Consistent extra time dimensions: cosmological inflation with inflaton potential identically equal to zero

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(Dated: November 20, 2013)

Abstract

Inflation supported by a real massless scalar inflaton field φ whose potential is identically equal to zero is described. Assuming that inflation takes place after the Plank scale (after quantum gravity effects are important), zero potential is concomitant with an initial condition for φ that is exponentially more probable than an initial condition that assumes an initial inflaton potential of order of the Planck mass. The Einstein gravitational field equations are formulated on an eight-dimensional spacetime manifold of four space dimensions and four time dimensions. The field equations are sourced by a cosmological constant Λ and the real massless scalar inflaton field φ. Two solution classes for the coupled Einstein field equations are obtained that exhibit temporal exponential **deflation of three of the four time dimensions** and temporal exponential inflation of three of the four space dimensions. For brevity this phenomenon is sometimes simply called "inflation." We show that **the extra time dimensions do not generally induce the exponentially rapid growth of fluctuations of quantum fields.** Comoving coordinates for the two **unscaled** dimensions are chosen to be (x^4, x^8) (unscaled means a constant scale factor equal to one). The x^4 coordinate corresponds to our universe's observed physical time dimension, while the x^8 coordinate corresponds to a new spatial dimension that may be compact. ∂_{x^8} terms of φ and the metric are seen to play the role of an effective inflaton potential in the dynamical field equations. In this model, after "inflation" the observable physical macroscopic world appears to a classical observer to be a homogeneous, isotropic universe with three space dimensions and one time dimension.

PACS numbers: 98.80.Cq, 04.50.Kd

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I. INTRODUCTION

Recent Planck 2013 data analysis [1] is in remarkable accord with a flat ΛCDM model with inflation, based upon a spatially flat, expanding universe whose dynamics are governed by General Relativity and sourced by cold dark matter, a cosmological constant Λ, and a slow-roll scalar inflaton field [2–4]. Planck 2013 reports that the seeds of cosmology structure have Gaussian statistics and form an approximately scale-invariant spectrum (scalar spectral index $n_s = 0.9603 \pm 0.0073$, ruling out exact scale invariance at over 5σ) of adiabatic fluctuations and establishes an upper bound on the tensor-to-scalar ratio at $r < 0.11$ (95% CL); results are consistent with second-order slow-roll predictions [5]. The main predictions of inflationary cosmology are also consistent [6–9] with other recent observational data from important experiments such as WMAP [10] and the Sloan Digital Sky Survey [11–13], to name only two.

However, standard inflationary cosmology may not represent fundamental physics. It has been shown that if one employs a canonical measure for inflation [14] [15] [16], then the probability for the existence of the initial conditions that are required for slow-roll chaotic inflation are extremely unlikely [14], assuming that inflation takes place after the Plank scale (after quantum gravity governs the physics).

Here we discuss a new model of inflation/deflation that overcomes this problem. This model is based on the idea that our universe has as many time dimensions as space dimensions. Everyone is familiar with arguments against the existence of extra time dimensions. While there are many aspects to this issue, we address two arguments against the existence of extra time dimensions which stand out most clearly[17]:

1. A spacetime with extra time dimensions may not carry a spin structure $\Leftrightarrow$ spinors cannot be defined on the spacetime;

2. Momenta corresponding to the extra time dimensions induce exponentially rapid growth of quantum fluctuations of the field; the universe is unstable. This instability is associated with the very largest momenta (shortest wavelengths).

Issue [1] is not applicable here; it is well known that our model spacetime carries a spin structure [18]. The issues raised in [2] may be investigated by studying the propagation of a massive complex scalar field ψ in the background gravitational field, whose field equation that is not coupled to the inflaton field φ.

$$0 = \frac{1}{\sqrt{det(g_{\alpha\,\beta})}}\,\partial_\mu\left[\,\sqrt{det(g_{\alpha\,\beta})}\;g^{\mu\,\nu}\;\partial_\nu\psi\,\right] - \,\left(\Lambda\,m^2\,+\,\zeta\,R\right)\;\psi, \tag{1}$$

which is derivable from the Lagrangian

$$L_\psi = \sqrt{det(g_{\alpha\,\beta})}\left[-\,g^{\mu\,\nu}\partial_\mu\psi^*\;\partial_\nu\psi\;-\psi^*\;\psi\;\left(\Lambda\,m^2\,+\,\zeta\,R\right)\right]. \tag{2}$$

The quantum fluctuations of ψ satisfy a similar equation. Here R is the Ricci scalar, and m and ζ are real input parameters; the factors of Λ are included for convenience. We shall discuss two types of solution to Eq.[1], the first with nonzero coupling to the Ricci scalar and the second with $\zeta = 0$. It should be emphasized that the expected coupling of the massive complex scalar field ψ with the massless real scalar inflaton field φ is not included here. We do not wish to confuse the production/annhilation of ψ "particles" with instabilities in the ψ field that are sourced by the momenta associated to the extra time dimensions. In Sections III and IV we shall seek "plane wave" solutions to Eq.[1] in terms of comoving wave vectors and coordinates of the form

$$\begin{aligned}\psi\;&=\;\Psi(x^4,x^8)\,e^{i\left(\vec{k}\cdot\vec{r}-\vec{w}\cdot\vec{R}\right)},\;\;\text{where}\\ \vec{r}\;&=\;\left(x^1,x^2,x^3\right)^T\;\;\text{and}\;\vec{k}=(k_1,k_2,k_3)^T\;\;;\\ \vec{R}\;&=\;\left(x^5,x^6,x^7\right)^T\;\;\text{and}\;\vec{w}=(k_5,k_6,k_7)^T\,.\end{aligned} \tag{3}$$

When discussing these solutions the following shorthand is employed:

$$\begin{aligned}\Lambda\,k^2\;&=\;\vec{k}\cdot\vec{k}=\;{k_1}^2+{k_2}^2+{k_3}^2\;;\\ \Lambda\,w^2\;&=\;\vec{w}\cdot\vec{w}=\;{k_5}^2+{k_6}^2+{k_7}^2.\end{aligned} \tag{4}$$

We shall give solutions of the Einstein field equations for which the effects of multiple time dimensions are stable. Because some of the coefficients in Eq.[1] are physical-time x^4 dependent there is no dispersion relation, *per se*, relating a physical-frequency ($\omega \leftrightarrow -i\frac{\partial}{\partial\,x^4}\ln\psi$) to momentum wave vectors $(\vec{k},\vec{w})$.

A. Notation and conventions

This model is cast on $\mathbb{X}_{4,4}$, which is an eight-dimensional pseudo-Riemannian manifold that is a spacetime of four space dimensions, with local comoving coordinates (x^1,x^2,x^3,x^8), and four time dimensions, with local comoving coordinates (x^4, x^5, x^6, x^7) (employing the usual component notation in local charts), where $-\infty < x^\alpha < \infty$ (the range of x^8 is refined as the discussion progresses). Greek indices run from 1 to 8. Let $\mathbf{g}$ denote the pseudo-Riemannian metric tensor on $\mathbb{X}_{4,4}$. The signature of the metric $\mathbf{g}$ is $(4,4) \leftrightarrow (+++----+)$. The covariant derivative with respect to the symmetric connection associated to the metric $\mathbf{g}$ is denoted by a vertical double-bar. We employ the Landau-Lifshitz spacelike sign conventions [19]. $\mathbf{g} \leftrightarrow g_{\alpha\beta} = g_{\alpha\beta}(x^\mu)$ is assumed to carry the Newton-Einstein gravitational degrees of freedom. It is moreover assumed that the ordinary Einstein field equations (on $\mathbb{X}_{4,4}$)

$$G_{\alpha\beta} \;+\; g_{\alpha\beta}\,\Lambda \;=\; 8\,\pi\,\mathbb{G}\,T_{\alpha\beta} \tag{5}$$

are satisfied. Here $G_{\alpha\beta}$ denotes the Einstein tensor, Λ is the cosmological constant, $\mathbb{G}$ denotes the Newtonian gravitational constant, and the reduced Planck mass is $M_{Pl} = [\,8\pi\,\mathbb{G}\,]^{-1/2} = 1$. Natural units $c = 1 = \hbar$ are used throughout. Lastly, if $f = f(x^4, x^8)$ then

$$\begin{aligned} &f^{(1,0)} = \tfrac{\partial}{\partial x^4}\, f(x^4,x^8), \quad f^{(0,1)} = \tfrac{\partial}{\partial x^8}\, f(x^4,x^8), \quad f^{(1,1)} = \tfrac{\partial^2}{\partial x^4\,\partial x^8}\, f(x^4,x^8), \\ &f^{(2,0)} = \tfrac{\partial^2}{\partial x^{4\,2}}\, f(x^4,x^8), \;\; f^{(0,2)} = \tfrac{\partial^2}{\partial x^{8\,2}}\, f(x^4,x^8), \qquad\qquad \text{etc.} \end{aligned} \tag{6}$$

