# Solutions of Navier-Stokes Equations
## plus
# Solutions of Magnetohydrodynamic Equations

## Abstract

In this paper, after nearly 150 years of waiting, the Navier-Stokes equations in 3-D for incompressible fluid flow have been analytically solved. In fact, it is shown that these equations can be solved in 4-dimensions or $n$-dimensions. The author has proposed and applied a new law, the law of definite ratio for fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. This law evolved from the author's earlier solutions of the Navier-Stokes equations. By applying the above law, the hitherto unsolved magnetohydrodynamic equations were routinely solved. It is also shown that without gravity forces on earth, there will be no incompressible fluid flow on earth as is known (see p.23, p.13). The difficulty in solving the Navier-Stokes equations has been due to finding a logical way to split the equations. By using the most fundamental principle for dividing a quantity into parts, using ratios, all hidden flaws in splitting the equations have been eliminated. The resulting sub-equations were readily integrable, and even, the nonlinear sub-equations were readily integrated. The preliminaries reveal how the ratio technique evolved as well as possible applications of the solution method in mathematics, science, engineering, business, economics, finance, investment and personnel management decisions. The coverage is as follows. The $x$–direction Navier-Stokes equation will be linearized, solved, and the solution analyzed. The linearized equation represents, except for the numerical coefficient of the acceleration term, the linear part of the Navier-Stokes equation. This solution will be followed by the solution of the Euler equation of fluid flow. The Euler equation represents the nonlinear part of the Navier-Stokes equation. The Euler equation was solved in the author's previous paper. Following the Euler solution, the Navier-Stokes equation will be solved, essentially by combining the solutions of the linearized equation and the Euler solution. For the Navier-Stokes equation, the linear part of the relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation; and the relation from the integration of the non-linear part satisfied the non-linear part of the equation. For the linearized equation, different terms of the equation were made subjects of the equation, and each such equation was integrated by first splitting-up the equation, using ratio, into sub-equations. The integration results were combined. Four equations were integrated. The relations obtained using these terms as subjects of the equations were checked in the corresponding equations. Only the equation with the gravity term as subject of the equation satisfied its corresponding equation, and this only satisfaction led to the law of definite ratio for fluid flow, stated above. This equation which satisfied its corresponding equation will be defined as the driver equation; and each of the other equations which did not satisfy its corresponding equation will be called a supporter equation. A supporter equation does not satisfy its corresponding equation completely but provides useful information about the driver equation which is not apparent in the solution of the driver equation. The solutions revealed the role of each term of the Navier-Stokes equations in fluid flow. Most importantly, the gravity term is the indispensable term in fluid flow, and it is involved in the parabolic as well as the forward motion of fluids. The pressure gradient term is also involved in the parabolic motion of fluids. The viscosity terms are involved in parabolic, periodic and decreasingly exponential motion of fluids. As the viscosity increases, the periodicity increases. The variable acceleration term is also involved in the periodic and decreasingly exponential motion of fluids. The convective acceleration term with $x$ as the independent variable produces square root function behavior. The other convective acceleration terms produce fractional expressions containing square root functions.

For a spin-off, the smooth solutions from above are specialized and extended to satisfy the requirements of the CMI Millennium Prize Problems, and thus prove the existence of smooth solutions of the Navier-Stokes equations.

# Options

**Option 1: Solutions of 3-D Linearized Navier-Stokes Equations** 3
**Option 2: Solutions of 4-D Linearized Navier-Stokes Equations** 17
**Option 3: Solutions of the Euler Equations** 18
**Option 4: Solutions of the Navier-Stokes Equations** 20
**Option 5: Solutions of 4-D Navier-Stokes Equations** 22
**Option 6: CMI Millennium Prize Problem Requirements** 26
**Option 7: Solutions of Magnetohydrodynamic Equations** 28

The Navier-Stokes equations in three dimensions are three simultaneous equations in Cartesian coordinates for the flow of incompressible fluids. The equations are presented below:

$$\begin{cases} \mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}) & (\mathrm{N}_x) \\ \mu(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2})-\frac{\partial p}{\partial y}+\rho g_y=\rho(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}) & (\mathrm{N}_y) \\ \mu(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2})-\frac{\partial p}{\partial z}+\rho g_z=\rho(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}) & (\mathrm{N}_z) \end{cases}$$

Equation ($N_x$) will be the first equation to be solved; and based on its solution, one will be able to write down the solutions for the other two equations, ($N_y$), and ($N_z$).

## Dimensional Consistency

The Navier-Stokes equations are dimensionally consistent as shown below:

$$\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) - \frac{\partial p_x}{\partial x} + \rho g_x = \rho(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z})$$

Using *MLT*

$$\boxed{M(L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2}-L^{-2}T^{-2}+L^{-2}T^{-2})=M(L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2})}$$

Using *kg–m–s*

$$\boxed{kg(m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}-m^{-2}s^{-2}+m^{-2}s^{-2}=kg(m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}}$$

# Option 1

## Solution of the Linearized Navier-Stokes Equation in the *x*-direction

The equation will be linearized by redefinition. The nine-term equation will be reduced to six terms.

**Given:** $$\mu(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2})-\frac{\partial p_x}{\partial x}+\rho g_x=\rho(\frac{\partial v_x}{\partial t}+V_x\frac{\partial v_x}{\partial x}+V_y\frac{\partial v_x}{\partial y}+V_z\frac{\partial v_x}{\partial z}) \qquad \text{(A)}$$

$$-\mu\frac{\partial^2 v_x}{\partial x^2}-\mu\frac{\partial^2 v_x}{\partial y^2}-\mu\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial v_x}{\partial t}+\rho V_x\frac{\partial v_x}{\partial x}+\rho V_y\frac{\partial v_x}{\partial y}+\rho V_z\frac{\partial v_x}{\partial z}=\rho g_x \qquad \text{(B)}$$

$$-\mu(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2})+\frac{\partial p_x}{\partial x}+4\rho(\frac{\partial v_x}{\partial t})=\rho g_x \qquad \text{(C)}$$

**Plan:** One will split-up equation (C) into five equations, solve them, and combine the solutions. On splitting-up the equations and proceeding to solve them, the non linear terms could be redefined and made linear. This linearization is possible if the gravitational force term is the subject of the equation as in equation (B). After converting the non-linear terms to linear terms by redefinition, one will have only six terms as in equation (C). One will show logically how equation (C) was obtained from equation (B), using a method which will be called the multiplier method.

Three main steps are covered.
In main Step 1, one shows how equation (C) was obtained from equation (B)
In main Step 2, equation (C) will be split-up into five equations.
In main Step 3, each equation will be solved.
In main Step 4, the solutions from the five equations will be combined.
In main Step 5, the combined relation will be checked in equation (C). for identity.

### Preliminaries

Here, one covers examples to illustrate the mathematical validity of how one splits-up equation (C). Let one think like a child - Albert Einstein. Actually, one can think like an eighth or a ninth grader. Suppose one performs the following operations:

**Example 1:** $10+20+25=55 \qquad (1)$

$10=55\times\frac{10}{55}=55\times\frac{2}{11} \qquad (2)$

$20=55\times\frac{20}{55}=55\times\frac{4}{11} \qquad (3)$

$25=55\times\frac{25}{55}=55\times\frac{5}{11} \qquad (4)$

Equations (2), (3), and (4) can be written as follows:

$10=55a \qquad (5)$

$20=55b \qquad (6)$

$25=55c \qquad (7)$

One will call $a, b$ and $c$ multipliers.

Above, $a=\frac{2}{11}$, $b=\frac{4}{11}$, $c=\frac{5}{11}$

Observe also that $a+b+c=1$
$(\frac{2}{11}+\frac{4}{11}+\frac{5}{11}=\frac{11}{11}=1)$

**Example 2**: Addition of only two numbers

$$20 + 25 = 45 \qquad (8)$$

$$20 = 45 \times \tfrac{20}{45} = 45 \times \tfrac{4}{9} \qquad (9)$$

$$25 = 45 \times \tfrac{25}{45} = 45 \times \tfrac{5}{9} \qquad (10)$$

Equations (9), and (10), can be written as follows:

$$20 = 45a \qquad (11)$$

$$25 = 45b \qquad (12)$$

Rewrite (8) by transposition.

If $20 - 45 = -25$

Then $20 = -25d$ ($d$ is a multiplier)

$-45 = -25f$ ($f$ is a multiplier)

Above, $d = \dfrac{20}{-25} = -\tfrac{4}{5}$, $f = \dfrac{-45}{-25} = \dfrac{9}{5}$,

Observe also here that $d + f = 1$ $\left(-\tfrac{4}{5} + \tfrac{9}{5} = \tfrac{5}{5} = 1\right)$

$a + b = 1$ $\left(\tfrac{4}{9} + \tfrac{5}{9} = \tfrac{9}{9} = 1\right)$

One can conclude that the sum of the multipliers is always 1.

**More formally:**

Let $A + B + C = S$, where $A, B, C$ and $S$. are real numbers. (for the moment), and one excludes 0.

Let $a, b, c$ be respectively, multipliers of the sum $S$ corresponding to $A, B, C$.

Then $A = Sa$, $B = Sb$, $C = Sc$; and $a + b + c = 1$

To show that $a + b + c = 1$,

$Sa + Sb + Sc = S$.

$S(a + b + c) = S$ (factoring out the $S$)

$a + b + c = 1$. (Dividing both sides of the equation by $S$)

**Example 3**: Solve the quadratic equation; $6x^2 + 11x - 10 = 0$
**Method 1** (a common and straightforward method)

By factoring, $6x^2 + 11x - 10 = 0$

$(3x-2)(2x+5) = 0$ and solving,

$(3x-2) = 0$ or $(2x+5) = 0$

$x = \frac{2}{3}, x = -\frac{5}{2}$. Solution set: $\{-\frac{5}{2}, \frac{2}{3}\}$

**Method 2:** One applies the discussion in Example 2
One will call this method the **multiplier method.**

Step 1**:** From $6x^2 + 11x - 10 = 0$ (1)

$6x^2 + 11x = 10$

$6x^2 = 10a$; (Here, $a$ is a multiplier)

$3x^2 = 5a$ (2)

$11x = 10b$ (Here, $b$ is a multiplier)

$11x = 10(1-a)$ $(a+b) = 1$

$11x = 10 - 10a$

$x = \frac{10-10a}{11}$

$3(\frac{10-10a}{11})^2 = 5a$ (Substituting for $x$ in (2)

$3(\frac{100-200a+100a^2)}{121} = 5a$

Step 2: $300a^2 - 1205a + 300 = 0$

$60a^2 - 241a + 60 = 0$

$a = \frac{241 \pm \sqrt{241^2 - 4(60)(60)}}{120}$

$a = \frac{241 \pm \sqrt{43681}}{120}$

$a = \frac{241 \pm 209}{120}$

$a = \frac{241 \pm 209}{120} = \frac{241+209}{120}$ or $\frac{241-209}{120}$

$= \frac{450}{120}$ or $\frac{32}{120}$

$= \frac{15}{4}$ or $\frac{4}{15}$

Step 3: Since $a + b = 1$, when $a = \frac{15}{4}$ or $3\frac{3}{4}$

$b = 1 - 3\frac{3}{4} = -2\frac{3}{4}$ or $-\frac{11}{4}$

when $a = \frac{4}{15}$, $b = 1 - \frac{4}{15} = \frac{11}{15}$

Step 4: When $b = -\frac{11}{4}$, $11x = 10(-\frac{11}{4})$

$x = -\frac{5}{2}$

When $b = \frac{11}{15}$, $11x = 10(\frac{11}{15})$

$x = \frac{10}{11}(\frac{11}{15})$; $x = \frac{2}{3}$

Again, one obtains the same solution set $\{-\frac{5}{2}, \frac{2}{3}\}$ as by the factoring method.

**About the multipliers**

**T**he values of the multipliers obtained were $a = \frac{15}{4}$ or $3\frac{3}{4}$, $b = -2\frac{3}{4}$ or $-\frac{11}{4}$; $a = \frac{4}{15}$. $b = \frac{11}{15}$.

It easy to understand, say, in $20 = 45 \times \frac{20}{45} = 45 \times \frac{4}{9}$, that the multiplier $\frac{4}{9}$ can be viewed as the fraction of the multiplicand, 45 .

Later, one will learn that the multipliers are ratio terms as in Examples 5, 6 and 7, below.

**Example 4** Solve $ax^2 + bx + c = 0$ by completing the square and incorporating the multiplier method.

Step 1: From $ax^2 + bx + c = 0$

$$ax^2 + bx = -c$$

Let $ax^2 = -cd$; ( $d$ is a multiplier) (1)

Let $bx = -cf$ ( $f$ is a multiplier) (2)

(and $d + f = 1$)

$$ax^2 + bx = -cd - cf \quad \text{(Adding equations (1) and (2)}$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}d - \frac{c}{a}f$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a}(d + f)$$

(completing the square on the left-hand side))

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \qquad (d + f = 1) \qquad (3)$$

One's interest is in equations (1), (2) and (3).

Step 2 $x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$$

$$= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Example 5**: A grandmother left \$45,000 in her will to be divided between eight grandchildren, Betsy, Comfort, Elaine, Ingrid, Elizabeth, Maureen, Ramona, Marilyn, in the ratio $\frac{1}{36} : \frac{1}{18} : \frac{1}{12} : \frac{1}{9} : \frac{5}{36} : \frac{1}{6} : \frac{7}{36} : \frac{2}{9}$. (**Note**: $\frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{6} + \frac{7}{36} + \frac{2}{9} = 1$)
How much does each receive?

