2 = e_5^2 - e_1^2 = e_9^2 - e_5^2 = e_7^2 - e_6^2 = e_2^2 - e_7^2$ .... eq 1.2

$e_1^2 - e_8^2 = e_6^2 - e_1^2 = e_5^2 - e_3^2 = e_7^2 - e_5^2 = e_9^2 - e_4^2 = e_2^2 - e_9^2$ .... eq 2.2

We can represent any three perfect squares in arithmetic progression by two numbers.
let us see How?

$$x^2 + z^2 = 2y^2$$
$$x < y < z$$
$$(z+y)(z-y) = (y+x)(y-x)$$

Let
$$a = z + y$$
$$b = z - y$$
$$c = y + x$$
$$d = y - x$$
$$ab = cd$$
$$a - b = c + d$$
$$a = b + c + d$$
$$b(b + c + d) = cd$$
$$c = \frac{b(b+d)}{d-b}$$

Let
$$f = d - b$$
$$c = \frac{b(2b+f)}{f} = \frac{2b^2}{f} + b$$
$$a = f + 2b + \frac{b(2b+f)}{f} = \frac{2b^2}{f} + 3b + f$$
$$x = \frac{c-d}{2} = \frac{\frac{2b^2}{f} - f}{2} = \frac{b^2}{f} - \frac{f}{2}$$

$$y = \frac{c+d}{2} = \frac{\frac{2b^2}{f}+2b+f}{2} = \frac{b^2}{f} + b + \frac{f}{2}$$

$$z = \frac{a+b}{2} = \frac{\frac{2b^2}{f}+3b+f+b}{2} = \frac{b^2}{f} + 2b + \frac{f}{2}$$

When We Represent the sets of three perfect squares in arithmetic progression by two numbers in magic square of squares in eq 1.2.

$$2e_1^2 = e_8^2 + e_6^2$$

$$e_8 = \frac{p^2}{q} - \frac{q}{2}$$

$$e_1 = \frac{p^2}{q} + p + \frac{q}{2}$$

$$e_6 = \frac{p^2}{q} + 2p + \frac{q}{2}$$

$$2e_5^2 = e_3^2 + e_7^2$$

$$e_3 = \frac{r^2}{s} - \frac{s}{2}$$

$$e_5 = \frac{r^2}{s} + r + \frac{s}{2}$$

$$e_7 = \frac{r^2}{s} + 2r + \frac{s}{2}$$

$$2e_9^2 = e_4^2 + e_2^2$$

$$e_4 = \frac{t^2}{u} - \frac{u}{2}$$

$$e_9 = \frac{t^2}{u} + t + \frac{u}{2}$$

$$e_2 = \frac{t^2}{u} + 2t + \frac{u}{2}$$

When we put back to 3x3 magic square

$(\frac{p^2}{q}+p+\frac{q}{2})^2$	$(\frac{t^2}{u}+2t+\frac{u}{2})^2$	$(\frac{r^2}{s}-\frac{s}{2})^2$
$(\frac{t^2}{u}-\frac{u}{2})^2$	$(\frac{r^2}{s}+r+\frac{s}{2})^2$	$(\frac{p^2}{q}+2p+\frac{q}{2})^2$
$(\frac{r^2}{s}+2r+\frac{s}{2})^2$	$(\frac{p^2}{q}-\frac{q}{2})^2$	$(\frac{t^2}{u}+t+\frac{u}{2})^2$

When we do rows, clumons and diagonals equalities calculation again we can get the following equations.

$$\frac{2p^3}{q}+3p^2+pq=\frac{2r^3}{s}+3r^2+rs=\frac{2t^3}{u}+3t^2+tu$$

.... eq 1.3.

$$2(\frac{2r^2}{s}-s)^2=(\frac{2p^2}{q}-q)^2+(\frac{2t^2}{u}-u)^2$$ ....

eq 2.3.

If there exist a magic square of squares, there is positive integers solution for the equations above. The first solution is premitive and there is no solution relatively less than them. We can make the equations a single Equation with four variables Let us see if the problem has another similar equation with relatively less variables, that can disqualify the existence of Magic Square of in order three.