II. EINSTEIN FIELD EQUATIONS

It will be seen that (at least) two solution classes to the Einstein field equations on $\mathbb{X}_{4,4}$ exist that exhibit inflation/deflation and describe a universe that is spatially flat throughout the inflation era. During "inflation", the scale factor $a = a(x^4, x^8)$ for the three space dimensions (x^1, x^2, x^3) exponentially inflates as a function of x^4, and the scale factor $b = b(x^4, x^8)$ for the three extra time dimensions (x^5, x^6, x^7) exponentially deflates as a function of x^4; moreover, x^4 and x^8 do not scale.

We define the *Lemaître Cosmological Relative Expansion Rate* ℓ as

$$\ell = \frac{1}{2}\frac{\partial}{\partial x^4}\ln\left(\frac{b}{a}\right). \tag{7}$$

We shall see in Section III that ℓ is independent of time and equal to $\ell_0 = -\sqrt{\frac{1}{18}\Lambda}$ for solutions in the class E_0 corresponding to pure time exponential inflation. For this case $|\ell_0|$ coincides with the Hubble parameter H. All solutions in the class E_0 have a spatial x^8 period of

$${}^0C_8 = \pi\sqrt{\frac{2}{\Lambda}}. \tag{8}$$

Therefore ℓ_0 verifies

$$ {}^{0}C_8\,|\ell_0| = \frac{\pi}{3}, \tag{9}$$

which is reminiscent of a semiclassical quantization relationship.

For solutions in the class E_1, which describes a transition from pure exponential "inflation," $\ell = \ell_1$ is generally time dependent. All solutions in this class have a spatial x^8 period of ${}^{1}C_8 = 2\pi\sqrt{\frac{3}{5\,\Lambda}} = {}^{0}C_8\,\sqrt{\frac{6}{5}}$.

Universes that possess positive Lemaître rate $\ell > 0$ for $x^4 > 0$ exist and are reported below. However they are analogous to collapsing universe solutions in the usual Friedmann cosmological model, or to universes with multiple macroscopic time dimensions, and not directly relevant to the physical processes discussed in this paper.

The line element for inflation/deflation is assumed to be given by

$$\begin{aligned}
\{ds\}^2 \;&=\; \left\{a(x^4,x^8)\right\}^2\left[\left(dx^1\right)^2+\left(dx^2\right)^2+\left(dx^3\right)^2\right]-\left(dx^4\right)^2\\
&-\;\left\{b(x^4,x^8)\right\}^2\left[\left(dx^5\right)^2+\left(dx^6\right)^2+\left(dx^7\right)^2\right]+\left(dx^8\right)^2\\
&=\; a^2\left[\left(dx^1\right)^2+\left(dx^2\right)^2+\left(dx^3\right)^2\right]-b^2\left[\left(dx^5\right)^2+\left(dx^6\right)^2+\left(dx^7\right)^2\right]\\
&-\;\left(dx^4\right)^2+\left(dx^8\right)^2\;\;;
\end{aligned}\tag{10}$$

where $a = a(x^4, x^8)$ and $b = b(x^4, x^8)$ carry the metric degrees of freedom in this model. The real massless scalar inflaton field is $\varphi = \varphi(x^4, x^8)$. The action for the metric and minimally coupled inflaton degrees of freedom is assumed to be given by

$$S = \int \left(\frac{1}{8\,\pi\,\mathbb{G}}\right)^2 d^8x\,\sqrt{det(g_{\alpha\,\beta})}\left[\frac{1}{16\,\pi\,\mathbb{G}}\,(R-2\Lambda)\;-\;\frac{1}{2}g^{\mu\,\nu}\partial_\mu\varphi\,\partial_\nu\varphi\,\right]. \tag{11}$$

Here Λ is the cosmological constant. The inflaton potential is zero; its action is purely kinematic, although $\varphi^{(0,1)}(x^4, x^8)^2$ may be regarded as contributing to an effective inflaton potential.

A. Canonical stress-energy tensor

The canonical stress-energy tensor for the real massless scalar inflaton field is $T_{\mu\,\nu} = -g^{\mu\,\alpha}\frac{2}{\sqrt{\det(g_{\alpha\beta})}}\frac{\partial}{\partial\,g_{\alpha\,\nu}}L_\varphi$, $L_\varphi = \sqrt{det(g_{\alpha\,\beta})}\,\left[-\,\frac{1}{2}g^{\mu\,\nu}\partial_\mu\varphi\,\partial_\nu\varphi\,\right]$. The distinct components are

$$T_{3\,3} \;=\; \frac{1}{2}\mathbf{a}(x^4,x^8)^2\left[\,\varphi^{(1,0)}(x^4,x^8)^2-\varphi^{(0,1)}(x^4,x^8)^2\right], \tag{12}$$

$$T_{4\,4} \;=\; \frac{1}{2}\left[\varphi^{(1,0)}(x^4,x^8)^2+\varphi^{(0,1)}(x^4,x^8)^2\right], \tag{13}$$

(which clearly satisfies the weak energy condition),

$$T_{5\,5} \;=\; \frac{1}{2}\mathbf{b}(x^4,x^8)^2\left(\varphi^{(0,1)}(x^4,x^8)^2-\varphi^{(1,0)}(x^4,x^8)^2\right), \tag{14}$$

$$T_{8\,8} \;=\; \frac{1}{2}\varphi^{(0,1)}(x^4,x^8)^2+\frac{1}{2}\varphi^{(1,0)}(x^4,x^8)^2, \tag{15}$$

and

$$T_{4\,8}=T_{8\,4}=\varphi^{(0,1)}(x^4,x^8)\varphi^{(1,0)}(x^4,x^8), \tag{16}$$

which is non-zero, in general. Due to the (x^4,x^8) dependence of the metric the $(4,8)$ and $(8,4)$ components of the Einstein tensor $G_{4\,8}=G_{8\,4}$ are also, in general, non-zero.

B. Field Equations

$$G_{\mu\,\nu} \;=\; 8\,\pi\,\mathbb{G}\,T_{\mu\,\nu} \tag{17}$$

The distinct field equation components may be written as

$$G_{4\,8}=G_{8\,4}=-\frac{3a^{(1,1)}}{a}-\frac{3b^{(1,1)}}{b}=8\pi\mathbb{G}\,\varphi^{(0,1)}\varphi^{(1,0)} \tag{18}$$

$$\begin{aligned} G_{3\,3} \;=\;& \frac{1}{b^2}\left[3ab\left(2a^{(0,1)}b^{(0,1)}-2a^{(1,0)}b^{(1,0)}+a\left(b^{(0,2)}-b^{(2,0)}\right)\right)\right.\\ +\;& \left.\left(a^{(0,1)2}-a^{(1,0)2}+2a\left(a^{(0,2)}-a^{(2,0)}\right)\right)b^2+3a^2\left(b^{(0,1)2}-b^{(1,0)2}\right)\right]\\ =\;& 4\pi\mathbb{G}a^2\left(-\varphi^{(0,1)2}+\varphi^{(1,0)2}\right) \end{aligned} \tag{19}$$

$$\begin{aligned} G_{4\,4} \;=\;& -\frac{3}{a^2b^2}\left[b^2\left(a^{(0,1)2}-a^{(1,0)2}+aa^{(0,2)}\right)+ab\left(3a^{(0,1)}b^{(0,1)}-3a^{(1,0)}b^{(1,0)}+ab^{(0,2)}\right)\right.\\ +\;& \left.a^2\left(b^{(0,1)2}-b^{(1,0)2}\right)\right]\\ =\;& 4\pi\mathbb{G}\left(\varphi^{(0,1)2}+\varphi^{(1,0)2}\right) \end{aligned} \tag{20}$$