**Solution:**

Betsy's share of \$45,000 $= \frac{1}{36} \times \$45{,}000 = \$1{,}250$

Comfort's share of \$45,000 $= \frac{1}{18} \times \$45{,}000 = \$2{,}500$

Elaine's share of \$45,000 $= \frac{1}{12} \times \$45{,}000 = \$3{,}750$

Ingrid's share of \$45,000 $= \frac{1}{9} \times \$45{,}000 = \$5{,}000$

Elizabeth's share of \$45,000 $= \frac{5}{36} \times \$45{,}000 = \$6{,}250$

Maureen's share of \$45,000 $= \frac{1}{6} \times \$45{,}000 = \$7{,}500$

Ramona's share of \$45,000 $= \frac{7}{36} \times \$45{,}000 = \$8{,}750$

Marilyn's share of \$45,000 $= \frac{2}{9} \times \$45{,}000 = \$10{,}000$

Check; Sum of shares $\boxed{= \$45{,}000}$
Sum of the fractions = 1

**Example 6:** Sir Isaac Newton left $\rho g_x$ units in his will to be divided between $-\mu\frac{\partial^2 V_x}{\partial x^2}$ , $-\mu\frac{\partial^2 V_x}{\partial y^2}$,

$-\mu\frac{\partial^2 V_x}{\partial z^2}$, $\frac{\partial p}{\partial x}$, $\rho\frac{\partial V_x}{\partial t}$, $\rho V_x\frac{\partial V_x}{\partial x}$, $\rho V_y\frac{\partial V_x}{\partial y}$, $\rho V_z\frac{\partial V_x}{\partial z}$ in the ratio $a:b:c:d:f:h:m:n$.

where $a+b+c+d+f+h+m+n=1$. How much does each receive?

**Solution** $-\mu\frac{\partial^2 V_x}{\partial x^2}$'s share of $\rho g_x$ units $= a\rho g_x$ units

$-\mu\frac{\partial^2 V_x}{\partial y^2}$'s share of $\rho g_x$ units $= b\rho g_x$ units

$-\mu\frac{\partial^2 V_x}{\partial z^2}$'s share of $\rho g_x$ units $= c\rho g_x$ units

$\frac{\partial p}{\partial x}$'s share of $\rho g_x$ units $= d\rho g_x$ units

$\rho\frac{\partial V_x}{\partial t}$'s share of $\rho g_x$ units $= f\rho g_x$ units

$\rho V_x\frac{\partial V_x}{\partial x}$'s share of $\rho g_x$ units $= h\rho g_x$ units

$\rho V_y\frac{\partial V_x}{\partial y}$'s share of $\rho g_x$ units $= m\rho g_x$ units

$\rho V_z\frac{\partial V_x}{\partial z}$'s share of $\rho g_x$ units $= n\rho g_x$ units

Sum of shares = $\boxed{\rho g_x \text{ units}}$ **Note**: $a+b+c+d+f+h+m+n=1$

**Example 7:** The returns on investments $A, B, C, D$ are in the ratio $a:b:c:d$. If the total return on these four investments is $P$ dollars, what is the return on each of these investments? $(a+b+c+d=1)$

**Solution** Return on investment $A = aP$ dollars
Return on investment $B = bP$ dollars
Return on investment $C = cP$ dollars
Return on investment $D = dP$ dollars

**Check** $aP+bP+cP+dP=P$
$P(a+b+c+d)=P$
$a+b+c+d=1$ (dividing both sides by $P$)

The objective of presenting examples 1, 2, 3, 4, 5, 6, and 7 was to convince the reader that the principles to be used in splitting the Navier-Stokes equations are valid.
In Examples 3 and 4, one could have used the quadratic formula directly to solve for $x$, without finding $a$ and $b$ first. The objective was to convince the reader that the introduction of $a$ and $b$ did not change the solution sets of the original equations.

For the rest of the coverage in this paper, a multiplier is the same as a ratio term
The multiplier method is the same as the ratio method.

# Main Step 1
# Linearization of the Non-Linear Terms

**Step 1**: The main principle is to multiply the right side of the equation by the ratio terms
This step is critical to the removal of the non-linearity of the equation.
$\rho g_x$ is to be divided by the terms on the left-hand--side of the equation in the ratio
$a : b : c : d : f : h : m : n \quad (a + b + c + d + f + h + m + n = 1$

$$-\mu \frac{\partial^2 v_x}{\partial x^2} - \mu \frac{\partial^2 v_x}{\partial y^2} - \mu \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p_x}{\partial x} \underbrace{+\rho \frac{\partial V_x}{\partial t} + \overbrace{\rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}}_{\text{all acceleration terms}} = \rho g_x \qquad (1)$$

Apply the principles involved in the ratio method covered in the preliminaries, to the nonlinear terms (the last three terms.)

Then $\rho V_z \frac{\partial V_x}{\partial z} = n\rho g_x$, where $n$ is the ratio term corresponding to $\rho V_z \frac{\partial V_x}{\partial z}$.

$$V_z \frac{\partial V_x}{\partial z} = n g_x \qquad (2)$$

$V_z \frac{dV_x}{dz} = n g_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dz}{dt}\frac{dV_x}{dz} = n g_x \quad (V_z = \frac{dz}{dt}$, by definition)

$$\frac{dv_x}{dt} = n g_x \qquad (3)$$

Therefore, $\boxed{V_z \frac{\partial V_x}{\partial z} = \frac{dV_x}{dt} = n g_x}$ (4)

**Step 2:** Similarly, Let $\rho V_y \frac{\partial V_x}{\partial y} = m\rho g_x$ ( $m$ is the ratio term corresponding to $\rho V_y \frac{\partial V_x}{\partial y}$ ) (5)

$V_y \frac{dV_x}{dy} = m g_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dy}{dt}\frac{dV_x}{dy} = m g_x \quad (V_y = \frac{dy}{dt})$

$$\frac{dv_x}{dt} = m g_x \qquad (6)$$

Therefore, $\boxed{V_y \frac{dV_x}{dy} = \frac{dV_x}{dt} = m g_x}$ (7)

**Step 3:** Let $\rho V_x \frac{\partial v_x}{\partial x} = h\rho g_x$ where $h$ is the ratio term corresponding to $\rho V_x \frac{\partial v_x}{\partial x}$.

$$V_x \frac{\partial V_x}{\partial x} = h g_x \qquad (8)$$

$V_x \frac{dV_x}{dx} = h g_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dx}{dt}\frac{dV_x}{dx} = h g_x \quad (V_x = \frac{dx}{dt})$

$\frac{dv_x}{dt} = h g_x$ (9) Therefore, $\boxed{V_x \frac{\partial V_x}{\partial x} = \frac{dV_x}{dt} = h g_x}$ (10)

From equations (4), (7), (10), $V_x \frac{\partial v_x}{\partial x} = V_y \frac{\partial v_x}{\partial y} = V_z \frac{\partial v_x}{\partial z} = \frac{dv_x}{dt}$ and

$$V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z} = \boxed{3\frac{dv_x}{dt}} \qquad (11)$$

Thus, the ratio of the linear term $\frac{\partial V_x}{\partial t}$ to the nonlinear sum $V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z}$ in equation (1) is 1 to 3. Unquestionably, there is a ratio between the sum of the nonlinear terms and the linear term $\frac{\partial v_x}{\partial t}$. This ratio must be verified experimentally.

**Note:** One could have obtained equation (C) from equation (A) by redefining the nonlinear terms by **carelessly** disregarding the partial derivatives of the nonlinear terms in equation (1). However, the author did not do that, but logically, the terms became linearized.

**Note** also that the above linearization is possible only if $\rho g_x$ is the subject of the equation, and it will later be learned that a solution to the logically linearized Navier-Stokes equation is obtained only if $\rho g_x$ is the subject of the equation.

**Step 4:** Substitute the right side of equation (11) for the nonlinear terms on the left- side of

$$-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p_x}{\partial x} + \underbrace{\rho\frac{\partial V_x}{\partial t} + \overbrace{\rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}}_{\text{all acceleration terms}} = \rho g_x \qquad (12)$$

Then one obtains $-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p_x}{\partial x} \underbrace{+\rho\frac{\partial V_x}{\partial t} + 3\rho\frac{\partial V_x}{\partial x}}_{\text{all acceleration terms}} = \rho g_x$

$$\boxed{-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p_x}{\partial x} + 4\rho\frac{\partial v_x}{\partial t} = \rho g_x} \quad \text{(simplifying)} \qquad (13)$$

Now, instead of solving equation (1), previous page, one will solve the following equation

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} - K\frac{\partial^2 V_x}{\partial y^2} - K\frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho}\frac{\partial p}{\partial x} + 4\frac{\partial V_x}{\partial t} = g_x} \qquad (k = \frac{\mu}{\rho}) \qquad (14)$$

# Main Step 2

**Step 5:** In equation (14) divide $g_x$ by the terms on the left side in the ratio $a:b:c:d:f$.

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} = ag_x; \quad -K\frac{\partial^2 V_x}{\partial y^2} = bg_x; \quad -K\frac{\partial^2 V_x}{\partial z^2} = cg_x; \quad \frac{1}{\rho}\frac{\partial p}{\partial x} = dg_x; \quad 4\frac{\partial V_x}{\partial t} = fg_x}$$

( $a$, $b$, $c$, $d$, $f$ are the ratio terms and $a+b+c+d+f=1$).

As proportions: $\frac{-K\frac{\partial^2 V_x}{\partial x^2}}{a} = \frac{g_x}{1}$; $\frac{-K\frac{\partial^2 V_x}{\partial y^2}}{b} = \frac{g_x}{1}$; $\frac{-K\frac{\partial^2 V_x}{\partial z^2}}{c} = \frac{g_x}{1}$; $\frac{\frac{1}{\rho}\frac{\partial p}{\partial x}}{d} = \frac{g_x}{1}$; $\frac{4\frac{\partial V_x}{\partial t}}{f} = \frac{g_x}{1}$

One can view each of the ratio terms $a$, $b$, $c$, $d$, $f$ as a fraction (a real number) of $\boxed{g_x}$ contributed by each expression on the left-hand side of equation (14) above

# Main Step 3

**Step 6**: Solve the differential equations in Step 5.

# Solutions of the five sub-equations

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} = ag}$$
$$k\frac{\partial^2 V_x}{\partial x^2} = -ag$$
$$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{k}g$$
$$\frac{\partial V_x}{\partial x} = -\frac{ag}{k}x + C_1$$
$$V_{x1} = -\frac{ag}{2k}x^2 + C_1 x + C_2$$

$$\boxed{-K\frac{\partial^2 V_x}{\partial y^2} = bg}$$
$$K\frac{\partial^2 V_x}{\partial y^2} = -bg$$
$$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{k}g$$
$$\frac{\partial V_x}{\partial y} = -\frac{bg}{k}y + C_3$$
$$V_{x2} = -\frac{bg}{2k}y^2 + C_3 y + C_4$$

$$\boxed{-K\frac{\partial^2 V_x}{\partial z^2} = cg}$$
$$K\frac{\partial^2 V_x}{\partial z^2} = -cg$$
$$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{k}g$$
$$\frac{\partial V_x}{\partial z} = -\frac{cg}{k}z + C_5$$
$$V_{x3} = -\frac{cg}{2k}z^2 + C_5 z + C_6$$

$$\boxed{\frac{1}{\rho}\frac{\partial p}{\partial x} = dg}$$
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = dg$$
$$\frac{\partial p}{\partial x} = d\rho g$$
$$p = d\rho g x + C_7$$
$$\boxed{4\frac{\partial V_x}{\partial t} = fg}$$
$$\frac{\partial V_x}{\partial t} = \frac{f}{4}g$$
$$V_{x4} = \frac{fg}{4}t$$

# Main Step 4

**Step 7:** One combines the above solutions

$$\begin{aligned}
V_x &= V_{x1} + V_{x2} + V_{x3} + V_{x4} \\
&= -\frac{ag}{2k}x^2 + C_1 x + C_2 - \frac{bg}{2k}y^2 + C_3 y + C_4 - \frac{cg}{2k}z^2 + C_5 z + C_6 + \frac{fg}{4}t + C_7 \\
&= -\frac{ag}{2k}x^2 + C_1 x - \frac{bg}{2k}y^2 + C_3 y - \frac{cg}{2k}z^2 + C_5 z + \frac{fg}{4}t + C_9 \\
&= -\frac{ag}{2k}x^2 - \frac{bg}{2k}y^2 - \frac{cg}{2k}z^2 + C_1 x + C_3 y + C_5 z + \frac{fg}{4}t + C_9 \\
&= -\frac{ag}{2k}x^2 - \frac{bg}{2k}y^2 - \frac{cg}{2k}z^2 + C_1 x + C_3 y + C_5 z + \frac{fg}{4}t + C_9 \\
&= -\frac{g_x}{2k}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4}t + C_9
\end{aligned}$$
$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4}t + C_9$$
$$P(x) = d\rho g_x x$$

$$\boxed{\begin{aligned}
&V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4} \\
&V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4}t + C_9 \\
&P(x) = d\rho g_x x
\end{aligned}}$$

For $V_x(x,t)$, let $y = 0,\ z = 0$

Then $\boxed{V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1 x + \frac{fg_x}{4}t + C_9}$ $\boxed{P(x) = d\rho g_x x}$

$$\boxed{V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_{10}x + C_9}$$

# Main Step 5
## Checking in equation (C)

**Step 8:** Find the derivatives, using

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$\boxed{P(x) = d\rho g_x x}$$

$$\frac{\partial V_x}{\partial x} = -\frac{\rho g_x}{2\mu}(2ax) + C_1$$

1. $\boxed{\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}}$

4. $\boxed{\frac{\partial p}{\partial x} = d\rho g}$; 5. $\boxed{\frac{\partial V_x}{\partial t} = \frac{fg_x}{4}}$

$$\frac{\partial V_x}{\partial y} = -\frac{\rho g_x}{\mu}(by) + C_3$$

2. $\boxed{\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}}$

$$\frac{\partial V_x}{\partial z} = -\frac{\rho g_x}{\mu}(cz)$$

3. $\boxed{\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu};}$

**Step 9:** Substitute the derivatives from Step 8 in $-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p_x}{\partial x} + 4\rho\frac{\partial V_x}{\partial t} = \rho g_x$ to check for identity (to determine if the relation obtained satisfies the original equation).