When we Represent the sets of three perfect squares in arithmetic progression by two numbers in magic square of squares in diffirent way eq 2.2.

$$2e_3^2=e_8^2+e_4^2$$

$$e_8=\frac{j^2}{k}-\frac{k}{2}$$

$$e_3=\frac{j^2}{k}+j+\frac{k}{2}$$

$$e_4=\frac{j^2}{k}+2j+\frac{k}{2}$$

$$2e_5^2=e_1^2+e_9^2$$

$$e_1=\frac{l^2}{m}-\frac{m}{2}$$

$$e_5=\frac{l^2}{m}+l+\frac{m}{2}$$

$$e_9 = \frac{l^2}{m} + 2l + \frac{m}{2}$$
$$2e_9^2 = e_4^2 + e_2^2$$
$$e_6 = \frac{n^2}{o} - \frac{o}{2}$$
$$e_7 = \frac{n^2}{o} + n + \frac{o}{2}$$
$$e_2 = \frac{n^2}{o} + 2n + \frac{o}{2}$$

When we put back to 3x3 magic square once again

$(\frac{j^2}{k} + j + \frac{k}{2})^2$	$(\frac{n^2}{o} + 2n + \frac{o}{2})^2$	$(\frac{l^2}{m} - \frac{m}{2})^2$
$(\frac{n^2}{o} - \frac{o}{2})^2$	$(\frac{l^2}{m} + l + \frac{m}{2})^2$	$(\frac{j^2}{k} + 2j + \frac{k}{2})^2$
$(\frac{l^2}{m} + 2l + \frac{m}{2})^2$	$(\frac{j^2}{k} - \frac{k}{2})^2$	$(\frac{n^2}{o} + n + \frac{o}{2})^2$

When we do all possible rows, clumons and diagonals equalities calculation once again we can get the following equations.

$\frac{2j^3}{k} + 3j^2 + jk = \frac{2l^3}{m} + 3l^2 + lm = \frac{2n^3}{o} + 3n^2 + no$ .... eq 1.4.

$2(\frac{2l^2}{m} - m)^2 = (\frac{2j2}{k} - k)^2 + (\frac{2n^2}{o} - o)^2$ .... eq 2.4.

$$t = e_2 - e_9$$
$$n = e_2 - e_7$$
$$e_7 > e_9 \rightarrow t > n$$

$$t = n + e_7 - e_9$$
$$n + u + e_7 - e_9 = n + o + e_6 - e_4$$
$$u + e_7 - e_9 = o + e_6 - e_4$$
$$e_6 - e_4 > e_7 - e_9 \rightarrow u > o$$

$$r = l + e_7 - e_9$$

$e_7 > e_9 \rightarrow r > l$

$r = l + e_7 - e_9$
$l + s + e_7 - e_9 = l + m + e_1 - e_3$
$s + e_7 - e_9 = m + e_1 - e_3$
$e_1 - e_3 > e_7 - e_9 \rightarrow s > m$

$j - e_1 - e_3 = p - e_6 - e_4$
$e_6 - e_4 > e_1 - e_3 \rightarrow p > j$

$p + q = p + k + 2(e_1 - e_3) - (e_6 - e_4)$
$q = k + 2(e_1 - e_3) - (e_6 - e_4)$
$2(e_1 - e_3) > (e_6 - e_4) \rightarrow q > k$

We know that $e_7^2 - e_9^2 = e_1^2 - e_3^2$ and $e_7 > e_1$ then $e_1 - e_3 = e_7 - e_9$
And we should bother if we are not clear for last two inequalities because we are going to use four of six variables for single equation and at least one of them is comparable.
i.e. $p + q = e_{-1} - e_8$ and $j + k = e_{-3} - e_8$ $p + q => j + k \rightarrow p > jorq > k$ finally

$j < p, k < q, l < r, m < s, n < t, ando < t$

Now the existence of Magic Square of squares in order three beholds two similar equations at the same time that contradicts each other in the logic of positive integer Solutions. This implies the magic square of square never Exist.
Q.E.D

References

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