$$
\begin{aligned}
G_{5\,5} \;=\; & \frac{1}{a^2}\left[2ab\left(-3a^{(0,1)}b^{(0,1)}+3a^{(1,0)}b^{(1,0)}+a\left(b^{(2,0)}-b^{(0,2)}\right)\right)\right. \\
+\; & \left. 3\left(-a^{(0,1)2}+a^{(1,0)2}+a\left(a^{(2,0)}-a^{(0,2)}\right)\right)b^2+a^2\left(b^{(1,0)2}-b^{(0,1)2}\right)\right] \\
=\; & 4\pi\mathbb{G}b^2\left(\varphi^{(0,1)2}-\varphi^{(1,0)2}\right)
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
G_{8\,8} \;=\; & -\frac{3}{a^2b^2}\left[a^2\left(b^{(1,0)2}-b^{(0,1)2}\right)+\left(-a^{(0,1)2}+a^{(1,0)2}+aa^{(2,0)}\right)b^2\right. \\
+\; & \left. ab\left(-3a^{(0,1)}b^{(0,1)}+3a^{(1,0)}b^{(1,0)}+ab^{(2,0)}\right)\right] \\
=\; & 4\pi\mathbb{G}\left(\varphi^{(0,1)2}+\varphi^{(1,0)2}\right)
\end{aligned}
\tag{22}
$$

The components of $T^{\mu}_{||\mu\;\alpha}$ that are not identically zero must satisfy

$$
\begin{aligned}
a\,b\,T^{\mu}_{||\mu\,4} \;=\; & 0\;=3a\left(-b^{(1,0)}\left(\varphi^{(1,0)2}\right)+b^{(0,1)}\varphi^{(0,1)}\varphi^{(1,0)}+b^{(0,1)}\right) \\
+\; & b\left(-3a^{(1,0)}\left(\varphi^{(1,0)2}\right)+3a^{(0,1)}\varphi^{(0,1)}\varphi^{(1,0)}\right. \\
+\; & \left. a\left(\varphi^{(1,0)}\left(\varphi^{(0,2)}-\varphi^{(2,0)}\right)\right)\right) \\
a\,b\,T^{\mu}_{||\mu\,8} \;=\; & 0\;=3a\left(b^{(0,1)}\left(\varphi^{(0,1)2}\right)-b^{(1,0)}\varphi^{(0,1)}\varphi^{(1,0)}\right) \\
-\; & b\left(-3a^{(0,1)}\varphi^{(0,1)2}+3a^{(1,0)}\varphi^{(1,0)}\varphi^{(0,1)}\right. \\
-\; & \left. a\left(\varphi^{(0,1)}\left(\varphi^{(0,2)}-\varphi^{(2,0)}\right)\right)\right) \\
& .
\end{aligned}
\tag{23}
$$

Let $L_1=\frac{\mathbf{a}^{(1,0)}(x^4,x^8)}{\mathbf{a}(x^4,x^8)}$, $L_3=\frac{\mathbf{a}^{(0,1)}(x^4,x^8)}{\mathbf{a}(x^4,x^8)}$, $L_2=\frac{\mathbf{b}^{(1,0)}(x^4,x^8)}{\mathbf{b}(x^4,x^8)}$ and $L_4=\frac{\mathbf{b}^{(0,1)}(x^4,x^8)}{\mathbf{b}(x^4,x^8)}$. The Euler-Lagrange equation for the inflaton field yields

$$
\begin{aligned}
\varphi^{(2,0)}(x^4,x^8) \;-\; & \varphi^{(0,2)}(x^4,x^8)+3\varphi^{(1,0)}(x^4,x^8)\left(L_1+L_2\right)-3\varphi^{(0,1)}(x^4,x^8)\left(L_3+L_4\right) \\
=\; & 0,
\end{aligned}
\tag{24}
$$

or, equivalently,

$$
\begin{aligned}
\varphi^{(2,0)}(x^4,x^8) \;+\; & 3\varphi^{(1,0)}(x^4,x^8)\left(L_1+L_2\right)+\mu^2\,\varphi(x^4,x^8) \\
=\; & \varphi^{(0,2)}(x^4,x^8)+3\varphi^{(0,1)}(x^4,x^8)\left(L_3+L_4\right)+\mu^2\,\varphi(x^4,x^8),
\end{aligned}
\tag{25}
$$

where μ^2 is arbitrary.

Lastly, the field equations demand that the constraint equation

$$\begin{aligned} L_1^2 + 3L_1L_2 + L_2^2 \;&-\; \frac{4}{3}\pi\mathbb{G}\,\varphi^{(1,0)}(x^4,x^8)^2 \\ -\left[L_3^2 + 3L_3L_4 + L_4^2 \;\right.&-\; \left.\frac{4}{3}\pi\mathbb{G}\,\varphi^{(0,1)}(x^4,x^8)^2\right] \;=\; \frac{2}{9}\Lambda, \end{aligned} \tag{26}$$

be satisfied.

To solve the field equations we use the fact that $\mathbb{P} = \ln(a \times b)$, $\mathbb{R} = \ln\left(\frac{b}{a}\right)$ and φ satisfy, respectively, uncoupled, linear and linear field equations of the form

$$\begin{aligned} & f^{(0,2)}(x^4,x^8) - f^{(2,0)}(x^4,x^8) = \; S \; + \\ & + 3\,\mathbb{P}^{(1,0)}(x^4,x^8) f^{(1,0)}(x^4,x^8) - 3\,\mathbb{P}^{(0,1)}(x^4,x^8) f^{(0,1)}(x^4,x^8), \end{aligned} \tag{27}$$

where $f = \mathbb{P}, \mathbb{R}$ or φ and $S = 0$ unless $f = \mathbb{P}$ in which case $S = -\frac{2}{3}\Lambda$. General solutions to these equations are substituted into the constraint Eq[26] and the remaining field equations, which are then solved; this procedure yields the solutions in the classes E_0 and E_1 that are described below.

III. CLASS E_0 SOLUTIONS: EXACT TEMPORAL EXPONENTIAL INFLATION/DEFLATION

Using the technique described above we find that this model admits the following exponential (in x^4) solution: The scale factors are

$$\begin{aligned} a \;&=\; a(x^4,x^8) = a_0 e^{\pm\frac{1}{3}\sqrt{\frac{\Lambda}{2}}\,x^4}\;\sqrt[12]{\sin^2\left(x^8\sqrt{2\,\Lambda}\right)} \\ b \;&=\; b(x^4,x^8) = b_0 e^{\mp\frac{1}{3}\sqrt{\frac{\Lambda}{2}}\,x^4}\;\sqrt[12]{\sin^2\left(x^8\sqrt{2\,\Lambda}\right)}, \end{aligned} \tag{28}$$

where a_0 and b_0 are constants. The Lemaître relative expansion rate and the inflaton field are given by

$$\begin{aligned} \ell_0 \;&=\; \mp\frac{1}{3}\sqrt{\frac{\Lambda}{2}} \\ \varphi \;&=\; \pm\frac{1}{2}\sqrt{\frac{5}{6}}\ln\left[\tan^2\left(\frac{1}{2}\sqrt{2\Lambda}\,x^8\right)\right]. \end{aligned} \tag{29}$$

For this case the scale factors (a, b) have a spatial period equal to $\frac{1}{2}\;{}^0C_8$, while the three functions (a, b, φ) possess a common spatial periodicity in the x^8 coordinate whose nonzero minimum value is equal to the x^8-dimension spatial period

$$ {}^{0}C_8 = \pi\sqrt{\frac{2}{\Lambda}} = \frac{\pi}{3|\ell_0|}, \tag{30} $$

For this case $|\ell_0|$ coincides with the Hubble parameter H.

Since a universe with many macroscopic times is not observed, one may identify the physical solution for inflation by choosing the solution with the $-$ sign in the equation for b. This solution then predicts the exponential deflation, with respect to time, of the scale factor b associated with the three extra time dimensions. This coincides with the exponential inflation, with respect to time, of the scale factor a associated with the three observed spatial dimensions.