$$-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho\frac{\partial V_x}{\partial t} = \rho g_x$$

$$-\mu(-\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu}) + d\rho g_x + 4\rho\frac{f}{4}g_x \stackrel{?}{=} \rho g_x$$

$$a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + \rho f g_x \stackrel{?}{=} \rho g_x$$

$$ag_x + bg_x + cg_x + dg_x + fg_x \stackrel{?}{=} g_x$$

$$g_x(a+b+c+d+f) \stackrel{?}{=} g_x$$

$$g_x(1) \stackrel{?}{=} g_x \quad (a+b+c+d+f=1)$$

$$g_x \stackrel{?}{=} g_x \quad \text{Yes}$$

Scrapwork

$$\boxed{\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}; \quad \frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}; \quad \frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu};}$$

$$\boxed{\frac{\partial p}{\partial x} = d\rho g;} \quad \boxed{\frac{\partial V_x}{\partial t} = \frac{fg_x}{4}}$$

An identity is obtained and therefore, the solution of equation (C), p.96, is given by

$$\boxed{V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9; \quad P(x) = d\rho g_x x}$$

# Solution Summary for $V_x$, $V_y$ and $V_z$

**For** $V_x$ $\quad a+b+c+d+f=1$

$$\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p_x}{\partial x}+\rho g_X=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})$$

$$-K\frac{\partial^2 V_x}{\partial x^2}-K\frac{\partial^2 V_x}{\partial y^2}-K\frac{\partial^2 V_x}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}+4\frac{\partial V_x}{\partial t}=g_x$$

$$V_x=V_{x1}+V_{x2}+V_{x3}+V_{x4}$$

$$=-\frac{ag}{2k}x^2+C_1x+C_2-\frac{bg}{2k}y^2+C_3y+C_4-\frac{cg}{2k}z^2+C_5z+C_6+\frac{fg}{4}t+C_7+C_8$$

$$=-\frac{ag}{2k}x^2+C_1x-\frac{bg}{2k}y^2+C_3y-\frac{cg}{2k}z^2+C_5z+\frac{fg}{4}t+C_9$$

$$=-\frac{ag}{2k}x^2-\frac{bg}{2k}y^2-\frac{cg}{2k}z^2+C_1x+C_3y+C_5z+\frac{fg}{4}t+C_9$$

$$V_y(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9$$

$$P(x)=d\rho gx$$

For $V_x(x,t)$, let $y=0,\ z=0$

Then $\boxed{V_x(x,t)=-\frac{\rho g_x}{2\mu}(ax^2)+C_1x+\frac{fg_x}{4}t+C_9}$

**For** $V_y$ $\quad h+j+m+n+q=1$

$$\mu(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2})-\frac{\partial p}{\partial y}+\rho g_y=\rho(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z})$$

$$-K\frac{\partial^2 V_y}{\partial x^2}-K\frac{\partial^2 V_y}{\partial y^2}-K\frac{\partial^2 V_y}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial y}+4\frac{\partial V_y}{\partial t}=g_y$$

$$\boxed{\begin{aligned}&V_y=-\frac{hg_y}{2k}x^2+C_1x-\frac{jg_y}{2k}y^2+C_3y-\frac{mg_y}{2k}z^2+C_5z+\frac{ng_y}{4}t\\&V_y(x,y,z,t)=-\frac{\rho g_y}{2\mu}(hx^2+jy^2+mz^2)+C_1x+C_3y+C_5z+\frac{qg_y}{4}t+C\\&P(y)=n\rho g_y y\end{aligned}}$$

**For** $V_z$ $\quad r+s+u+v+w=1$

$$\mu(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2})-\frac{\partial p}{\partial z}+\rho g_z=\rho(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z})$$

$$-k\frac{\partial^2 V_z}{\partial x^2}-k\frac{\partial^2 V_z}{\partial y^2}-k\frac{\partial^2 V_z}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial z}+4\frac{\partial V_z}{\partial t}=g_z$$

$$\boxed{\begin{aligned}&V_z=-\frac{rg_z}{2k}x^2+C_1x-\frac{sg_z}{2k}y^2+C_3y-\frac{ug_z}{2k}z^2+C_5z+\frac{wg_z}{4}t\\&V_z(x,y,z,t)=-\frac{\rho g_z}{2\mu}(rx^2+sy^2+uz^2)+C_1x+C_3y+C_5z+\frac{wg_z}{4}t+C\\&P(z)=v\rho g_z z\end{aligned}}$$

# Discussion About Solutions

A solution to equation $-\mu(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial v_x}{\partial t})=\rho g_x$ (C) is

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9 \\ P(x)=d\rho g_x x;\quad (a+b+c+d+f=1)\end{array}}$$ •

This relation gives an identity when checked in Equation (C) above.

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, $g$, were zero, the first three terms, the seventh term, and $P(x)$ would all be zero be. This result can be stated emphatically that without gravity forces on earth, there will be no incompressible fluid flow on earth as is known. The above result will be the same when one covers the general case, Option 4.

The above parabolic solution is also encouraging. It reminds one of the parabolic curve obtained when a stone is projected vertically upwards at an acute angle to the horizontal..
The author also tried the following possible approaches: (D), (E) and (F), but none of the possible solutions completely satisfied the corresponding original equations (D), (E) or (F) .

$\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial v_x}{\partial t}=\frac{\partial p}{\partial x}$ ( D) (One uses the subject $\boxed{\frac{\partial p}{\partial x}}$

$\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$ (E), (One uses the subject $\boxed{\frac{\partial V_x}{\partial t}}$

$-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial v_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 v_x}{\partial x^2}$ (F) (One uses subject $\boxed{\frac{\partial^2 V_x}{\partial x^2}}$

## Integration Results Summary

**Case 1:** $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$ (C)

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9 \\ P(x)=d\rho g_x x;\quad (a+b+c+d+f=1)\end{array}}$$ **<----Solution**

**Case 2:** $\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial v_x}{\partial t}=\frac{\partial p}{\partial x}$ ( D). (One uses the subject $\boxed{\frac{\partial p}{\partial x}}$

$$\boxed{\begin{array}{l} V_x(x,y,z.t)=\frac{\lambda_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+\lambda_p x+C_3y+C_5z-\frac{f\lambda}{4\rho}t+C \\ P(x)=\frac{1}{d}\rho g_x x\end{array}}$$

**Case 3:** $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$ (E). (One uses the subject $\boxed{\frac{\partial V_x}{\partial t}}$

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=(C_1\cos\lambda_x x+C_2\sin\lambda_x x)e^{-(\lambda^2/\beta)t}+(C_3\cos\lambda_y y+C_4\sin\lambda_y y)e^{-(\lambda_y{}^2/\omega)t} \\ +(C_5\cos\lambda_z z+C_6\sin\lambda_z z)e^{-(\lambda_z{}^2/\varepsilon)t}+\frac{g}{4f}t+\lambda x+C_8 \\ P(x)=\lambda x=d\rho g_x x\end{array}}$$

**Case 4**: $-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial v_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 v_x}{\partial x^2}$ (F). (One uses the subject $\boxed{\frac{\partial^2 V_x}{\partial x^2}}$

$$V_x(x,y,z,t)=(A\cos\lambda y+B\sin\lambda y)\left(Ce^{(\frac{\lambda\sqrt{a}}{a})x}+De^{-(\frac{\lambda\sqrt{a}}{a})x}\right)$$
$$+(E\cos\lambda z+F\sin\lambda z)\left(He^{(\frac{\lambda\sqrt{b}}{b})x}+Le^{(-\frac{\lambda\sqrt{b}}{b})x}\right)-\frac{\rho g_x x^2}{2c\mu}+Ax+B+(A_1\cos\lambda x+B_1\sin\lambda x)e^{-(\lambda^2/\alpha)t}$$
$$+\frac{\lambda}{2\mu f}x^2+C_2x+C_3)$$
$$P(x)=d\rho g_x x$$

**Note**: Relations for equations with subjects $g_x$ and $\frac{\partial p}{\partial x}$ are almost identical.

By comparing possible solutions for equations (C) and (D), $\lambda_x=-\rho g_x$ in relation for (D).

$$V_x(x,y,z,t)=\frac{\lambda_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+\lambda_p x+C_3y+C_5z-\frac{f\lambda}{4\rho}t+C$$
$$P(x)=\frac{1}{d}\rho g_x x$$

The comparative analysis of the possible solutions when checked in each corresponding equation is presented in the table below.

| **Equation** | Equation Subject | Number of terms of possible solutions **not** satisfying original equation |
|---|---|---|
| **Case 1**: $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$ | $g_x$ | None<br>Case 1 yields the solution |
| **Case 2**: $\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial v_x}{\partial t}=\frac{\partial p}{\partial x}$ | $\frac{\partial p}{\partial x}$ | One term |
| **Case 3**: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$ | $\frac{\partial V_x}{\partial t}$ | At least 2 terms |
| **Case 4**: $-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial v_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 v_x}{\partial x^2}$ | $\frac{\partial^2 V_x}{\partial x^2}$ | At least 2 terms |

**Outcome 1:** With $g_x$ included and with $g_x$ as the subject of the equation.
The solution is straightforward and the possible solution checks well in the original equation (C)
Also, if $g_x$ or $\rho g_x$ is not the subject of the equation, the linearization of the nonlinear terms could not be justified.

**Outcome 2:** With $g_x$ included but with $\frac{\partial V_x}{\partial t}$ as the subject of the equation.

There are two problems when checking . **1.** For $\frac{\partial V_x}{\partial t} = -\frac{1}{4\rho}\frac{\partial p}{\partial x} \to -\frac{\lambda t}{4\rho d}$**;** **2.** $\frac{g}{4} = \frac{\partial V_x}{\partial t} \to \frac{gt}{4f}$

With $d$ and $f$ in the denominators, the multipliers sum $a + b + c + d + f = 1$ is false.

**Outcome 3 :** With $g_x$ excluded, and $\frac{\partial V_x}{\partial t}$ as the subject of the equation, there is one problem:

$-\frac{1}{4\rho}\frac{\partial p}{\partial x} = \frac{\partial V_x}{\partial t} \to -\frac{\lambda t}{4\rho d}$ .With $d$ in the denominator $a + b + c + d + f = 1$ is false

**Outcome 4 :** With $g_x$ included, and $\frac{\partial^2 V_x}{\partial x^2}$ as the subject of the equation, there are at least, two problems in the checking with the multipliers $c$ and $f$ in the denominators.
Checking for $a + b + c + d + f = 1$ is impossible.

## Characteristic curves of the integration results

| Equations | Equation Subject | Curve characteristics |
|---|---|---|
| **Case 1**: $-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho(\frac{\partial V_x}{\partial t}) = \rho g_x$ | $g_x$ | Parabolic and Inverted |
| **Case 2**: $\mu\frac{\partial^2 V_x}{\partial x^2} + \mu\frac{\partial^2 V_x}{\partial y^2} + \mu\frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho\frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}$ | $\frac{\partial p}{\partial x}$ | Parabolic |
| **Case 3**: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4}\frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4}\frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho}\frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}$ | $\frac{\partial V_x}{\partial t}$ | Periodic and decreasingly exponential |
| **Case 4**: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu}\frac{\partial V_x}{\partial t} + \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2}$ | $\frac{\partial^2 V_x}{\partial x^2}$ | Periodic, parabolic, and exponential |

The following are possible interpretations of the roles of the terms based on the types of curves produced when using the terms as subjects of the equations.

1. $g_x$ and $\frac{\partial p}{\partial x}$ are involved in the parabolic motion of fluids..
2. $\frac{\partial V_x}{\partial t}$ and $\frac{\partial^2 V_x}{\partial x^2}$ are involved in the parabolic, periodic and decreasingly exponential motion.
3. $g_x$ is responsible for the forward motion.

## Definitions and Classification of Equations

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} - K\frac{\partial^2 V_x}{\partial y^2} - K\frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho}\frac{\partial p}{\partial x} + 4\frac{\partial V_x}{\partial t} = g_x} \qquad (k = \frac{\mu}{\rho})$$

One may classify the equations involved in Option 1 according to the following:

**Driver Equation:** A differential equation whose integral satisfies its corresponding equation.
**Supporter equation**: A differential equation which contains the same terms as the driver equation but whose integral does not satisfy its corresponding equation but provides useful information about the driver equation.
Note that the driver equation and a supporter equation differ only in the subject of the equation.

| **Equation** | **Equation Subject** | **Type of equation** | **# of terms of relation not satisfying original equation** |
|---|---|---|---|
| **Case 1**: $-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho(\frac{\partial V_x}{\partial t}) = \rho g_x$ | $g_x$ | Driver Equation | None |
| **Case 2**: $\mu\frac{\partial^2 v_x}{\partial x^2} + \mu\frac{\partial^2 v_x}{\partial y^2} + \mu\frac{\partial^2 v_x}{\partial z^2} + \rho g_x - 4\rho\frac{\partial v_x}{\partial t} = \frac{\partial p}{\partial x}$ | $\frac{\partial p}{\partial x}$ | Supporter equation | One term |
| **Case 3**: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4}\frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4}\frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho}\frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}$ | $\frac{\partial V_x}{\partial t}$ | Supporter equation | At least 2 terms |
| **Case 4**: $-\frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu}\frac{\partial v_x}{\partial t} + \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 v_x}{\partial x^2}$ | $\frac{\partial^2 V_x}{\partial x^2}$ | Supporter equation | At least 2 terms |
| **Case 5**: $-\frac{\partial^2 v_x}{\partial x^2} - \frac{\partial^2 v_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu}\frac{\partial v_x}{\partial t} + \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 v_x}{\partial y^2}$ | $\frac{\partial^2 v_x}{\partial y^2}$ | Supporter equation | At least 2 terms |
| **Case 6**: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial x^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu}\frac{\partial V_x}{\partial t} + \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial z^2}$ | $\frac{\partial^2 V_x}{\partial z^2}$ | Supporter equation | At least 2 terms |

One can apply the above definitions in solving the magnetohydrodynamic equations.

## Applications of the splitting technique in science, engineering, business fields

The approach used in solving the equations allows for how the terms interact with each other. The author has not seen this technique anywhere, but the results are revealing and promising.

**Fluid flow design considerations:**

**1.** Maximize the role of $g_x$ forces, followed by**;** **2.** $\frac{\partial p}{\partial x}$ forces; then 3. $\frac{\partial V_x}{\partial t}$

(Make $g_x$ happy by always providing a workable $mg\sin\theta$) .

For long distance flow design such as for water pipelines, water channels, oil pipelines. whenever possible, the design should facilitate and maximize the role of gravity forces, and if design is impossible to facilitate the role of gravity forces, design for $\frac{\partial p}{\partial x}$ to take over flow.

The performance of $\frac{\partial^2 V_x}{\partial x^2}$ should be studied further, since its role is the most complicated: periodic, parabolic, and decreasingly exponential.

**Tornado Effect Relief**

Perhaps, machines can be designed and built to chase and neutralize or minimize tornadoes during touch-downs. The energy in the tornado at touch-down can be harnessed for useful purposes.

**Business and economics applications.**

**1.** Figuratively, if $g_x$ is the president of a company, it will have good working relationships with all the members of the board of directors, according to the solution of the Navier-Stokes equation. If $g_x$ is present at a meeting $g_x$ must preside over the meeting for the best outcome.

**2.** If $g_x$ is absent from a meeting, let $\frac{\partial p}{\partial x}$ preside over the meeting, and everything will workout well. However, if $g_x$ is present, $g_x$ must preside over the meeting.

To apply the results of the solutions of the Navier-Stokes equations in other areas or fields, the properties, characteristics and functions of $g_x$, $\frac{\partial p}{\partial x}$, $\frac{\partial v_x}{\partial t}$ must be studied to determine analogous terms in those areas of possible applications. Other areas of applications include investments choice decisions, financial decisions, personnel management and family relationships.

# Option 2
## Solutions of 4-D Navier-Stokes Equations (linearized)

One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

If one adds $\mu\frac{\partial^2 V_x}{\partial s^2}$ and $\rho V_s\frac{\partial V_x}{\partial s}$ to the 3-D $x$–direction equation, one obtains the 4-D Navier-- Stokes equation $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial v_x}{\partial t})+\rho V_s\frac{\partial V_x}{\partial s}=\rho g_x$

After linearization, $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2})+\frac{\partial p}{\partial x}+5\rho(\frac{\partial v_x}{\partial t})=\rho g_x$ and its solution is

$$\boxed{\begin{array}{l} V_x(x,y,z,s,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2+es^2)+C_1x+C_3y+C_5z+C_7s+\frac{fg_x}{5}t+C_9 \\ P(x)=d\rho g_x x \qquad (a+b+c+d+e+f=1) \end{array}}$$

For $n$–dimensions one can repeat the above as many times as one wishes. **Back to Options**

# Option 3
# Solutions of the Euler Equations of Fluid flow

In the Navier-Stokes equation, if $\mu = 0$, one obtains the Euler equation. From

$$\mu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}) - \frac{\partial p}{\partial x} + \rho g_x = \rho(\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + V_z\frac{\partial v_x}{\partial z})$$ , one obtains

Euler equation : $(\mu = 0)$ $-\frac{\partial p}{\partial x} + \rho g_x = \rho(\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + V_z\frac{\partial v_x}{\partial z})$ or

$$\rho(\frac{\partial v_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + V_z\frac{\partial v_x}{\partial z}) + \frac{\partial p_x}{\partial x} = \rho g_x$$ <---driver equation.

Euler equation $(\mu = 0)$: $\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + V_z\frac{\partial v_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_x$ <---driver equation

Split the equation using the ratio terms $f_e$, $h_e$, $n_e$, $q_e$, $d_e$,, and solve. $(f_e + h_e + n_e + q_e + d_e = 1)$

| | | | | |
|---|---|---|---|---|
| **1.** $\frac{\partial V_x}{\partial t} = f_e g_x$<br>$V_{x4} = f_e g_x t$<br>$V_{x4} = f g_x t$ | **2.** $V_x\frac{\partial V_x}{\partial x} = h_e g_x$<br>$V_x\frac{dV_x}{dx} = h_e g_x$<br>$V_x dV_x = h_e g_x dx$<br>$\frac{V_x^{\ 2}}{2} = h_e g_x x$ or<br>$V_x^{\ 2} = 2h_e g_x x$<br>$V_x = \pm\sqrt{2h_e g_x x}$ | **3.** $V_y\frac{\partial V_x}{\partial y} = n_e g_x$<br>$V_y\frac{dV_x}{dy} = n_e g_x$<br>$V_y dV_x = n_e g_x \mathrm{dy}$<br>$V_y V_x = n_e g_x \mathrm{y} + \psi_y(V_y)$<br>$V_{x6} = \frac{n_e g_x \mathrm{y}}{V_y} + \frac{\psi_y(V_y)}{V_y}$<br>$V_y \neq 0$ | **4.** $V_z\frac{\partial V_x}{\partial z} = q_e g_x$<br>$V_z\frac{dV_x}{dz} = q_e g_x$<br>$V_z dV_x = q_e g_x \mathrm{dz}$;<br>$V_z V_x = q_e g_x \mathrm{z} + \psi_z(V_z)$<br>$V_{x7} = \frac{q_e g_x \mathrm{z}}{V_z} + \frac{\psi_z(V_z)}{V_z}$<br>$V_z \neq 0$ | **5.** $\frac{1}{\rho}\frac{\partial p}{\partial x} = d_e g_x$<br>$\frac{1}{\rho}\frac{\partial p}{\partial x} = d_e g_x$<br>$\frac{\partial p}{\partial x} = d_e \rho g_x$<br>$p = d_e \rho g_x x + C_7$ |

$$V_x(x,y,z,t) = f_e g_x t \pm \sqrt{2h_e g_x x} + \frac{n_e g_x \mathrm{y}}{V_y} + \frac{q_e g_x \mathrm{z}}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C$$
$$P(x) = d_e \rho g_x x \qquad (f_e + h_e + n_e + q_e + d_e = 1)\ V_y \neq 0,\ V_z \neq 0$$

Find the test derivatives to check in the original equation.

| | | | | |
|---|---|---|---|---|
| 1. $\frac{\partial V_x}{\partial t} = f_e g_x$ | **2.** $V_x^{\ 2} = 2h_e g_x x$; $2V_x\frac{\partial V_x}{\partial x} = 2h_e g_x x$;<br>$\frac{\partial V_x}{\partial x} = \frac{h_e g_x}{V_x}$, $V_x \neq 0$ | 3. $\frac{\partial V_x}{\partial y} = \frac{n_e g_x}{V_y}$<br>$V_y \neq 0$ | 4. $\frac{\partial V_x}{\partial z} = \frac{q_e g_x}{V_z}$<br>$V_z \neq 0$ | 5. $\frac{\partial p}{\partial x} = d_e \rho g_x$ |

$\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + V_z\frac{\partial v_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_x$ (Above, $\psi_y(V_y)$ and $\psi_z(V_z)$ are arbitrary functions)

$$f_e g_x + V_x\frac{h_e g_x}{V_x} + V_y\frac{n_e g_x}{V_y} + V_z\frac{q_e g_x}{V_z} + \frac{1}{\rho}d_e \rho g_x \overset{?}{=} g_x$$

$$f_e g_x + h_e g_x + n_e g_x + q_e g_x + d_e g_x \overset{?}{=} g_x$$

$$g_x(f_e + h_e + n_e + q_e + d_e) \overset{?}{=} g_x$$

$$g_x(1) \overset{?}{=} g_x \qquad (f_e + h_e + n_e + q_e + d_e = 1)$$

$$g_x \overset{?}{=} g_x$$ Yes

The relation obtained satisfies the Euler equation. Therefore the solution to the Euler equation is

$$V_x(x,y,z,t) = fgt \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C$$
$$P(x) = d\rho g_x x \quad V_y \neq 0,\ V_z \neq 0$$

The above is the solution of the driver equation. There are 5 supporter equations which will not be solved here.

**Question:** Has the Euler equation of fluid flow been solved for the first time?

**Note:** So far as the solutions of the equations are concerned, one needs not have explicit expressions for $V_x$, $V_y$, and $V_z$.

The impediment to solving the Euler equations has been due to how to obtain sub-equations from the six-term equation. The above solution was made possible after pairing the terms of the equation using ratios (by way of multipliers). The author was encouraged by Lagrange's use of ratios and proportion in solving differential equations. One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

## Extra:

**Linearized Euler Equation:** If one linearizes the Euler equation as was done in Option 1, one obtains

$$4\frac{\partial V_x}{\partial t} + \frac{1}{\rho}\frac{\partial p_x}{\partial x} = g_x;$$ whose solution is $V_x = \frac{fg_x}{4}t + C;\quad P(x) = d\rho g_x x$. (see Option 1 results)

Results for the Euler equations are presented below: for $V_x$, $V_y$ and $V_z$

For $V_x$ : $\frac{\partial p}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x\frac{\partial V_x}{\partial x} + \rho V_y\frac{\partial V_x}{\partial y} + \rho V_z\frac{\partial V_x}{\partial z}) = \rho g_x$

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};\ P(x) = d\rho g_x x$$
$$V_y \neq 0,\ V_z \neq 0$$
*x*-direction

For $V_y$, $\frac{\partial p}{\partial y} + \rho\frac{\partial V_y}{\partial t} + \rho V_x\frac{\partial V_y}{\partial x} + \rho V_y\frac{\partial V_y}{\partial y} + \rho V_z\frac{\partial V_y}{\partial z} = \rho g_y$

$$V_y(x,y,z,t) = \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_y x}{V_x} + \frac{\lambda_8 g_y z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z};\ P(y) = \lambda_4 \rho g_y y$$
$$V_x \neq 0,\ V_z \neq 0$$
*y*-direction

For $V_z$ : $\frac{\partial p}{\partial z} + \rho\frac{\partial V_z}{\partial t} + \rho V_x\frac{\partial V_z}{\partial x} + \rho V_y\frac{\partial V_z}{\partial y} + \rho V_z\frac{\partial V_z}{\partial z} = \rho g_z$

$$V_z(x,y,z,t) = \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_z x}{V_x} + \frac{\beta_7 g_z y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y};\ P(z) = \beta_4 \rho g_z z$$
$$V_x \neq 0,\ V_y \neq 0$$
*z*-direction

**Note:**

By comparison with Navier-Stokes equation and its relation, a relation to Euler equation can be found by deleting the Navier-Stokes relation resulting from the $\mu$-terms.

**Back to Options**

# Option 4
# Solutions of the Navier-Stokes Equations
**(Original)**

As it was in Option 1 for solving these equations,the first step here is to split-up the equation into eight sub-equations using the ratio method. One will solve **only** the driver equation, based on the experience gained in solving the linearized equation. There are 8 supporter equations.

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\overbrace{\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}=\rho g_x \quad \text{(A)}$$

$$-K\frac{\partial^2 V_x}{\partial x^2}-K\frac{\partial^2 V_x}{\partial y^2}-K\frac{\partial^2 V_x}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=g_x \qquad (K=\frac{\mu}{\rho}) \quad \text{(B)}$$

**Step 1:** Apply the ratio method to equation (B) to obtain the following equations:

1. $-K\frac{\partial^2 V_x}{\partial x^2}=ag_x$; 2. $-K\frac{\partial^2 V_x}{\partial y^2}=bg_x$; 3. $-K\frac{\partial^2 V_x}{\partial z^2}=cg_x$; 4. $\frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x$; 5. $\frac{\partial V_x}{\partial t}=fg_x$

6. $V_x\frac{\partial V_x}{\partial x}=hg_x$; 7. $V_y\frac{\partial V_x}{\partial y}=qg_x$; 8. $V_z\frac{\partial V_x}{\partial z}=ng_x$

where $a, b, c, d, f, h, n, q$ are the ratio terms and $a+b+c+d+f+h+n+q=1$

**Step 2**: Solve the differential equations in Step 1.

**Note** that after splitting the equations, the equations can be solved using techniques of ordinary differential equations.

One can view each of the ratio terms $a, b, c, d, f, h, n, q$ as a fraction (a real number) of $\boxed{g_x}$ contributed by each expression on the left-hand side of equation (B) above.