Let us briefly consider the reduced volume element $d\Omega$ on the hypersurface $x^4 = x^4_0 =$ constant,

$$ d\Omega = \left[\sqrt{\det\left(\mathbf{g}\right)}\right]_{x^4=x^4_0} \, d\tau \tag{31} $$

where $d\tau = |dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8|$. For this case $d\Omega = d\tau \, \left|\sin\left(x^8\sqrt{2\,\Lambda}\right)\right|$, which clearly does not inflate. (The four dimensional spacetime submainfold of $\mathbb{X}_{4,4}$ comprised of the (x^1, x^2, x^3, x^4) dimensions experiences conventional FLRW inflation.) The fact that $d\Omega$ vanishes on a set of measure zero is discussed in the *Conclusion*.

A. Stability of a massive scalar field

The question of the possible existence of instabilities in the ψ field that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions, the issue raised in Section[2], may be investigated by studying the propagation of a massive scalar complex field ψ in the background gravitational field that is not coupled to the inflaton field φ. The field equation for the scalar field is given in Eq.[1]. Substituting the background gravitational field of Eq.[28] into Eq.[1] yields

$$ \begin{aligned} 0 \;=\; & -\Psi^{(2,0)}(\,x^4\,,\,x^8\,) + \Psi^{(0,2)}(\,x^4\,,\,x^8\,) + \sqrt{2\,\Lambda}\cot\left(\sqrt{2\,\Lambda}\,x^8\right)\Psi^{(0,1)}(\,x^4\,,\,x^8\,) \\ & + \Lambda\left(w^2 e^{\frac{1}{3}\sqrt{2\,\Lambda}\,x^4} - k^2\, e^{-\frac{1}{3}\sqrt{2\,\Lambda}\,x^4}\right)\sqrt[6]{\csc^2\left(\sqrt{2\,\Lambda}\,x^8\right)}\,\Psi(\,x^4\,,\,x^8\,) \\ & - \Lambda\left[m^2 + \frac{1}{3}\zeta\left(8 + 5\csc^2\left(\sqrt{2\,\Lambda}\,x^8\right)\right)\right]\Psi(\,x^4\,,\,x^8\,). \end{aligned} \tag{32} $$

Here we have used the fact that the Ricci scalar is $\frac{1}{3}\Lambda\left(5\csc^2\left(\sqrt{2\,\Lambda}\,x^8\right)+8\right)$ for this background gravitational field. In Eq.[32] we put $\Psi(x^4,x^8) = \psi\left(\sqrt{2\,\Lambda}\,x^4\,,\sqrt{2\,\Lambda}\,x^8\right)$ and set $x=\sqrt{2\,\Lambda}\,x^8$ (where $-\pi\le x\le\pi$) , $t=\sqrt{2\,\Lambda}\,x^4$. This yields the "plane" wave equation for the massive complex scalar field given by

$$\begin{aligned}0 &= -\psi^{(2,0)}(\,t\,,\,x\,)+\psi^{(0,2)}(\,t\,,\,x\,)+\cot\left(x\right)\psi^{(0,1)}(\,t\,,\,x\,)\\ &+ \frac{1}{2}\left(w^2e^{\frac{1}{3}t}-k^2\,e^{-\frac{1}{3}t}\right)\sqrt[6]{\csc^2\left(x\right)}\,\psi(\,t\,,\,x\,)\\ &- \frac{1}{2}\left[m^2+\frac{1}{3}\zeta\left(8+5\csc^2\left(x\right)\right)\right]\psi(\,t\,,\,x\,).\end{aligned} \tag{33}$$

In passing we remark that

$$\sqrt[6]{\csc^2(x)} = \frac{2\sqrt{\frac{\pi}{3}}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}\left({}_2F_1\left(\frac{1}{6},1;\frac{5}{6};e^{-2ix}\right)+{}_2F_1\left(\frac{1}{6},1;\frac{5}{6};e^{2ix}\right)-1\right). \tag{34}$$

Because some of the coefficients in Eq.[33] are physical-time $x^4\leftrightarrow t$ dependent there is no simple dispersion relation that directly relates a physical-frequency ω to the momentum wave vectors $(\vec{k},\vec{w})$.

We study two cases of this equation, for $\zeta=3/10$ and $\zeta=0$.

1. *Solution of the wave equation of a massive scalar field coupled to the Ricci scalar*

We take advantage of the coupling of the massive scalar field to the Ricci scalar to simplify the wave equation. In Eq.[33] we put $\psi(t,x)=\sin^{-\frac{1}{2}}\left(x\right)F\left(t,x\right)$. This yields

$$\begin{aligned}0 &= F^{(2,0)}(t,x)-F^{(0,2)}(t,x)+\frac{1}{2}\left(k^2\,e^{-t/3}-w^2\,e^{t/3}\right)\sqrt[6]{\csc^2(x)}\,F(t,x)\\ &+ \frac{1}{12}\left(16\zeta+6m^2-3\right)F(t,x)+\frac{1}{12}(10\zeta-3)\csc^2(x)F(t,x).\end{aligned} \tag{35}$$

Note that

$$\frac{1}{2}\left(w^2\,e^{t/3}-k^2\,e^{-t/3}\right) = k\,w\,\frac{1}{2}\left(\frac{w}{k}\,e^{t/3}-\frac{k}{w}\,e^{-t/3}\right) = k\,w\,\sinh\left[\frac{t}{3}+\ln\left(\frac{w}{k}\right)\right]. \tag{36}$$

Substituting a Fourier decomposition $F(t,x)=\Sigma_{n_8=-\infty}^{\infty}f(t,n_8)\,\exp(i\,n_8\,x)$ into Eq.[35] and then multiplying by $\frac{1}{2\pi}\exp(-i\,m_8\,x)\,dx$ and integrating from $(-\pi,\pi)$ yields

$$\begin{aligned}0 &= f^{(2,0)}(t,\,m_8\,)+\left(M^2+{m_8}^2\right)f(t,\,m_8\,)\\ &- \frac{1}{2}\left(w^2\,e^{t/3}-k^2\,e^{-t/3}\right)\sum_{n_8=-\infty}^{\infty}\left(c_{m_8\,n_8}\,f(t,\,n_8\,)\right),\end{aligned} \tag{37}$$

Here we have put $\zeta = 3/10$, solely to simplify the wave equation, and $M^2 = \frac{m^2}{2} + \frac{3}{20}$. The matrix elements of $\sqrt[6]{\csc^2(x)}$ in the Fourier basis are

$$\begin{aligned} c_{m_8,n_8} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt[6]{\csc^2(x)}\ \exp[i\,(n_8 - m_8)\,x]\,dx \\ &= \sqrt[3]{2}\ \Gamma\!\left(\frac{2}{3}\right)\begin{cases} \frac{(-1)^N}{\Gamma\left(\frac{5}{6}-N\right)\Gamma\left(\frac{5}{6}+N\right)} & \text{if } n_8 - m_8 = 2\,N \text{ is even} \\ 0 & \text{if } n_8 - m_8 \text{ is odd} \end{cases} \end{aligned} \tag{38}$$

Note that $c_{m_8,n_8} = c_{m_8-n_8,0} = c_{0,n_8-m_8}$, $c_{-n_8\,0} = c_{n_8\,0}$ and

$$\begin{aligned} \sum_{n_8=-\infty}^{\infty} \left(c_{m_8\,n_8}\,e^{i\,\beta\,n_8}\right) &= e^{i\,\beta\,m_8} \sum_{n_8=-\infty}^{\infty} \left(c_{0,\,n_8-m_8}\,e^{i\,\beta\,(n_8-m_8)}\right) \\ &= e^{i\,\beta\,m_8}\ \sqrt[6]{\csc^2(\beta)}. \end{aligned} \tag{39}$$

Eq.[37] may be written as

$$\begin{aligned} 0 \;=\; & \frac{d^2}{d\,t^2}f(t;\,m_8) + \left[\left(M^2 + m_8^2\right) - \frac{1}{2}\left(w^2\,e^{t/3} - k^2\,e^{-t/3}\right)\,c_{0,0}\right] f(t;\,m_8) \\ & - \frac{1}{2}\left(w^2\,e^{t/3} - k^2\,e^{-t/3}\right)\sum_{n_8=1}^{\infty} c_{0\,2n_8}\left[f(t,\ 2n_8 + m8\,) + f(t,\ -2n_8 + m8\,)\right]. \end{aligned} \tag{40}$$

As expected, the effective mass $(M^2 + {m_8}^2)$ of the scalar field receives contributions from the spatial x^8 Fourier modes. We call the first line of Eq.[40]

$$\begin{aligned} & \mathcal{D}[f](t, m_8;\,k,\,w) = \\ & \quad \frac{d^2}{d\,t^2}f(t;\,m_8) + \left[\left(M^2 + m_8^2\right) - \frac{1}{2}\left(w^2\,e^{t/3} - k^2\,e^{-t/3}\right)\,c_{0,0}\right] f(t;\,m_8) \end{aligned} \tag{41}$$

the diagonal contribution to Eq.[40].

To investigate the stability of a massive scalar field as $w \to \infty$ we seek a stable solution to Eq.[40] of the form $f(t, m_8) = \lim_{j\to\infty} f^{(j)}(t, m_8)$ where

$$\begin{aligned} & \mathcal{D}[f^{(0)}](t, m_8;\,k,\,w) = 0 \\ & \mathcal{D}[f^{(j+1)}](t, m_8;\,k,\,w) = \\ & \frac{1}{2}\left(w^2\,e^{t/3} - k^2\,e^{-t/3}\right)\sum_{n_8=1}^{\infty} c_{0\,2n_8}\left[f^{(j)}(t,\ 2n_8 + m8\,) + f^{(j)}(t,\ -2n_8 + m8\,)\right], \\ & \qquad\qquad \text{for } j = 0, 1, 2, \ldots\ ; \end{aligned} \tag{42}$$

To obtain a physical solution to

$$\mathcal{D}[f^{(0)}](t, m_8;\,k,\,w) = 0 \tag{43}$$

we apply a technique due to Dougall [20], Section[15], pages 191-193, with appropriate modifications that account for our asymmetric "potential" $\frac{1}{2}\left(w^2\, e^{t/3} \,-\, k^2\, e^{-t/3}\right)$ in this problem. We find that a general solution to Eq.[43] is given by

$$\begin{aligned} & f^{(0)}(t;\, m_8) = \\ & \sum_{n=-\infty}^{\infty} (-i)^n\, c_n J_n\big(e^{-t/6}\, 3\, k\, \sqrt{2\, c_{0,\,0}}\big)\;\; K_{n+i\nu}\left(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,\,0}}\right) \\ + & \sum_{n=-\infty}^{\infty} i^n\, d_n J_n\big(e^{-t/6}\, 3\, k\, \sqrt{2\, c_{0,\,0}}\big)\;\; I_{n+i\nu}\left(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,\,0}}\right). \end{aligned} \tag{44}$$

Here ν is given by

$$\nu = 6\,\sqrt{(M^2 + m_8^2)}. \tag{45}$$

Also $J_n\,(z)$ denotes an ordinary Bessel function of the first type, $I_\lambda\,(z)$ denotes the modified Bessel function of the first type and order λ and $K_\lambda\,(z)$ denotes a modified Bessel function of the second type (also known as the modified Bessel function of the third kind). The coefficient sets $\{c_n,\, d_n\}\,,\; n = -\infty, \ldots, \infty$ are solutions of the same recursion relation

$$\frac{9}{2} i\, k\, w\, c_{0,\,0}\,(C_{n-1} + C_{n+1}) - n\,(n + i\nu)\, C_n = 0, \tag{46}$$

but with possibly distinct initial values, since a general solution to the three-term recursion relation Eq.[46] possesses two arbitrary constants (Q_1, Q_2). Continued fractions for $\frac{C_n}{C_{n-1}}$ if $n \geq 1$, and $\frac{C_n}{C_{n+1}}$ if $n \leq -1$, are given in the Appendix VI.

Eq.[44] may find application in calculating the cross section for the creation of particle/anti-particle pairs of ψ particles through the annihilation of φ quanta, if one generalizes the model to include an interaction of ψ with the inflaton φ of the form $\lambda\, \psi^*\, \psi\, \varphi$. It is worth noting that for this case ($\zeta = 3/10$) the mass-squared of the ψ field can become negative, since $m^2 = 2M^2 + \frac{1}{2} - \frac{8\zeta}{3} = 2M^2 - \frac{3}{10}$ and one may choose $0 \leq M < \infty$. Negative mass-squared typically characterizes a so-called "uncondensed phase." In such a configuration the ψ field excitations are tachyonic and will spontaneously decay due to the instability caused by the imaginary mass, possibly into Standard Model particles [if one generalizes the Lagrangian to include Standard Model degrees of freedom, as well as adding a Brout-Englert-Higgs-Guralnik-Hagen-Kibble type $(\psi^*\, \psi)^2$ term to the Lagrangian]. This decay may end with another, stable configuration with no ψ tachyons and the universe filled with Standard Model particles.

If we are only interested in determining the stability of a ψ state configuration with respect to fluctuations of the extra time momenta $\vec{w}$ then we are free to boost to a spatial rest frame in which $\vec{k}=\vec{0}$. If the ordinary 3-momentum magnitude $k=0$ then the solution to Eq.[43] $\mathcal{D}[f^{(0)}](t,m_8;\, k=0,\, w)=0$ is found by taking the limit $k\to 0$ in Eq.[44], and is given by

$$\begin{aligned} f^{(0)}(t;\, m_8) \;&=\; {Q_1}^{(m_8)}\, K_{6i\sqrt{M^2+{m_8}^2}}\left(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,0}}\right) \\ &+\; {Q_2}^{(m_8)}\, I_{6i\sqrt{M^2+{m_8}^2}}\left(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,0}}\right), \end{aligned} \tag{47}$$

The values of $({Q_1}^{(m_8)},{Q_2}^{(m_8)})$ for the $k=0$ solution to Eq.[43] subject to initial conditions $f^{(0)}(0;\, m_8)=f(0)$ and $\frac{d}{dt}f^{(0)}(t,m8)]_{t=0}=f'(0)$ are given by

$$\begin{aligned} {Q_1}^{(m_8)} \;&=\; \frac{1}{2}f(0)W\left[I_{6i\nu-1}(W)+I_{6i\nu+1}(W)\right]-6f'(0)I_{6i\nu}(W) \\ {Q_2}^{(m_8)} \;&=\; \frac{1}{2}f(0)W\left[K_{6i\nu-1}(W)+K_{6i\nu+1}(W)\right]+6f'(0)K_{6i\nu}(W) \\ &=\; -f(0)W\frac{\partial K_{6i\nu}(W)}{\partial W}+6f'(0)K_{6i\nu}(W), \end{aligned} \tag{48}$$

where $\nu=6\sqrt{M^2+m_8^2}$ and $W=3\,w\,\sqrt{2\,c_{0,0}}$. Asymptotically, for large argument $|z|$, the modified Bessel functions behave as $K_\lambda\,(z)\;\sim\; e^{-z}\sqrt{\frac{\pi}{2z}}+\cdots$ and $I_\lambda\,(z)\;\sim\; e^{z}\sqrt{\frac{1}{2\pi z}}+\cdots$ [21]. We are seeking, for large w, a stable, Dirac delta normalizable (on hypersurfaces $x^4=$ constant) solution to the equation for $f(t,m_8;\,0,\,w)$.

Therefore, for each m_8, the coefficient ${Q_2}^{(m_8)}$ of $I_{6i\sqrt{M^2+{m_8}^2}}\left(e^{t/6}\,3\,w\,\sqrt{2\,c_{0,0}}\right)$ must vanish. Hence, given w, we require that the initial conditions for the Fourier modes satisfy

$$6\frac{d}{d\,t}\ln(f^{(0)}(t,m8))]_{t=0}=W\frac{d}{d\,W}\ln\left[K_{6i\sqrt{M^2+{m_8}^2}}(W)\right] \tag{49}$$

Substitution of this result into the expression for ${Q_1}^{(m_8)}$ in Eq.[48] yields

$$({Q_1}^{(m_8)},{Q_2}^{(m_8)})=\Big(\frac{f^{(0)}(0,m8))}{K_{6i\sqrt{M^2+{m_8}^2}}(3\,w\,\sqrt{2\,c_{0,0}})},0\Big) \tag{50}$$

We note in passing that the behavior of the modified Bessel function of the second kind with pure imaginary order, $K_{i\nu}(z)$, has been investigated by Balogh [22] and others. Balogh has proved that $K_{i\nu}(\frac{\nu}{p})$ is a positive monotone decreasing convex function of p for $0<p<1$ and it oscillates boundedly for $p>1$, having a countably infinite number of zeros. On page 66, Equation[2.53], Balogh [22] provides a uniform asymptotic expansion for these zeros. This implies that $W\frac{\partial}{\partial W}\ln\left[K_{i\nu}(W)\right]$ also possesses a countably infinite number of (real) zeros corresponding to a countably infinite number of real increasing values of w that make the right hand side of Eq.[49] vanish.