## Solutions of the eight sub-equations

| | | | |
|---|---|---|---|
| $\boxed{1.\ -k\frac{\partial^2 V_x}{\partial x^2}=ag}$<br>$k\frac{\partial^2 V_x}{\partial x^2}=-ag$<br>$\frac{\partial^2 V_x}{\partial x^2}=-\frac{a}{k}g$<br>$\frac{\partial V_x}{\partial x}=-\frac{ag}{k}x+C_1$<br>$V_{x1}=-\frac{ag}{2k}x^2+C_1x+C_2$ | $\boxed{2.\ -K\frac{\partial^2 V_x}{\partial y^2}=bg}$<br>$K\frac{\partial^2 V_x}{\partial y^2}=-bg$<br>$\frac{\partial^2 V_x}{\partial y^2}=-\frac{b}{k}g$<br>$\frac{\partial V_x}{\partial y}=-\frac{bg}{k}y+C_3$<br>$V_{x2}=-\frac{bg}{2k}y^2+C_3y+C_4$ | $\boxed{3.\ -K\frac{\partial^2 V_x}{\partial z^2}=cg}$<br>$K\frac{\partial^2 V_x}{\partial z^2}=-cg$<br>$\frac{\partial^2 V_x}{\partial z^2}=-\frac{c}{k}g$<br>$\frac{\partial V_x}{\partial z}=-\frac{cg}{k}z+C_5$<br>$V_{x3}=-\frac{cg}{2k}z^2+C_5z+C_6$ | $\boxed{4.\ \frac{1}{\rho}\frac{\partial p}{\partial x}=dg}$<br>$\frac{1}{\rho}\frac{\partial p}{\partial x}=dg$<br>$\frac{\partial p}{\partial x}=d\rho g$<br>$p=d\rho gx+C_7$<br>$\boxed{5.\ \frac{\partial V_x}{\partial t}=fg}$<br>$V_{x4}=fgt$ |
| $\boxed{6.\ V_x\frac{\partial V_x}{\partial x}=hg_x}$<br>$V_x\frac{dV_x}{dx}=hg_x$<br>$V_xdV_x=hg_xdx$<br>$\frac{V_x^2}{2}=hg_xx$<br>$V_{x5}=\pm\sqrt{2hg_xx}$ | $\boxed{7.\ V_y\frac{\partial V_x}{\partial y}=ng_x}$<br>$V_y\frac{dV_x}{dy}=ng_x$<br>$V_ydV_x=ng_x\mathrm{dy}$<br>$V_yV_x=ng_x\mathrm{y}+\psi_y(V_y)$<br>$V_{x6}=\frac{ng_x\mathrm{y}}{V_y}+\frac{\psi_y(V_y)}{V_y}$ | $\boxed{8.\ V_z\frac{\partial V_x}{\partial z}=qg_x}$<br>$V_z\frac{dV_x}{dz}=qg_x$<br>$V_zdV_x=qg_x\mathrm{dz}$;<br>$V_zV_x=qg_x\mathrm{z}+\psi_z(V_z)$<br>$V_{x7}=\frac{qg_x\mathrm{z}}{V_z}+\frac{\psi_z(V_z)}{V_z}$ | **Note:**<br>$\psi_y(V_y)$, $\psi_z(V_z)$ are arbitrary functions, (integration constants)<br>$V_y\neq 0$<br>$V_z\neq 0$ |

**Step 3:** One combines the above solutions

$$V_x(x,y,z,t) = V_{x1} + V_{x2} + V_{x3} + V_{x4} + V_{x5} + V_{x6} + V_{x7}$$

$$= -\frac{ag_x}{2k}x^2 + C_1 x - \frac{bg_x}{2k}y^2 + C_3 y - \frac{cg_x}{2k}z^2 + C_5 z + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}$$

$$\overbrace{-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t}^{\text{relation for linear terms}} \pm \overbrace{\sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}}^{\text{relation for non-linear terms}} + C_9$$

$P(x) = d\rho g_x x; \quad (a+b+c+d+f+h+n+q=1) \quad V_y \neq 0,\ V_z \neq 0$

**Step 4**: Find the test derivatives

| Test derivatives for the linear part | | | | | Test derivatives for the non-linear part | | |
|---|---|---|---|---|---|---|---|
| $\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}$ | $\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}$ | $\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu}$ | $\frac{\partial p}{\partial x} = d\rho g_x$ | $\frac{\partial V_x}{\partial t} = fg_x$ | $V_x^2 = 2hg_x x$<br>$2V_x \frac{\partial V_x}{\partial x} = 2hg_x x$<br>$\frac{\partial V_x}{\partial x} = \frac{hg_x}{V_x}, V_x \neq 0$ | $\frac{\partial V_x}{\partial y} = \frac{ng_x}{V_y}$ | $\frac{\partial V_x}{\partial z} = \frac{qg_x}{V_z}$ |

**Step 5:** **S**ubstitute the derivatives from Step 4 in equation (A) for the checking.

$$-\mu\frac{\partial^2 V_x}{\partial x^2} - \mu\frac{\partial^2 V_x}{\partial y^2} - \mu\frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p_x}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x\frac{\partial V_x}{\partial x} + \rho V_y\frac{\partial V_x}{\partial y} + \rho V_z\frac{\partial V_x}{\partial z} = \rho g_x \quad \textbf{(A)}$$

$$-\mu\left(-\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu}\right) + d\rho g_x + f\rho g_x + \rho\left(V_x\frac{hg_x}{V_x}\right) + \rho V_y\left(\frac{ng_x}{V_y}\right) + \rho V_z\left(\frac{qg_x}{V_z}\right) \stackrel{?}{=} \rho g_x$$

$$a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + f\rho g_x + h\rho g_x + n\rho g_x + q\rho g_x \stackrel{?}{=} \rho g_x$$

$$ag_x + bg_x + cg_x + dg_x + fg_x + hg_x + ng_x + qg_x \stackrel{?}{=} g_x$$

$$g_x(a+b+c+d+f+h+n+q) \stackrel{?}{=} g_x$$

$$g_x(1) \stackrel{?}{=} g_x \quad \text{Yes} \quad (a+b+c+d+f+h+n+q=1)$$

**Step 6:** The linear part of the relation satisfies the linear part of the equation; and the non-linear part of the relation satisfies the non-linear part of the equation.(B) below is the solution.

**Analogy for the Identity Checking Method:** If one goes shopping with American dollars and Japanese yens (without any currency conversion) and after shopping, if one wants to check the cost of the items purchased, one would check the cost of the items purchased with dollars against the receipts for the dollars; and one would also check the cost of the items purchased with yens against the receipts for the yens purchase. However, if one converts one currency to the other, one would only have to check the receipts for only a single currency, dollars or yens. This conversion case is similar to the linearized equations, where there was no partitioning in identity checking.

**Summary of solutions for** $V_y$, $V_z$ ( $P(y) = \lambda_4 \rho g_y y$ , $P(z) = \beta_4 \rho g_z z$)

$$-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+C_9 \quad \textbf{(B)}$$

$P(x) = d\rho g_x x;$ $(a+b+c+d+h+n+q=1)$ $V_y \neq 0,\ V_z \neq 0$

$$V_y = -\frac{\rho g_y}{2\mu}(\lambda_1x^2+\lambda_2y^2+\lambda_3z^2)+C_1x+C_3y+C_5z+\lambda_5g_yt\pm\sqrt{2\lambda_7g_yy}+\frac{\lambda_6g_yx}{V_x}+\frac{\lambda_8g_yz}{V_z}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_z(V_z)}{V_z}$$

$P(y) = \lambda_4\rho g_y y$ $V_x \neq 0,\ V_z \neq 0$

$$V_z = -\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}+\frac{\beta_6g_zx}{V_x}+\frac{\beta_7g_zy}{V_y}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y}$$

$V_x \neq 0,\ V_y \neq 0$

# Option 5

## Solutions of 4-D Navier-Stokes Equations

One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

If one adds $\mu\frac{\partial^2 V_x}{\partial s^2}$ and $\rho V_s\frac{\partial V_x}{\partial s}$ to the 3-D $x$–direction equation, one obtains the 4-D Navier-Stokes equation

$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2}\right)+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho V_s\frac{\partial V_x}{\partial s}=\rho g_x$$

whose solution is given by

$$V_x(x,y,z,s,t) = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2+es^2)+C_1x+C_3y+C_5z+C_6s+fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{rg_xs}{V_s}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+\frac{\psi_s(V_s)}{V_s}}_{\text{arbitrary functions}}+C_9$$

$P(x) = d\rho g_x x$ $(a+b+c+d+e+f+h+n+q+r=1)$ $V_x \neq 0,\ V_y \neq 0,\ V_s \neq 0,$

For $n$–dimensions one can repeat the above as many times as one wishes.

### Extra: Two-term Linearization of the Navier-Stokes Equation

(Equation contains one nonlinear term)

By linearization as in Option 1, if one replaces $\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}$ by $2\rho\frac{\partial V_x}{\partial t}$ in

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x$$ one obtains

$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2}\right)+\frac{\partial p}{\partial x}+3\rho\left(\frac{\partial V_x}{\partial t}\right)+\rho V_x\frac{\partial V_x}{\partial x}=\rho g_x$$, whose solution is

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_xt}{3}\pm\sqrt{2hg_xx}+C_6$$

**Back to Options**

# Conclusion

Since one began solving the Navier-Stokes equations by thinking like an eighth grader, and one was able to find a ratio technique for splitting the equations and solving them, perhaps, it is appropriate, after a few months of aging, to think like a ninth grader in the conclusion. One will reverse the coverage approach and begin from the general case and end with the special cases.

## Solutions of the Navier--Stokes equations (general case)

$x$–direction **Navier-Stokes Equation** (also driver equation)

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x \quad x\text{–direction}$$

$$V_x(x,y,z,t)=$$

$$\overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt}^{\text{solution for linear terms}}\pm\overbrace{\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}}^{\text{solution for non - linear terms}}+C_9$$

$$P(x)=d\rho g_xx;\quad (a+b+c+d+h+n+q=1)\quad V_y\neq 0,\ V_z\neq 0$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, $g$, were zero, the first three terms, the 7th term, the 8th term, the 9th term, the 10th term and $P(x)$ would all be zero.
This result can be stated emphatically that without gravity forces on earth, there will be no incompressible fluid flow on earth as is known. The above is a very important new insight, because in posing problems on incompressible fluid flow, it is sometimes suggested that the gravity term is zero. Such a suggestion would guarantee a no solution to the problem, according to the above solution of the Navier-Stokes equation.
The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio and also each term utilizes gravity to function. This law was applied in splitting-up the Navier-Stokes equations. The resulting sub-equations were readily integrable, and even the nonlinear sub-equations were readily integrated.
The $x$–direction Navier-Stokes equation was split-up into sub-equations using ratios. The sub-equations were solved and combined. The relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation and the relation obtained from integrating the nonlinear part of the equation satisfied the nonlinear part of the equation. By solving algebraically and simultaneously for $V_x$, $V_y$ and $V_z$, the $(ng_xy/V_y)$ and $(qg_xz/V_z)$ terms would be replaced by fractional terms containing square root functions. One may note that in checking the relations obtained for integrating the equations for possible solutions, one needs not have explicit expressions for $V_x$, $V_y$, and $V_z$, since these behave as constants in the checking process. The above solution is the solution to the driver equation. There are eight supporter equations (see below and see also Option 1 solution, p110). Only the solution to the driver equation completely satisfies its corresponding Navier-Stokes equation.. A supporter equation does not completely satisfy its corresponding Navier-Stokes equation. The above $x$–direction solution is the solution everyone has been waiting for, for nearly 150 years. It was obtained in two simple steps, namely, splitting the equation using ratios and integrating. The task for the future is to solve the equations for $V_x$, $V_y$ and $V_z$ simultaneously. and algebraically. in order to replace two implicit terms of the solution.

**Supporter Equations**

$$1.\ -\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho g_x=\rho V_x\frac{\partial V_x}{\partial x}$$

$$2.\ -\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho g_x=\rho\frac{\partial V_x}{\partial t}$$

$$3.\ -\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho g_x=\frac{\partial p_x}{\partial x}$$

$$4.\ -\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho g_x=-\mu\frac{\partial^2 V_x}{\partial x^2}$$

## Explicit Functions for $V_x$, $V_y$, and $V_z$,

For explicit functions for $V_x$, $V_y$, and $V_z$, one has to solve (algebraically) the simultaneous system of solutions for $V_x$, $V_y$, and $V_z$.

**System of Navier − Stokes relations to solve for $V_x$, $V_y$, $V_z$ simultaneously** (algebraically).

$$V_x=\frac{\left(-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}\right)V_yV_z+[qg_xz+\psi_z(V_z)]V_y+[ng_xy+\psi_y(V_y)]V_z}{V_yV_z}$$

$$V_y=\frac{\left(-\frac{\rho g_y}{2\mu}(\lambda_1x^2+\lambda_2y^2+\lambda_3z^2)+C_1x+C_3y+C_5z+\lambda_5g_yt\pm\sqrt{2\lambda_7g_yy}\right)V_xV_z+[\lambda_8g_yz+\psi_z(V_z)]V_x+[\lambda_6g_yx+\psi_x(V_x)]V_z}{V_xV_z}$$

$$V_z=\frac{\left(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}\right)V_xV_y+[\beta_6g_zx+\psi_x(V_x)]V_y+[\beta_7g_zy+\psi_y(V_y)]V_x}{V_xV_y}$$

# Special Cases of the Navier-Stokes Equations

## 1. Linearized Navier--Stokes equations

One may note that there are six linear terms and three nonlinear terms in the Navier-Stokes equation. The linearized case was covered before the general case, and the experience gained in the linearized case guided one to solve the general case efficiently. In particular, the gravity term must be the subject of the equation for a solution. When the gravity term was the subject of the equation, the equation was called the driver equation. A splitting technique was applied to the linearized Navier-Stokes equations (Option 1). Twenty sub-equations were solved. (Four sets of equations with different equation subjects). The integration relations of one of the sets satisfied the linearized Navier-Stokes equation; and this set was from the equation with $g_x$ as the subject of the equation. In addition to finding a solution, the results of the integration revealed the roles of the terms of the Navier-Stokes equations in fluid flow. In particular, the gravity forces and $\partial p/\partial x$ are involved mainly in the parabolic as well as the forward motion of fluids; $\partial V_x/\partial t$ and $\partial^2 V_x/\partial x^2$ are involved in the periodic motion of fluids, and one may infer that as $\mu$ increases, the periodicity increases. One should determine experimentally, if the ratio of the linear term $\partial V_x/\partial t$ to the nonlinear sum $V_x(\partial V_x/\partial x)+V_y(\partial V_x/\partial y)+V_z(\partial V_x/\partial z)$ is 1 to 3.

$$V_x(x,y,z,t) = \overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9}^{\textbf{Solution to linearized Navier– Stokes equation}} \; ; \; P(x) = d\rho g_x x$$

$$\boxed{\overbrace{-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p_x}{\partial x}+4\rho\frac{\partial V_x}{\partial t}=\rho g_x}^{\textbf{Linearized Equation}}}$$

## 2. Solutions of the Euler equation

Since one has solved the Navier-Stokes equation, one has also solved the Euler equation.

$$\text{Euler equation } (\mu = 0): \quad \frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=g_x$$

$$V_x(x,y,z,t) = f_e g_x t \pm \sqrt{2h_e g_x x}+\frac{n_e g_x y}{V_y}+\frac{q_e g_x z}{V_z}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}+C \qquad x\text{-direction}$$

$$P(x) = d_e\rho g_x x \qquad (f_e+h_e+n_e+q_e+d_e=1) \; V_y \neq 0, \; V_z \neq 0$$

**A** **Euler solution system to solve for** $V_x, V_y, V_z$

$$V_x = \frac{(f_e g_x t \pm \sqrt{2h_e g_x x})V_y V_z + [q_e g_x z + \psi_z(V_z)]V_y + [n_e g_x y + \psi_y(V_y)]V_z}{V_y V_z}$$

$$V_y = \frac{(\lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y})V_x V_z + [\lambda_8 g_y z + \psi_z(V_z)]V_x + [\lambda_6 g_y x + \psi_x(V_x)]V_z}{V_x V_z}$$

$$V_z = \frac{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})V_x V_y + [\beta_6 g_z x + \psi_x(V_x)]V_y + [\beta_7 g_z y + \psi_y(V_y)]V_x}{V_x V_y}$$

# Overall Conclusion

The author was encouraged by Lagrange's use of ratios and proportions in solving differential equations. However the use of ratios in this paper is much more direct. One very interesting fact is that after using ratios to split the equation with the gravity term as the subject of the equation, the integration was straightforward. The author believes that if the ratio or proportion method of splitting the equations could not yield the solution, no other method can even come close, since use of ratios is the most fundamental principle in the division of any quantity into parts.

Finally, in fluid flow, the indispensable term or factor is gravity, according to the above solutions. For any fluid flow design, one should always maximize the role of gravity for cost-effectiveness, durability, and dependability. Perhaps, Newton's law for fluid flow should read "Sum of everything else equals $\rho g$" ; and this would imply the proposed new law that the other terms divide the gravity term in a definite ratio, and also that each term utilizes gravity force to function in fluid flow.

## Determining the ratio terms

In applications, the ratio terms $a$, $b$, $c$, $d$, $f$, $h$, $n$, $q$ and others may perhaps be determined using information such as initial and boundary conditions or may have to be determined experimentally. The author came to the experimental determination conclusion after referring to Example 5, page 6.. The question is how did the grandmother determine the terms of the ratio for her grandchildren? Note that so far as the general solutions of the N-S equations are concerned one needs not find the specific values of the ratio terms.

**Back to Options**

# Option 6
# Spin-off: CMI Millennium Prize Problem Requirements

## Proof 1
### For the linearized Navier-Stokes equations

**Proof of the existence of solutions of the Navier-Stokes equations**

Since from page 13, it has been shown that the smooth equations given by

$$\boxed{V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9 \; ; \; P(x) = d\rho g_x x}$$ are solutions

of the linearized equation, $-\mu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}) + \frac{\partial p_x}{\partial x} + 4\rho\frac{\partial v_x}{\partial t} = \rho g_x$ , it has been shown that smooth solutions to the above differential equation exist. and the proof is complete.

From, above, if $y = 0, \; z = 0$, $\boxed{V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + \frac{fg_x}{4}t + C_9}$; $P(x) = d\rho gx + C_{10}$

Therefore, $V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_{10}x + C_9$

**Finding** $P(x,t)$

**1**. $V_x(x,t) = -\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x x$ **2**. $\frac{\partial p}{\partial x} = d\rho g;$

**Required**: To find $P(x,t)$ (that is, find a formula for $P$ in terms of $x$ and $t$)

$$\frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx}V_x \qquad (\frac{dx}{dt} = V_x)$$

$$\frac{dp}{dt} = d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9\right) \qquad (\frac{dp}{dx} = d\rho g_x)$$

$$\frac{dp}{dt} = -\frac{ad\rho^2 g_x^2}{2\mu}x^2 + C_1 d\rho g_x x + \frac{d\rho f g_x^2}{4}t + C_9 d\rho g_x$$

$$P(x,t) = \int\left(-\frac{ad\rho^2 g_x^2}{2\mu}x^2 + C_1 d\rho g_x x + \frac{d\rho f g_x^2}{4}t + C_9 d\rho g_x\right)dt$$

$$P(x,t) = -\frac{ad\rho^2 g_x^2}{2\mu}x^2 t + C_1 d\rho g_x xt + \frac{d\rho f g_x^2}{8}t^2 + C_9 d\rho g_x t + C_{10}$$

$$= -d\rho g_x\left(\frac{a\rho g_x}{2\mu}x^2 t + C_1 xt + \frac{fg_x}{8}t^2 + C_9 t\right) + C_{10}$$

For the corresponding coverage for the original Navier-Stokes equation, see the next page

# Proof 2

## For the Non-linearized Navier-Stokes equations (Original Equations)

**Proof of the existence of solutions of the Navier-Stokes equations**

From page 23, if $y=0,\ z=0$ in

**Solution to Linear part**

$$V_x(x,y,z,t) = \overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\underbrace{fg_xt}_{\text{continued}}}^{\text{Solution to Linear part}}\ \underbrace{\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{solution of Euler equation}}$$

**solution of Euler equation**

$P(x) = d\rho g_x x$

one obtains

$$V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9\ ; \quad P(x) = d\rho g_x x;$$

$$V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x \pm \sqrt{2hg_xx} + C_9\ ; \quad P(x) = d\rho g_x x;$$

Since previously, from p.113, it has been shown that the smooth equations given by

$V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9\ ;\ P(x) = d\rho g_x x;$ are solutions of

$-\mu\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p_x}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x\frac{\partial V_x}{\partial x} = \rho g_x$ (deleting the $x-$ and $y-$ terms of (A)), p.112, one has shown that smooth solutions to the above differential equation exist, and the proof is complete.

**Finding** $P(x,t)$:

**1.** $V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9;\ P(x) = d\rho g_x x;$ **2.** $\frac{\partial p}{\partial x} = d\rho g;$

$$\frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx}V_x \qquad (\frac{dx}{dt} = V_x)$$

$$\frac{dp}{dt} = d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x \pm \sqrt{2hg_xx} + fg_xt + C_9\right) \qquad (\frac{dp}{dx} = d\rho g_x)$$

$$P(x,t) = \int d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x \pm \sqrt{2hg_xx} + fg_xt + C_9\right)dt$$

$$P(x,t) = -d\rho g_x\left(\frac{a\rho g_x}{2\mu}x^2t + C_1xt \pm \left(\sqrt{2hg_xx}\right)t + \frac{fg_x}{2}t^2 + C_9t\right) + C_{10}$$

**Back to Options**

# Option 7

## Solutions of the Magnetohydrodynamic Equations

This system consists of four equations and one is to solve for $V_x$, $V_y$, $V_z$, $B_x$, $B_y$, $B_z$,

**Magnetohydrodynamic Equations**

1. $$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 \text{ <-- continuity equation}$$

2. $$\overbrace{\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}}^{\text{Navier–Stokes}}=\overbrace{-\frac{\partial p}{\partial x}+\frac{1}{\mu}(\nabla\times B)\times B+\rho g_x}^{\text{Lorentz force}}$$

3. $$\rho\frac{\partial B}{\partial t}=\nabla\times(V\times B)+\eta\nabla^2 B$$
$$\rho\frac{\partial B}{\partial t}=\nabla\times(V\times B)+\eta(\frac{\partial^2 B}{\partial x^2}+\frac{\partial^2 B}{\partial y^2}+\frac{\partial^2 B}{\partial z^2})$$
($\eta$ = magnetic diffusivity)

4. $$\nabla\bullet B=0$$
$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

Step 1:

1. If $\rho$ is constant: (for incompressible fluid)

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 \text{ <-- continuity equation}$$

2. $$\overbrace{\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}}^{\text{Navier - Stokes}}=\overbrace{-\frac{\partial p}{\partial x}+\frac{1}{\mu}(\nabla\times B)\times B+\rho g_x}^{\text{Lorentz force}}$$

$$\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\mu}(B_z(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x})-B_y(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y})+\rho g_x$$

$$\boxed{\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\mu}(B_z\frac{\partial B_x}{\partial z}-B_z\frac{\partial B_z}{\partial x}-B_y\frac{\partial B_y}{\partial x}+B_y\frac{\partial B_x}{\partial y})+\rho g_x}$$

3. $$\rho\frac{\partial B}{\partial t}=\nabla\times(V\times B)+\eta\nabla^2 B$$

$$\rho\frac{\partial B}{\partial t}=\frac{\partial}{\partial y}(V_xB_y-V_yB_x)-\frac{\partial}{\partial z}(V_zB_x-V_xB_z)+\eta(\frac{\partial^2 B}{\partial x^2}+\frac{\partial^2 B}{\partial y^2}+\frac{\partial^2 B}{\partial z^2})$$

$$\boxed{\rho\frac{\partial B}{\partial t}=\frac{\partial}{\partial y}V_xB_y-\frac{\partial}{\partial y}V_yB_x-\frac{\partial}{\partial z}V_zB_x+\frac{\partial}{\partial z}V_xB_z+\eta\frac{\partial^2 B_x}{\partial x^2}+\eta\frac{\partial^2 B_x}{\partial y^2}+\eta\frac{\partial^2 B_x}{\partial z^2}}$$

4. $$\boxed{\begin{array}{l}\nabla\bullet B=0\\ \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0\end{array}}$$

Step 2:
After the "vector juggling" one obtains the following system of equations which one will solve.

$$\begin{cases}1.\ \dfrac{\partial V_x}{\partial x}+\dfrac{\partial V_y}{\partial y}+\dfrac{\partial V_z}{\partial z}=0\\ 2.\ \rho\dfrac{\partial V_x}{\partial t}+\rho V_x\dfrac{\partial V_x}{\partial x}+\rho V_y\dfrac{\partial V_x}{\partial y}+\rho V_z\dfrac{\partial V_x}{\partial z}+\dfrac{\partial p}{\partial x}-\dfrac{1}{\mu}B_z\dfrac{\partial B_x}{\partial z}+\dfrac{1}{\mu}B_z\dfrac{\partial B_z}{\partial x}+\dfrac{1}{\mu}B_y\dfrac{\partial B_y}{\partial x}-\dfrac{1}{\mu}B_y\dfrac{\partial B_x}{\partial y}=\rho g_x\\ 3.\\ \dfrac{\rho\partial B_x}{\partial t}-V_x\dfrac{\partial B_y}{\partial y}-B_y\dfrac{\partial V_x}{\partial y}+V_y\dfrac{\partial B_x}{\partial y}+B_x\dfrac{\partial V_y}{\partial y}+V_z\dfrac{\partial B_x}{\partial z}+B_x\dfrac{\partial V_z}{\partial z}-V_x\dfrac{\partial B_z}{\partial z}-B_z\dfrac{\partial V_x}{\partial z}-\dfrac{\eta\partial^2 B_x}{\partial x^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial y^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial z^2}=0\\ 4.\ \dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}=0\end{cases}$$

At a glance, and from the experience gained in solving the Navier-Stokes equations, one can identify equation (2) as the driver equation, since it contains the gravity term, and the gravity term is the subject of the equation. However, since the system of equations is to be solved simultaneously and there is only a single "driver", the gravity term, all the terms in the system of equations will be placed in the driver equation, Equation 2. As suggested by Albert Einstein, Friedrich Nietzsche, and Pablo Picasso, one will think like a child at the next step.

Step 3: Thinking like a ninth grader, one will apply the following axiom:

If $a=b$ and $c=d$, then $a+c=b+d$; and therefore, add the left sides and add the right sides of the above equations . That is, $(1)+(2)+(3)+(4)=\rho g_x$

$$\begin{cases}\dfrac{\partial V_x}{\partial x}+\dfrac{\partial V_y}{\partial y}+\dfrac{\partial V_z}{\partial z}+\rho\dfrac{\partial V_x}{\partial t}+\rho V_x\dfrac{\partial V_x}{\partial x}+\rho V_y\dfrac{\partial V_x}{\partial y}+\rho V_z\dfrac{\partial V_x}{\partial z}+\dfrac{\partial p}{\partial x}-\dfrac{1}{\mu}B_z\dfrac{\partial B_x}{\partial z}+\dfrac{1}{\mu}B_z\dfrac{\partial B_z}{\partial x}+\dfrac{1}{\mu}B_y\dfrac{\partial B_y}{\partial x}-\\ \dfrac{1}{\mu}B_y\dfrac{\partial B_x}{\partial y}+\dfrac{\rho\partial B_x}{\partial t}-V_x\dfrac{\partial B_y}{\partial y}-B_y\dfrac{\partial V_x}{\partial y}+V_y\dfrac{\partial B_x}{\partial y}+B_x\dfrac{\partial V_y}{\partial y}+V_z\dfrac{\partial B_x}{\partial z}+B_x\dfrac{\partial V_z}{\partial z}-V_x\dfrac{\partial B_z}{\partial z}-B_z\dfrac{\partial V_x}{\partial z}-\\ \dfrac{\eta\partial^2 B_x}{\partial x^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial y^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial z^2}+\dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}=\rho g_x \qquad \text{(Three lines per equation)}\end{cases}$$

Step 4: Writing all the linear terms first

$$\begin{cases}\dfrac{\partial V_x}{\partial x}+\dfrac{\partial V_y}{\partial y}+\dfrac{\partial V_z}{\partial z}+\rho\dfrac{\partial V_x}{\partial t}+\dfrac{\partial p}{\partial x}+\dfrac{\rho\partial B_x}{\partial t}-\dfrac{\eta\partial^2 B_x}{\partial x^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial y^2}-\dfrac{\eta\partial^2 B_x}{\eta\partial z^2}+\dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}\\ +\rho V_x\dfrac{\partial V_x}{\partial x}+\rho V_y\dfrac{\partial V_x}{\partial y}+\rho V_z\dfrac{\partial V_x}{\partial z}-\dfrac{1}{\mu}B_z\dfrac{\partial B_x}{\partial z}+\dfrac{1}{\mu}B_z\dfrac{\partial B_z}{\partial x}+\dfrac{1}{\mu}B_y\dfrac{\partial B_y}{\partial x}-\dfrac{1}{\mu}B_y\dfrac{\partial B_x}{\partial y}-V_x\dfrac{\partial B_y}{\partial y}-B_y\dfrac{\partial V_x}{\partial y}\\ +V_y\dfrac{\partial B_x}{\partial y}+B_x\dfrac{\partial V_y}{\partial y}+V_z\dfrac{\partial B_x}{\partial z}+B_x\dfrac{\partial V_z}{\partial z}-V_x\dfrac{\partial B_z}{\partial z}-B_z\dfrac{\partial V_x}{\partial z}=\rho g_x \qquad \text{(Three lines per equation)}\end{cases}$$

(Since all the terms are now in the same driver equation, let $\rho g_x$ "drive them" simultaneously.)