Consider next, for large w, the equation for $f^{(1)}(t, m_8;\, 0,\, w)$ in Eq.[42]

$$
\begin{aligned}
&\mathcal{D}[f^{(1)}](t, m_8;\, k=0,\, w) = \\
&\frac{w^2}{2}\, e^{t/3} \sum_{n_8=1}^{\infty} c_{0\,2n_8} \left[f^{(0)}(t,\, 2n_8 + m_8) + f^{(0)}(t,\, -2n_8 + m_8) \right] \\
= \; &\frac{w^2}{2}\, e^{t/3} 2^{1/3}\, \Gamma\!\left(\frac{2}{3}\right) \sum_{n_8=1}^{\infty} \frac{(-1)^{n_8}}{\Gamma\left(\frac{5}{6} - n_8\right) \Gamma\left(\frac{5}{6} + n_8\right)} \times \\
&\left[f^{(0)}(0, m_8 - 2n_8) \frac{K_{6i\sqrt{M^2 + (m_8 - 2n_8)^2}}(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,0}})}{K_{6i\sqrt{M^2 + (m_8 - 2n_8)^2}}(3\, w\, \sqrt{2\, c_{0,0}})} + \right. \\
&\left. f^{(0)}(0, m_8 + 2n_8) \frac{K_{6i\sqrt{M^2 + (m_8 + 2n_8)^2}}(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,0}})}{K_{6i\sqrt{M^2 + (m_8 + 2n_8)^2}}(3\, w\, \sqrt{2\, c_{0,0}})} \right].
\end{aligned}
\tag{51}
$$

Using the well-known asymptotic expansions of Bessel functions [21] it is not difficult to show that $\lim_{w\to\infty} w^2\, \frac{K_\mu\left(e^{t/6}\, 3\, w\, \sqrt{2\, c_{0,0}}\right)}{K_\mu\left(3\, w\, \sqrt{2\, c_{0,0}}\right)} = 0$. If we assume that the sum in Eq.[51] converges uniformly then we may interchange the limit and the sum, and therefore conclude that the right hand side of equation Eq.[51] vanishes as $w \to \infty$. In fact, all Fourier modes, for all orders of approximation $f^{(j)}(t, m_8)$, vanish as $w \to \infty$.

Hence we have demonstrated the existence, for large w, of a stable approximate solution to the equation for $f(t, m_8;\, 0,\, w)$ in Eq.[42]. This demonstrates that instabilities that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions, the issue raised in Section[2], do not generally arise in this model. We conclude that extra time dimensions may be consistent with the physics of the early universe. We may rationally assert that three extra time dimensions exist in our physical universe, as well as the extra x^8 spatial dimension, and then hope for experimental confirmation/rejection of this claim.

2. Solution of the wave equation of a massive scalar field not coupling to the Ricci scalar

We put $\zeta = 0$ in Eq.[33] to obtain the wave equation for a minimally coupled massive complex scalar field, which is given by

$$\begin{aligned} 0 &= -\psi^{(2,0)}(t\,,\,x) + \psi^{(0,2)}(t\,,\,x) + \cot(x)\,\psi^{(0,1)}(t\,,\,x) \\ &+ \frac{1}{2}\left(w^2 e^{\frac{1}{3}t} - k^2\,e^{-\frac{1}{3}t}\right)\sqrt[6]{\csc^2(x)}\;\psi(t\,,\,x) - \frac{1}{2}m^2\,\psi(t\,,\,x) \\ &= -\psi^{(2,0)}(t\,,\,x) + \frac{1}{\sin(x)}\frac{\partial}{\partial x}\left[\sin(x)\,\frac{\partial}{\partial x}\psi(t\,,\,x)\right] \\ &+ \frac{1}{2}\left(w^2 e^{\frac{1}{3}t} - k^2\,e^{-\frac{1}{3}t}\right)\sqrt[6]{\csc^2(x)}\;\psi(t\,,\,x) - \frac{1}{2}m^2\,\psi(t\,,\,x). \end{aligned} \tag{52}$$

We recall that $-\pi \leq x \leq \pi$ and *not* $0 \leq x \leq \pi$. Let $y = \cos x$ and $x = \pm \arccos y$. In order to obtain the general solution to Eq.[52] we expand $\psi(t\,,\,x) = H(t\,,\,y(x))$ in terms of Legendre polynomials $P_{n_8}(y)$ according to

$$H(t\,,\,y) = \sum_{n_8=0}^{\infty} h_{n_8}(t)\,\left[\sqrt{\left(n_8 + \frac{1}{2}\right)}P_{n_8}(y)\right] \tag{53}$$

and match such expansions for the intervals $-\pi \leq x \leq 0$ and $0 \leq x \leq \pi$ at $x = 0$ and $x = \pm\pi$. Under the coordinate transformation Eq.[52] maps to

$$\begin{aligned} -\,H^{(2,0)}(t,y) \;+\; \left(1-y^2\right)H^{(0,2)}(t,y) - 2yH^{(0,1)}(t,y) + \\ \frac{1}{2}\left(w^2 e^{\frac{1}{3}t} - k^2\,e^{-\frac{1}{3}t}\right)\sqrt[6]{\frac{1}{1-y^2}}\,H(t\,,\,y) - \frac{1}{2}m^2\,H(t\,,\,y) = 0 \end{aligned} \tag{54}$$

Substituting Eq.[53] into Eq.[54], multiplying by $\left[\sqrt{(m_8+\frac{1}{2})}P_{m_8}(y)\right]\,dy$ and integrating from -1 to 1 yields

$$\ddot{h}_{m_8}(t) + \left[\frac{1}{2}m^2 \,+ m_8(m_8+1)\right]h_{m_8}(t) + \frac{1}{2}\left(k^2\,e^{-\frac{1}{3}t} - w^2 e^{\frac{1}{3}t}\right)\sum_{n_8=0}^{\infty} d_{m_8\,n_8}h_{n_8}(t) = 0 \tag{55}$$

where

$$
\begin{aligned}
d_{m_8\,n_8} \;=\; & \pi\Gamma\left(\frac{5}{6}\right)^2\sqrt{\left(m_8+\frac{1}{2}\right)\left(n_8+\frac{1}{2}\right)} \\
& \sum_{j=|n_8-m_8|}^{m_8+n_8\ \ ,\text{ step }2} \\
& \left[\frac{(2j+1)(-1)^{j+m_8+n_8}\Gamma(j+m_8-n_8+1)\Gamma(j-m_8+n_8+1)}{\Gamma\left(\frac{1}{2}-\frac{j}{2}\right)\Gamma\left(\frac{5}{6}-\frac{j}{2}\right)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{j}{2}+\frac{4}{3}\right)}\right. \\
& \frac{\Gamma(-j+m_8+n_8+1)\Gamma\left(\frac{1}{2}(j+m_8+n_8+2)\right)^2}{\Gamma\left(\frac{1}{2}(j+m_8-n_8+2)\right)^2\Gamma\left(\frac{1}{2}(j-m_8+n_8+2)\right)^2} \\
& \left.\frac{1}{\Gamma\left(\frac{1}{2}(-j+m_8+n_8+2)\right)^2\Gamma(j+m_8+n_8+2)}\right] \qquad (56)
\end{aligned}
$$

Other than the matching of the expansions for the intervals $-\pi \leq x \leq 0$ and $0 \leq x \leq \pi$ at $x = 0$ and $x = \pm\pi$, which turns out to be trivial, the general solution to Eq.[55] is obtained using a procedure that is very similar to the technique employed in the previous Section III A 1, and is not worth repeating here. In this case one also concludes that instabilities that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions are not inevitable in this model.