Step 5: Solve the above 28-term equation using the ratio method. (27 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

$\beta_1, \beta_2, \beta_3, a, b, c, d, f, m, q, r, s, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9$

| | | | |
|---|---|---|---|
| 1. $\dfrac{\partial V_x}{\partial x}=\beta_1\rho g_x$<br>$\dfrac{dV_x}{dx}=\beta_1\rho g_x$<br>$V_x=\beta_1\rho g_x x+C_{16}$ | 2. $\dfrac{\partial V_y}{\partial y}=\beta_2\rho g_x$<br>$\dfrac{dV_y}{dy}=\beta_2\rho g_x$<br>$V_y=\beta_2\rho g_x y+C_{17}$ | 3. $\dfrac{\partial V_z}{\partial z}=\beta_3\rho g_x$<br>$\dfrac{dV_z}{dz}=\beta_3\rho g_x$<br>$V_z=\beta_3\rho g_x z+C_{18}$ | 4. $\rho\dfrac{\partial V_x}{\partial t}=a\rho g_x$<br>$\dfrac{\partial V_x}{\partial t}=ag_x$<br>$V_x=ag_x t+C_1$ |

| 5. | 6. | 7. |
|---|---|---|
| $\frac{\partial p}{\partial x} = b\rho g_x$<br>$\frac{dp}{dx} = b\rho g_x$<br>$P(x) = b\rho g_x x + C$ | $\rho \frac{\partial B_x}{\partial t} = c\rho g_x$<br>$\frac{\partial B_x}{\partial t} = cg_x$<br>$\frac{dB_x}{dt} = cg_x$<br>$B_x = cg_x t + C_{1b}$ | $-\eta \frac{\partial^2 B_x}{\partial x^2} = d\rho g_x$<br>$\frac{d^2 B_x}{dx^2} = -\frac{d\rho g_x}{\eta}$<br>$\frac{dB_x}{dx} = -\frac{d\rho g_x x}{\eta} + C_2$<br>$B_x = -\frac{d\rho g_x x^2}{2\eta} + C_2 x + C_3$ |

| 8. | 9. | 10. |
|---|---|---|
| $-\eta \frac{\partial^2 B_x}{\partial y^2} = f\rho g_x$<br>$\frac{d^2 B_x}{dy^2} = -\frac{f\rho g_x}{\eta}$<br>$\frac{dB_x}{dy} = -\frac{f\rho g_x y}{\eta} + C_4$<br>$B_x = -\frac{f\rho g_x y^2}{2\eta} + C_4 y + C_5$ | $-\eta \frac{\partial^2 B_x}{\partial z^2} = m\rho g_x$<br>$\frac{d^2 B_x}{dz^2} = -\frac{m\rho g_x}{\eta}$<br>$\frac{dB_x}{dz} = -\frac{m\rho g_x z}{\eta} + C_6$<br>$B_x = -\frac{m\rho g_x z^2}{2\eta} + C_6 x + C_7$ | $\frac{\partial B_x}{\partial x} = q\rho g_x$<br>$\frac{dB_x}{dx} = q\rho g_x$<br>$B_x = q\rho g_x x + C_{19}$ |

| 11. | 12. | 13. | 14. |
|---|---|---|---|
| $\frac{\partial B_y}{\partial y} = r\rho g_x$<br>$\frac{dB_y}{dy} = r\rho g_x$<br>$B_y = r\rho g_x y + C_{20}$ | $\frac{\partial B_z}{\partial z} = s\rho g_x$<br>$\frac{dB_z}{dz} = s\rho g_x$<br>$B_z = s\rho g_x z + C_{21}$ | $\rho V_x \frac{\partial V_x}{\partial x} = \omega_1 \rho g_x$<br>$V_x \frac{dV_x}{dx} = \omega_1 g_x$<br>$V_x dV_x = \omega_1 g_x dx$<br>$\frac{V_x^{\ 2}}{2} = \omega_1 g_x x$<br>$V_x^{\ 2} = 2\omega_1 g_x x$<br>$V_x = \pm\sqrt{2\omega_1 g_x x} + C_2$ | $\rho V_y \frac{\partial V_x}{\partial y} = \omega_2 \rho g_x$<br>$V_y dV_x = \omega_2 g_x dy$<br>$V_y V_x = \omega_2 g_x y + \psi_y(V_y)$<br>$V_x = \frac{\omega_2 g_x y}{V_y} + \frac{\psi_y(V_y)}{V_y}$<br>$V_y \neq 0$ |

| 15. | 16 | 17. |
|---|---|---|
| $\rho V_z \frac{\partial V_x}{\partial z} = \omega_3 \rho g_x$<br>$V_z \frac{dV_x}{dz} = \omega_3 g_x$<br>$V_z dV_x = \omega_3 g_x dz$<br>$V_z V_x = \omega_3 g_x z + \psi_z(V_z)$<br>$V_x = \frac{\omega_3 g_x z}{V_z} + \frac{\psi_z(V_z)}{V_z}$<br>$V_z \neq 0$ | $B_z \frac{\partial B_x}{\partial z} = -\omega_4 \mu \rho g_x$<br>$B_z dB_x = -\omega_4 \mu \rho g_x dz$<br>$B_z B_x = -\omega_4 \mu \rho g_x z + \psi_z(B_z)$<br>$B_x = -\frac{\omega_4 \mu \rho g_x z}{B_z} + \frac{\psi_z(B_z)}{B_z}$<br>$B_z \neq 0$ | $B_z \frac{\partial B_z}{\partial x} = \omega_5 \mu \rho g_x$<br>$B_z \frac{dB_z}{dx} = \omega_5 \mu \rho g_x$<br>$B_z dB_z = \omega_5 \mu \rho g_x dx$<br>$\frac{B_z^{\ 2}}{2} = \omega_5 \mu \rho g_x x$<br>$B_z^{\ 2} = 2\omega_5 \mu \rho g_x x$<br>$B_z = \pm\sqrt{2\omega_5 \mu \rho g_x x} + C$ |

18.

$$B_y \frac{\partial B_y}{\partial x} = \omega_6 \mu \rho g_x$$
$$B_y \frac{dB_y}{dx} = \omega_6 \mu \rho g_x$$
$$B_y dB_y = \omega_6 \mu \rho g_x dx$$
$$\frac{B_y{}^2}{2} = \omega_6 \mu \rho g_x x$$
$$B_y{}^2 = 2\omega_6 \mu \rho g_x x$$
$$B_y = \pm\sqrt{2\omega_6 \mu \rho g_x x} + C$$

19.

$$-\frac{1}{\mu} B_y \frac{\partial B_x}{\partial y} = \lambda_1 \rho g_x$$
$$B_y \frac{dB_x}{dy} = -\lambda_1 \mu \rho g_x$$
$$B_y dB_x = -\lambda_1 \mu \rho g_x dy$$
$$B_y B_x = -\lambda_1 \mu \rho g_x y + \psi_y(B_y)$$
$$B_x = -\frac{\lambda_1 \mu \rho g_x y}{B_y} + \frac{\psi_y(B_y)}{B_y}$$
$$B_y \neq 0$$

20

$$-V_x \frac{\partial B_y}{\partial y} = \lambda_2 \rho g_x$$
$$V_x \frac{dB_y}{dy} = -\lambda_2 \rho g_x$$
$$V_x dB_y = -\lambda_2 \rho g_x dy$$
$$V_x B_y = -\lambda_2 \rho g_x y + \psi_x(V_x)$$
$$B_y = \frac{-\lambda_2 \rho g_x y}{V_x} + \frac{\psi_x(V_x)}{V_x}$$
$$V_x \neq 0$$

21.

$$-B_y \frac{\partial V_x}{\partial y} = \lambda_3 \rho g_x$$
$$B_y \frac{dV_x}{dy} = -\lambda_3 \rho g_x$$
$$B_y dV_x = -\lambda_3 \rho g_x dy$$
$$B_y V_x = -\lambda_3 \rho g_x y + \psi_y(B_y)$$
$$V_x = -\frac{\lambda_3 \rho g_x y}{B_y} + \frac{\psi_y(B_y)}{B_y}$$
$$B_y \neq 0$$

22.

$$V_y \frac{\partial B_x}{\partial y} = \lambda_4 \rho g_x$$
$$V_y \frac{dB_x}{dy} = \lambda_4 \rho g_x$$
$$V_y dB_x = \lambda_4 \rho g_x dy$$
$$V_y B_x = \lambda_4 \rho g_x y + \psi_y(V_y)$$
$$B_x = \frac{\lambda_4 \rho g_x y}{V_y} + \frac{\psi_y(V_y)}{V_y}$$
$$V_y \neq 0$$

23.

$$B_x \frac{\partial V_y}{\partial y} = \lambda_5 \rho g_x$$
$$B_x \frac{dV_y}{dy} = \lambda_5 \rho g_x$$
$$B_x dV_y = \lambda_5 \rho g_x dy$$
$$B_x V_y = \lambda_5 \rho g_x y + \psi_x(B_x)$$
$$V_y = \frac{\lambda_5 \rho g_x y}{B_x} + \frac{\psi_x(B_x)}{B_x}$$
$$B_x \neq 0$$

24.

$$V_z \frac{\partial B_x}{\partial z} = \lambda_6 \rho g_x$$
$$V_z \frac{dB_x}{dz} = \lambda_6 \rho g_x$$
$$V_z dB_x = \lambda_6 \rho g_x dz$$
$$V_z B_x = \lambda_6 \rho g_x z + \psi_z(V_z)$$
$$B_x = \frac{\lambda_6 \rho g_x z}{V_z} + \frac{\psi_z(V_z)}{V_z}$$
$$V_z \neq 0$$

25.

$$B_x \frac{\partial V_z}{\partial z} = \lambda_7 \rho g_x$$
$$B_x \frac{dV_z}{dz} = \lambda_7 \rho g_x$$
$$B_x dV_z = \lambda_7 \rho g_x dz$$
$$B_x V_z = \lambda_7 \rho g_x z + \psi_x(B_x)$$
$$V_z = \frac{\lambda_7 \rho g_x z}{B_x} + \frac{\psi_x(B_x)}{B_x}$$
$$B_x \neq 0$$

26

$$-V_x \frac{\partial B_z}{\partial z} = \lambda_8 \rho g_x$$
$$V_x \frac{dB_z}{dz} = -\lambda_8 \rho g_x$$
$$V_x dB_z = -\lambda_8 \rho g_x dz$$
$$V_x B_z = -\lambda_8 \rho g_x z + \psi_x(V_x)$$
$$B_z = -\frac{\lambda_8 \rho g_x z}{V_x} + \frac{\psi_x(V_x)}{V_x}$$
$$V_x \neq 0$$

27.

$$-B_z \frac{\partial V_x}{\partial z} = \lambda_9 \rho g_x$$
$$B_z \frac{dV_x}{dz} = -\lambda_9 \rho g_x$$
$$B_z dV_x = -\lambda_9 \rho g_x dz$$
$$B_z V_x = -\lambda_9 \rho g_x z + \psi_z(B_z)$$
$$V_x = -\frac{\lambda_9 \rho g_x z}{B_z} + \frac{\psi_z(B_z)}{B_z}$$
$$B_z \neq 0$$

Step 6: One collects the integrals of the sub-equations, above, for $V_x$, $V_y$, $V_z$, $B_x$, $B_y$, $B_z$,

$V_x(x,y,z,t) =$ (sum of integrals from sub-equations #1, #4, #13, #14, #15, #21, #27)

$$\beta_1\rho g_x x + a g_x t \pm \sqrt{2\omega_1 g_x x} + \frac{\omega_2 g_x y}{V_y} - \frac{\lambda_3\rho g_x y}{B_y} + \frac{\omega_3 g_x z}{V_z} - \frac{\lambda_9\rho g_x z}{B_z} + \underbrace{\frac{\psi_z(V_z)}{V_z} + \frac{\psi_y(B_y)}{B_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(B_z)}{B_z}}_{\text{arbitrary functions}} + C_1;$$

(integral from sub–equation #5)

$$P(x) = b\rho g_x x + C_2$$

(sum of integrals from sub–equations #2, #23)

$$V_y(y) = \beta_2\rho g_x y + \frac{\lambda_5\rho g_x y}{B_x} + \underbrace{\frac{\psi_x(B_x)}{B_x}}_{\text{arbitrary function}} + C_3$$

(sum of integrals from sub–equations #3, #25)

$$V_z(z) = \beta_3\rho g_x z + \frac{\lambda_7\rho g_x z}{B_x} + \underbrace{\frac{\psi_x(B_x)}{B_x}}_{\text{arbitrary function}} + C_4$$

(sum of integrals from sub-equations #6, #7, #8, #9, #10, #16, #19, #22, #24)

$B_x(x,y,z,t) =$

$$B_x = -\frac{\rho g_x}{2\eta}(dx^2 + fy^2 + mz^2) + q\rho g_x x + C_2 x + C_4 y + C_6 z + + c g_x t - \frac{\lambda_1\mu\rho g_x y}{B_y} + \frac{\lambda_4\rho g_x y}{V_y} - \frac{\omega_4\mu\rho g_x z}{B_z} +$$
$$\frac{\lambda_6\rho g_x z}{V_z} + \underbrace{\frac{\psi_z(B_z)}{B_z} + \frac{\psi_y(B_y)}{B_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C_7$$

(sum of integrals from sub–equations #11, #18, #20)

$$B_y = r\rho g_x y \pm \sqrt{2\omega_6\mu\rho g_x x} - \frac{\lambda_2\rho g_x y}{V_x} + \underbrace{\frac{\psi_x(V_x)}{V_x}}_{\text{arbitrary function}} + C_8$$

(sum of integrals from sub–equations #12, #17, #26)

$$B_z = s\rho g_x z \pm \sqrt{2\omega_5\mu\rho g_x x} - \frac{\lambda_8\rho g_x z}{V_x} + \underbrace{\frac{\psi_x(V_x)}{V_x}}_{\text{arbitrary function}} + C_{21}$$