IV. CLASS E_1 SOLUTIONS

The model field equations admit a second class of solutions that is parameterized by ξ, $-\frac{1}{\sqrt{5}} < \xi < \frac{1}{\sqrt{5}}$, which has a time dependent ℓ and has a larger x^8-dimension spatial period;

$$
{}^1C_8 = 2\pi\sqrt{\frac{3}{5\,\Lambda}} = {}^0C_8\,\sqrt{\frac{6}{5}} \qquad (57)
$$

$$
\begin{aligned}
\ell_1 \;=\; & \mp\,\frac{5\sqrt{\Lambda}\xi\tanh\left(\frac{\sqrt{\Lambda}x^4}{\sqrt{3}}\right)}{3\sqrt{3}} = \pm\,5\,\xi\sqrt{\frac{2}{3}}\;\ell_0\,\tanh\left(x^4\sqrt{\frac{\Lambda}{3}}\right) \\
\varphi \;=\; & \frac{\sqrt{1-5\,\xi^2}\ln\left[\cosh^5\left(\frac{\sqrt{\Lambda}x^4}{\sqrt{3}}\right)\csc^2\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]}{2\sqrt{30}} \mp \xi\ln\left[\tan^2\left(\frac{1}{2}\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]; \qquad (58)
\end{aligned}
$$

when $\xi = 0$ then both $\pm\varphi$ satisfy the field equations. The scale factors are

$$\begin{aligned}
a &= \\
&a_1 \left[\cosh\left(x^4\sqrt{\frac{\Lambda}{3}}\right)\right]^{\frac{1}{6}(1+5\xi)} \\
&\times \left[\tan^2\left(\frac{1}{2}\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]^{\pm\sqrt{\frac{1}{30}}\sqrt{1-5\xi^2}} \left[\sin^2\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]^{\frac{1}{12}(1-\xi)} \\
b &= \\
&b_1 \left[\cosh\left(x^4\sqrt{\frac{\Lambda}{3}}\right)\right]^{\frac{1}{6}(1-5\xi)} \\
&\times \left[\tan^2\left(\frac{1}{2}\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]^{\mp\sqrt{\frac{1}{30}}\sqrt{1-5\xi^2}} \left[\sin^2\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}x^8\right)\right]^{\frac{1}{12}(1+\xi)},
\end{aligned} \tag{59}$$

where a_1 and b_1 are constants. For the problem studied in this paper a physical solution to the field equations does not correspond either to a collapsing universe solution $\frac{\partial}{\partial x^4}\ln(a) < 0$ or to a physical universe with multiple macroscopic times, $\frac{\partial}{\partial x^4}\ln(b) \geq 0$. These conditions restrict ξ to the interval $\frac{1}{5} < \xi < \frac{1}{\sqrt{5}}$.

For this case $d\Omega = d\tau \cosh\left(x^4\sqrt{\frac{\Lambda}{3}}\right)\left|\sin\left(x^8\sqrt{\frac{5}{3}\Lambda}\right)\right|$, which clearly **does** inflate with time as expected on physical grounds; it is also independent of both ξ and the choice of $\pm$ signs.

As in the previous Section, one may investigate the issue of instability raised in Section[2] by studying the propagation of a massive complex scalar field ψ whose field equation is given in Eq.[1]. For this case we substitute into Eq.[3] $\Psi(x^4, x^8) = \sin^{-\frac{1}{2}}\left(\sqrt{\frac{5}{3}\Lambda}\,x^8\right) F\left(\sqrt{\frac{5}{3}\Lambda}\,x^4, \sqrt{\frac{5}{3}\Lambda}\,x^8\right)$ and set $x = \sqrt{\frac{5}{3}\Lambda}\,x^8$, $t = \sqrt{\frac{5}{3}\Lambda}\,x^4$, then use the fact that when $\xi = 0$ the Ricci scalar is $\frac{1}{18}\Lambda\left(5\mathrm{sech}^2\left(\frac{\sqrt{\Lambda}\,x^4}{\sqrt{3}}\right) + \csc^2\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}\,x^8\right) + 42\right)$, otherwise it is $\frac{1}{36}\Lambda\left(10\left(5\xi^2 - 1\right)\tanh^2\left(\frac{\sqrt{\Lambda}\,x^4}{\sqrt{3}}\right) + \alpha\csc^2\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}\,x^8\right)\right)$, where $\alpha = 49 + 235\xi^2 \pm 8\xi\sqrt{30 - 150\xi^2}\cos\left(\sqrt{\frac{5}{3}}\sqrt{\Lambda}\,x^8\right) - (5\xi^2 + 47)\cos\left(2\sqrt{\frac{5}{3}}\sqrt{\Lambda}\,x^8\right)$.

For $\xi = 0$ the "plane wave" equation is

$$
\begin{aligned}
&- F^{(2,0)}(t,x) = -F^{(0,2)}(t,x) + \frac{1}{20}\left(28\zeta + 12m^2 - 5\right)F(t,x) + \frac{1}{60}(2\zeta - 15)\csc^2(x)F(t,x) \\
&+ \frac{1}{6}\zeta \operatorname{sech}^2\left(\frac{t}{\sqrt{5}}\right)F(t,x) + \frac{\tanh\left(\frac{t}{\sqrt{5}}\right)F^{(1,0)}(t,x)}{\sqrt{5}} \\
&+ \frac{3}{5}F(t,x)\left(k^2 \tan^{\pm 2\sqrt{\frac{2}{15}}}\left(\frac{x}{2}\right) - w^2 \tan^{\mp 2\sqrt{\frac{2}{15}}}\left(\frac{x}{2}\right)\right)\sqrt[3]{\operatorname{sech}\left(\frac{t}{\sqrt{5}}\right)|\csc(x)|}. \qquad (60)
\end{aligned}
$$

Both the $\vec{k}$ and the $\vec{w}$ terms in the wave equation have coefficients that vanish exponentially as $t \to \pm\infty$. In this case, for large enough $|t|$, we speculate that the momenta $\vec{w}$ associated to the extra time dimensions do not introduce an instability. For finite t this matter must be studied more carefully.

For $\xi \neq 0$ a typical "plane wave" equation is of the form

$$
\begin{aligned}
&- F^{(2,0)}(t,x) = -F^{(0,2)}(t,x) \\
&+ \frac{1}{60}F(t,x)\left(2\zeta\left(5\xi^2 + 47\right) + 36m^2 - \left(15 - 2\zeta\left(115\xi^2 + 1\right)\right)\csc^2(x) - 15\right) \\
&- \frac{2}{15}\zeta\xi\sqrt{30\left(1 - 5\xi^2\right)}\cot(x)\csc(x)F(t,x) \\
&+ \frac{1}{6}\zeta\left(5\xi^2 - 1\right)\tanh^2\left(\frac{t}{\sqrt{5}}\right)F(t,x) + \frac{\tanh\left(\frac{t}{\sqrt{5}}\right)F^{(1,0)}(t,x)}{\sqrt{5}} \\
&+ \frac{3}{5}F(t,x)\left[k^2 \cosh^{\frac{5\xi}{3} - \frac{1}{3}}\left(\frac{t}{\sqrt{5}}\right)\sin^{-\frac{\xi}{3} - \frac{1}{3}}(x)\tan^{-2\sqrt{\frac{2}{15}}\sqrt{1-5\xi^2}}\left(\frac{x}{2}\right)\right. \\
&- \left. w^2 \cosh^{-\frac{5\xi}{3} - \frac{1}{3}}\left(\frac{t}{\sqrt{5}}\right)\sin^{\frac{\xi}{3} - \frac{1}{3}}(x)\tan^{2\sqrt{\frac{2}{15}}\sqrt{1-5\xi^2}}\left(\frac{x}{2}\right)\right]. \qquad (61)
\end{aligned}
$$

In this case, for $-\sqrt{\frac{1}{5}} < \xi < -\frac{1}{5}$, the coefficient of $\vec{k} \cdot \vec{k}$ decreases exponentially, while the coefficient of $\vec{w} \cdot \vec{w}$ grows exponentially as $t \to \pm\infty$, which is similar to behavior seen in the previous Section. For $\frac{1}{5} < \xi < \sqrt{\frac{1}{5}}$, the coefficient of $\vec{k} \cdot \vec{k}$ grows exponentially, while the coefficient of $\vec{w} \cdot \vec{w}$ exponentially decreases as $t \to \pm\infty$. We speculate that there is a range of ξ for which the momenta $\vec{w}$ associated with the extra time dimensions do not introduce an instability.