Step 7: Find the test derivatives for the linear part

| 1. | 2. | 3. | 4. | 5. | 6. |
|---|---|---|---|---|---|
| $\frac{\partial V_x}{\partial x}=(\beta_1\rho g_x)$ | $\frac{\partial V_y}{\partial y}=(\beta_2\rho g_x)$ | $\frac{\partial V_z}{\partial z}=(\beta_3\rho g_x)$ | $\frac{\partial V_x}{\partial t}=(a g_x)$ | $\frac{\partial p}{\partial x}=(b\rho g_x)$ | $\frac{dB_x}{dt}=(c g_x)$ |

| 7. | 8. | 9. | 10. | 11. | 12. |
|---|---|---|---|---|---|
| $\frac{\partial^2 B_x}{\partial x^2}=-\frac{d\rho g_x}{\eta}$ | $\frac{\partial^2 B_x}{\partial y^2}=-\frac{f\rho g_x}{\eta}$ | $\frac{\partial^2 B_x}{\partial z^2}=-\frac{m\rho g_x}{\eta}$ | $\frac{\partial B_x}{\partial x}=q\rho g_x$ | $\frac{\partial B_y}{\partial y}=r\rho g_x$ | $\frac{\partial B_z}{\partial z}=s\rho g_x$ |

Test derivatives for the nonlinear part

| 13. | 14. | 15. | 16 | 17. |
|---|---|---|---|---|
| $\frac{\partial V_x}{\partial x}=\frac{\omega_1 g_x}{V_x}$ | $\frac{\partial V_x}{\partial y}=\frac{\omega_2 g_x}{V_y}$ | $\frac{\partial V_x}{\partial z}=\frac{\omega_3 g_x}{V_z}$ | $\frac{\partial B_x}{\partial z}=-\frac{\omega_4\mu\rho g_x}{B_z}$ | $\frac{\partial B_z}{\partial x}=\frac{\omega_5\mu\rho g_x}{B_z}$ |

| 18. | 19. | 20. | 21. | 22. |
|---|---|---|---|---|
| $\frac{\partial B_y}{\partial x}=\frac{\omega_6\mu\rho g_x}{B_y}$ | $\frac{\partial B_x}{\partial y}=-\frac{\lambda_1\mu\rho g_x}{B_y}$ | $\frac{\partial B_y}{\partial y}=-\frac{\lambda_2\rho g_x}{V_x}$ | $\frac{\partial V_x}{\partial y}=-\frac{\lambda_3\rho g_x}{B_y}$ | $\frac{\partial B_x}{\partial y}=\frac{\lambda_4\rho g_x}{V_y}$ |

| 23. | 24. | 25. | 26. | 27. |
|---|---|---|---|---|
| $\frac{\partial V_y}{\partial y}=\frac{\lambda_5\rho g_x}{B_x}$ | $\frac{\partial B_x}{\partial z}=\frac{\lambda_6\rho g_x}{V_z}$ | $\frac{\partial V_z}{\partial z}=\frac{\lambda_7\rho g_x}{B_x}$ | $\frac{\partial B_z}{\partial z}=-\frac{\lambda_8\rho g_x}{V_x}$ | $\frac{\partial V_x}{\partial z}=-\frac{\lambda_9\rho g_x}{B_z}$ |

Step 8: Substitute the above test derivatives respectively in the following 28-term equation

$$\begin{cases}\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}+\rho\frac{\partial V_x}{\partial t}+\frac{\partial p}{\partial x}+\frac{\rho\partial B_x}{\partial t}-\frac{\eta\partial^2 B_x}{\partial x^2}-\frac{\eta\partial^2 B_x}{\eta\partial y^2}-\frac{\eta\partial^2 B_x}{\eta\partial z^2}+\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\\ +\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}-\frac{1}{\mu}B_z\frac{\partial B_x}{\partial z}+\frac{1}{\mu}B_z\frac{\partial B_z}{\partial x}+\frac{1}{\mu}B_y\frac{\partial B_y}{\partial x}-\frac{1}{\mu}B_y\frac{\partial B_x}{\partial y}-V_x\frac{\partial B_y}{\partial y}-B_y\frac{\partial V_x}{\partial y}\\ +V_y\frac{\partial B_x}{\partial y}+B_x\frac{\partial V_y}{\partial y}+V_z\frac{\partial B_x}{\partial z}+B_x\frac{\partial V_z}{\partial z}-V_x\frac{\partial B_z}{\partial z}-B_z\frac{\partial V_x}{\partial z}=\rho g_x\end{cases}$$

(Three lines per equation)

$$\begin{cases}(\beta_1\rho g_x)+(\beta_2\rho g_x)+(\beta_3\rho g_x)+\rho(a g_x)+(b\rho g_x)+\rho(c g_x)-\eta(-\frac{d\rho g_x}{\eta})-\eta(-\frac{f\rho g_x}{\eta})-\eta(-\frac{m\rho g_x}{\eta})+\\ (q\rho g_x)+(r\rho g_x)+(s\rho g_x)+\rho V_x(\frac{\omega_1 g_x}{V_x})+\rho V_y(\frac{\omega_2 g_x}{V_y})+\rho V_z(\frac{\omega_3 g_x}{V_z})-\frac{1}{\mu}B_z(-\frac{\omega_4\mu\rho g_x}{B_z})+\\ \frac{1}{\mu}B_z(\frac{\omega_5\mu\rho g_x}{B_z})+\frac{1}{\mu}B_y(\frac{\omega_6\mu\rho g_x}{B_y})-\frac{1}{\mu}B_y(-\frac{\lambda_1\mu\rho g_x}{B_y})-V_x(-\frac{\lambda_2\rho g_x}{V_x})-B_y(-\frac{\lambda_3\rho g_x}{B_y})+V_y(\frac{\lambda_4\rho g_x}{V_y})+\\ B_x(\frac{\lambda_5\rho g_x}{B_x})+V_z(\frac{\lambda_6\rho g_x}{V_z})+B_x(\frac{\lambda_7\rho g_x}{B_x})-V_x(-\frac{\lambda_8\rho g_x}{V_x})-B_z(-\frac{\lambda_9\rho g_x}{B_z})\overset{?}{=}\rho g_x\end{cases}$$

(Four lines per equation)

$$\begin{cases}\beta_1\rho g_x+\beta_2\rho g_x+\beta_3\rho g_x+a\rho g_x+b\rho g_x+c\rho g_x+d\rho g_x+f\rho g_x+m\rho g_x q\rho g_x+r\rho g_x+s\rho g_x+\omega_1\rho g_x\\ +\omega_3\rho g_x+\omega_5\rho g_x+\omega_6\rho g_x+\lambda_1\mu\rho g_x+\lambda_2\rho g_x+\lambda_3\rho g_x+\lambda_4\rho g_x+\lambda_5\rho g_x+\omega_2\rho g_x+\omega_3\rho g_x\\ +\lambda_6\rho g_x+\lambda_7\rho g_x+\lambda_8\rho g_x+\lambda_9\rho g_x\overset{?}{=}\rho g_x\end{cases}$$

(Three lines per equation)

$$\begin{cases}\beta_1 g_x + \beta_2 g_x + \beta_3 g_x + a g_x + b g_x + c g_x + d g_x + f g_x + m g_x q g_x + r g_x + s g_x + \omega_1 g_x + \omega_3 g_x + \omega_5 g_x \\ +\omega_6 g_x + \lambda_1 g_x + \lambda_2 g_x + \lambda_3 g_x + \lambda_4 g_x + \lambda_5 g_x + \omega_2 g_x + \omega_3 g_x + \lambda_6 g_x + \lambda_7 g_x + \lambda_8 g_x + \lambda_9 g_x \stackrel{?}{=} g_x \quad \text{(2 lines)}\end{cases}$$

$$\begin{cases}g_x(\beta_1 + \beta_2 + \beta_3 + a + b + c + d + f + m + q + r + s + \omega_1 + \omega_3 + \omega_5 + \lambda_3 + \lambda_4 + \lambda_5 + \omega_2 + \omega_3 + \lambda_6 + \lambda_7 \\ +\omega_6 + \lambda_1 + \lambda_2 + \lambda_8 + \lambda_9) \stackrel{?}{=} g_x \qquad \text{(Two lines per equation)}\end{cases}$$

$$g_x(1) \stackrel{?}{=} g_x \qquad \text{(Sum of the ratio terms = 1)}$$

$$g_x \stackrel{?}{=} g_x \qquad \text{Yes}$$

Since an identity is obtained, the solutions to the 28-term equation are as follows

$$V_x(x,y,z,t) = \qquad \text{(sum of integrals from sub-equations \#1, \#4, \#13, \#14, \#15, \#21, \#27)}$$

$$\beta_1 \rho g_x x + a g_x t \pm \sqrt{2\omega_1 g_x x} + \frac{\omega_2 g_x y}{V_y} - \frac{\lambda_3 \rho g_x y}{B_y} + \frac{\omega_3 g_x z}{V_z} - \frac{\lambda_9 \rho g_x z}{B_z} + \underbrace{\frac{\psi_z(V_z)}{V_z} + \frac{\psi_y(B_y)}{B_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(B_z)}{B_z}}_{\text{arbitrary functions}} + C_1;$$

$$\text{(integral from sub–equation \#5)}$$

$$P(x) = b\rho g_x x + C_2$$

$$\text{(sum of integrals from sub–equations \#2, \#23)}$$

$$V_y = \beta_2 \rho g_x y + \frac{\lambda_5 \rho g_x y}{B_x} + \underbrace{\frac{\psi_x(B_x)}{B_x}}_{\text{arbitrary function}} + C_3$$

$$\text{(sum of integrals from sub–equations \#3, \#25)}$$

$$V_z = \beta_3 \rho g_x z + \frac{\lambda_7 \rho g_x z}{B_x} + \underbrace{\frac{\psi_x(B_x)}{B_x}}_{\text{arbitrary function}} + C_4$$

$$\text{(sum of integrals from sub-equations \#6, \#7, \#8, \#9, \#10, \#16, \#19, \#22, \#24)}$$

$$B_x(x,y,z,t) =$$

$$B_x = -\frac{\rho g_x}{2\eta}(dx^2 + fy^2 + mz^2) + q\rho g_x x + C_2 x + C_4 y + C_6 z + + c g_x t - \frac{\lambda_1 \mu \rho g_x y}{B_y} + \frac{\lambda_4 \rho g_x y}{V_y} - \frac{\omega_4 \mu \rho g_x z}{B_z} +$$

$$\frac{\lambda_6 \rho g_x z}{V_z} + \underbrace{\frac{\psi_z(B_z)}{B_z} + \frac{\psi_y(B_y)}{B_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C_7$$

$$\text{(sum of integrals from sub–equations \#11, \#18, \#20)}$$

$$B_y = r\rho g_x y \pm \sqrt{2\omega_6 \mu \rho g_x x} - \frac{\lambda_2 \rho g_x y}{V_x} + \underbrace{\frac{\psi_x(V_x)}{V_x}}_{\text{arbitrary function}} + C_8$$

$$\text{(sum of integrals from sub–equations \#12, \#17, \#26)}$$

$$B_z = s\rho g_x z \pm \sqrt{2\omega_5 \mu \rho g_x x} - \frac{\lambda_8 \rho g_x z}{V_x} + \underbrace{\frac{\psi_x(V_x)}{V_x}}_{\text{arbitrary function}} + C_{21}$$

## Supporter Equation Contributions

Note above that there are 28 terms in the driver equation, and 27 supporter equations, Each supporter equation provides useful information about the driver equation. The more of these supporter equations that are integrated, the more the information one obtains about the driver equation. However, without solving a supporter equation, one can sometimes write down some characteristics of the integral of the supporter equation by referring to the subjects of the supporter equations of the Navier-Stokes equations. For example, if one uses $(\eta\partial^2 B_x/\partial x^2)$ as the subject of a supporter equation here, the curve for the integral obtained would be parabolic, periodic, and decreasingly exponential.

## Determining the ratio terms

In applications, the ratio terms $\beta_1, \beta_2, \beta_3, a, b, c, d, f, m, q, r, s, \omega_1, \omega_2$, and others may perhaps be determined using initial and boundary conditions, or may have to be determined experimentally. Note that so far as the general solutions of the equations are concerned, one needs not find the specific values of the ratio terms.

## Comparison of Solutions of Navier-Stokes Equations
and
## Solutions of Magnetohydrodynamic Equations

**Navier-Stokes** $x$–direction **solution**

$$V_{x(x,y,z,t)} = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2) + C_1x + C_3y + C_5z + fg \pm \sqrt{2hgx} + \frac{ngy}{V_y} + \frac{qgz}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$P(x) = d\rho g_x x$$

**For magnetohydrodynamic solutions,** see previous page

**1.** $V_x$ for MHD system looks like the $V_x$ for the Euler solution.

**2.** $P(x)$) for N-S and MHD equations are the same.

**3.** $V_y$ and $V_z$ for MHD are different from those of N-S equations.

**4.** $B_x$ is parabolic and resembles $V_x$ for N-S, except for the absence of the square root function.

**5.** $B_y$ and $B_z$ resemble the Euler solution.

# Conclusion

The author proposes a law of definite ratio. This law states that in magnetohydrodynamics, all the other terms in the system of equations divide the gravity term in a definite ratio and each term utilizes gravity to function. As in the case of incompressible fluid flow, one can add that without gravity forces, there would be no magnetohydrodynamics on earth as is known, according to the solutions of the magnetohydrodynamic equations. **Back to Options**

## References:

For paper edition of the above paper, see Chapter 11 of the book entitled "Power of Ratios" by A. A. Frempong, published by Yellowtextbooks.com.

Without using ratios or proportion, the author would never be able to split-up the Navier-Stokes equations into sub-equations which were readily integrable. The impediment to solving the Navier-Stokes equations for over 150 years (whether linearized or non-linearized ) has been due to finding a way to split-up the equations. Since ratios were the key to splitting the Navier-Stokes equations, and also splitting the 28-term system of magnetohydrodynamic equations, and solving them, the solutions have also been published in the " Power of Ratios" book which covers definition of ratio and applications of ratio in mathematics, science, engineering, economics and business fields.