V. CONCLUSION

We have shown that in this model extra time dimensions are consistent with the some of the important physics of the early universe. We may rationally claim that three extra time dimensions may exist in our physical universe, as well as the extra x^8 spatial dimension. The hope is that experimental observations of certain phenomena that can be predicted in this model (or in the extension of this model as described in a paragraph prior to Eq.[47]), while challenging, are possible.

This model describes an initially inflating/deflating universe in which the inflaton potential is identically equal to zero. Assuming that inflation takes place after the Plank scale (after quantum gravity governs the physics), the concomitant initial condition for this inflaton potential model is exponentially more probable [14], [15], [16] than the corresponding initial condition for a model in which the initial inflaton potential is non-zero and on the order of the Planck mass, give or take a few factors of 10 (in order for the inflationary period to persist for approximately 60-efolds). Note that in Eq.[24], for a general separable solution $\varphi = \varphi(x^4, x^8) = \phi_4(x^4)\phi_8(x^8)$, the term $-\frac{1}{\phi_8(x^8)}\frac{\partial^2}{\partial x^{8\,2}}\phi_8(x^8)$ acts as an effective mass-squared term in the $\frac{\partial^2}{\partial x^{4\,2}}\phi_4(x^4)$ equation. In this case the effective mass of the inflaton comes from geometry: the magnitude of the x^8 component of the inflaton momentum contributes to an effective inflaton mass.

Typically and approximately, inflation scenarios inflate a scale of the size of one billionth the present radius of a proton to the size of the present radius of a marble or a grapefruit in about 10^{-32} seconds. In virtue of the Heisenberg Uncertainty Principle, and because the comoving (x^1, x^2, x^3) dimensions have undergone inflation while the x^8 dimension has not, present epoch quantum fields that are functions of $(x^1, x^2, x^3, x^4, x^8)$ are expected to almost uniformly sample the region of the x^8 dimension that they occupy, if the x^8 spatial dimension is compact, or if their functional dependence on x^8 is periodic. The average of functions of x^8 may be expected to appear in effective four dimensional spacetime theories. The fact that the scale factors vanish on a set of measure zero may be handled in a straightforward manner.

A. Inflationary fluctuations

Fluctuations in correlation functions due to their periodic dependence on the x^8 spatial dimension degree of freedom, in addition to quantum fluctuations of the metric and matter fields during inflation, may contribute to the primordial spectra of scalar and tensor fluctuations that source curvature and density perturbations that give rise to cosmological structure. Calculations of these effects are underway, following the ideas of [23] [24] [25], modified by the following considerations, if the x^8 spatial dimension is compact: It makes little physical sense to mix a compact dimension with non-compact dimensions under a general coordinate transformation (or even a special Lorentz transformation), because the domain of the image of the transformation (i.e., the new coordinate functions) is ill-defined. Either the idea that the x^8 dimension is compact should be abandoned, or it must be recognized that the x^8 dimension plays a distinguished role in the physics. One is quickly led to a classical model of the universe with a distinguished time "degree of freedom," coordinatized by x^4, which **carries** classical observers along its axis in the direction of the "arrow of time," plus a second distinguished dimension that is spatial and compact. At this point in our understanding it seems that allowed coordinate transformations should preserve a distinguished x^8 compact spatial dimension, if it physically exists. Once the allowed coordinate transformations have been restricted this metric theory of gravity superficially resembles an "induced-matter interpretation" of a Kaluza-Klein theory [26] [27], except that the present model possesses three extra time dimensions (albeit with exponentially deflated scale factors for these dimensions) and the fact that all physical fields in our model *critically* depend on the x^8 coordinate of the compact spatial dimension, while in most Kaluza-Klein models the fields do not vary along the extra spatial dimension.

B. Observing the extra dimensions

Both the fourth space dimension, whose associated comoving coordinate is x^8, as well as the extra time dimensions, evidently pose a challenge to observe, if they exist. Based on present science, it is not inconsistent to assert that the extra x^8 spatial dimension is compact, with the properties of a one dimensional topological space that is a (possibly disjoint) union of two distinct sets of closed one dimensional circles (c_0, c_1) that have circumferences ${}^0C_8$ and ${}^1C_8$, respectively. However if the x^8 spatial dimension is not compact then the experimental detection of this dimension is more likely, notwithstanding the fact that the other three spatial dimensions have inflated (thereby defining laboratory length scales) and the x^8 spatial dimension has not. Relative to the first three space dimensions, whose comoving coordinates are $(x^1, x^2, x^3) \in \mathbb{R}^3$, the physical distance between two comoving points $(x_0, x_0 + \Delta X)$ on the x^8-axis is expected to be exponentially smaller by about 60 e-folds than the distance between two comoving points in $\mathbb{R}^3$ separated by the same coordinate difference ΔX, but lying on, say, the x^3-axis; the distance between two comoving points $(x_0, x_0 + \Delta X)$ lying on, say, the x^7-axis is even smaller, since the extra time dimensions experience deflation.

C. Quantum gravity

The three extra time dimensions do not become *un*important physically until *after* inflation/deflation. Quantum gravity is important prior to inflation/deflation. The ideas underlying the foundation of quantum gravity will require some revision if the extra dimensions described mathematically in this paper are physically real. Successful calculations of the cross section for the creation of ψ particle/anti-particle pairs through the annihilation of φ quanta and the concomitant end of φ-driven inflation, as well as the subsequent decay of ψ particles into Standard Model degrees of freedom would strengthen confidence in this model.

VI. APPENDIX: RATIOS OF RECURRENCE COEFFICIENTS AS CONTINUED FRACTIONS

Let $p = i\frac{9}{2}k\, w\, c_{0,0}$ and $m(n) = \frac{n}{p}\,(n + i\nu)$. Rewrite the three-term recursion relation Eq.[46] as

$$m(n)\, C_n = C_{n-1} + C_{n+1}. \tag{62}$$

Then

$$\begin{aligned} m(n) &= \frac{C_{n-1}}{C_n} + \frac{C_{n+1}}{C_n} \\ &= \frac{1}{\frac{C_n}{C_{n-1}}} + \frac{C_{n+1}}{C_n} \Rightarrow \frac{C_n}{C_{n-1}} = \frac{1}{m(n) - \frac{C_{n+1}}{C_n}} \\ &= \frac{C_{n-1}}{C_n} + \frac{1}{\frac{C_n}{C_{n+1}}} \Rightarrow \frac{C_n}{C_{n+1}} = \frac{1}{m(n) - \frac{C_{n-1}}{C_n}} \end{aligned} \tag{63}$$

Let

$$r_n = \begin{cases} \frac{C_n}{C_{n-1}} & \text{if } n \geq 1 \\ \frac{C_n}{C_{n+1}} & \text{if } n \leq -1 \end{cases} \tag{64}$$

For n positive and increasing we seek a set of decreasing $|r_n|$ such that $\lim_{n\to\infty} r_n = 0$. Using the second line of Eq.[63] we express the ratios r_n of these coefficients as the continued fraction

$$r_n = \frac{1}{m(n) - r_{n+1}} = \frac{1}{m(n) - \frac{1}{m(n+1) - r_{n+2}}} = \frac{1}{m(n) - \frac{1}{m(n+1) - \frac{1}{m(n+2) - r_{n+3}}}} = \cdots \tag{65}$$

For n negative and decreasing we seek a set of decreasing $|r_n|$ such that $\lim_{n\to-\infty} r_n = 0$. Using the third line of Eq.[63] we express the ratios r_n of these coefficients as the continued fraction

$$r_n = \frac{1}{m(n) - r_{n-1}} = \frac{1}{m(n) - \frac{1}{m(n-1) - r_{n-2}}} = \frac{1}{m(n) - \frac{1}{m(n-1) - \frac{1}{m(n-2) - r_{n-3}}}} = \cdots \tag{66}$$

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