THE ISO-DUAL TESSERACT

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Abstract

In this work, we deploy Santilli's iso-dual iso-topic lifting and Inopin's holographic ring (IHR) topology as a platform to introduce and assemble a tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space. For this, we prove that such an "iso-dual tesseract" can be constructed by following a procedure of simple, flexible, topologically-preserving instructions. Moreover, these novel results are significant because the tesseract's state and structure are directly inferred from the one initial cube (rather than two distinct cubes), which identifies a new iso-geometrical inter-connection between Santilli's exterior and interior dynamical systems.

Keywords: Santilli iso-number; Inopin holographic ring; Iso-geometry; Tesseract.

1 Introduction

Everybody knows what the square is: a square is a 2D object in 2D space with 4 equal edges, 4 equal angles, and 4 vertices. Most people know what the cube is: a cube is a 3D object in 3D space—the 3D analog of the square—with 12 equal edges, 6 square faces, and 8 vertices, where 3 edges meet at each vertex. But few people know what the *tesseract* is: a tesseract is a 4D object in 4D space—the 4D analog of the cube—with 32 edges, 24 faces, and 16 vertices, where 4 edges meet at each vertex. Basically, the tesseract is to the cube just as the cube is to the square.

To date, there are numerous *geometrical* procedures of tesseract construction that operate with conventional *mathematics*. However, in this paper, we disclose the first *iso-geometrical* procedure of tesseract construction that operates with Santilli's new *iso-mathematics* [1, 2, 3, 4, 5, 6].

To introduce and illustrate this notion, lets consider one approach to build a tesseract. First, we know that the cube has 8 vertices and the tesseract has 16 vertices, therefore a tesseract has *two* times as many vertices as a cube. For this method, this value of *two* is of interest to us—but why? Well, suppose that *two* distinct cubes are positioned in a 3D space, where the sum of the vertices of these two cubes is 16. These resulting 16 vertices indicate that a tesseract can be assembled from the two cubes by introducing 8 additional edges to inter-connect the 8 vertex pairs in a pairwise fashion. Now lets take this one step further: what if one could build a tesseract from a *one* cube instead of two? In conventional mathematics, this question may seem irrelevant because the 8 vertices of a single cube is insufficient to synthesize a tesseract of 16 vertices. However, in the realm of Santilli's *iso-mathematics* [1, 2, 3, 4, 5, 6], this question becomes legitimate when we consider the concept of *iso-duality*.

In this paper, we attack the said inquiry and prove that it is possible to build a tesseract from one *initial cube* by iso-topically lifting [1, 2, 3, 4, 5, 6] its 8 vertices to simultaneously infer an *exterior cube* and an *interior cube* to generate the required 16 vertices, where the double-projected cubes are iso-dual and are both iso-morphic, inter-locking, and synchronized to the initial cube. Consequently, the 16 generated vertices are inter-connected in a pairwise fashion to iso-mathematically synthesize the *iso-tesseract*. Thus, for this investigation, Section 2 presents the main procedure and results, while Section 3 recapitulates the significance of our discovery and suggests future modes of exploration along this research trajectory.

2 Procedure

In this main section, we launch our exploration by instantiating the dual 4D space-time IHR topology [6, 7, 8, 9] so we can subsequently generalize it to encompass the exterior and interior IHR iso-duality [10] and thereby assemble the iso-dual tesseract from one cube through a step-by-step process.

2.1 Preparation: initializing the dual 4D space-time IHR topology

Here, we prepare for the iso-dual tesseract construction of Section 2.2 by first recalling the dual 4D space-time IHR topology [6, 7, 8, 9, 10] via the following procedure:

1. First, given eq. (18) of [6] we identify $Y \equiv \mathbb{T}$ as the set of all triplex numbers, the Euclidean triplex space, and the *dual 3D Cartesian-spherical coordinate-vector state space*. Here, a triplex number $\vec{y} \in Y$ is a *dual 3D Cartesian-spherical coordinate-vector state* that is expressed via eq. (17) of [6] as

$$y = \vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z = (\vec{y}) = (|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S = (\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C, \quad \forall \vec{y} \in Y, \tag{1}$$

   where $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C$ is a *3D Cartesian coordinate-vector state* in the *3D Cartesian coordinate-vector state space* Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C \in Y_C$, while simultaneously $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S$ is a *3D spherical coordinate-vector state* in the *3D spherical coordinate-vector state space* Y_S so $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S \in Y_S$, such that Y_C and Y_S are dual, iso-morphic, synchronized, and interlocking in Y [6]. Hence, eq. (1) satisfies with the constraints imposed by eqs. (19–28) of [6]—see Figures 4 and 5 of [6].

2. Second, given eq. (33) of [6] we recall that

$$T_r^2 = \{\vec{y} \in Y : |\vec{y}| = r\}, \tag{2}$$

   where $T_r^2 \subset Y$ is the *2-sphere IHR* of amplitude-radius $r > 0$ that is centered on the origin $O \in Y$ and is iso-metrically embedded in Y; T_r^2

is the multiplicative group of all non-zero triplex numbers with the amplitude-radius r, which is simultaneously dual to the two triplex sub-spaces [6, 7, 8]: the "micro sub-space 3-brane" $Y_{-} \subset Y$ and the "macro sub-space 3-brane" $Y_{+} \subset Y$—see Figure 7 of [6]. Here, we note that the *1-sphere IHR* $T_r^1 \subset T_r^2$ of amplitude-radius $r > 0$ (and curvature $\frac{1}{r}$) from eq. (16) of [6] is the great circle of T_r^2.

At this point, we've initialized Inopin's dual 4D space-time IHR topology [6, 7, 8, 9, 10] and are therefore prepared to assemble the iso-dual tesseract of Section 2.2.

2.2 Engagement: constructing the iso-dual tesseract

Here, equipped with the dual 4D space-time IHR topology of Section 2.1, we introduce, define, and assemble the proposed iso-dual tesseract via the following procedure:

1. First, we recall that in conventional mathematics the number 1 is the multiplicative identity which satisfies the original number field axioms [11]. Thus, in general, the number 1 plays a crucial and diverse role throughout the various branches of mathematics such as, for example, normalization in statistics. Therefore, we begin by setting the amplitude-radius $r = 1$ for T_r^1 and T_r^2.

2. Second, we construct the *initial cube* from 8 triplex vertices that are confined to T_r^2. For this cube, we define the underlying set of 8 triplex vertices as

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2} \subset T_r^2 \subset Y \quad (3)$$

such that

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V_{T_r^2}^{\uparrow} \subset V_{T_r^2} \subset T_r^2 \subset Y \quad (4)$$

are the "top vertices" for the "top square surface" and

$$\{\bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2}^{\downarrow} \subset V_{T_r^2} \subset T_r^2 \subset Y \quad (5)$$

are the "bottom vertices" for the "bottom square surface", which comply with the *cubic vertex triplex amplitude-radius constraints*

$$\begin{aligned} 1 \;\equiv\; r \;\equiv\; |\vec{a}_r| \;\equiv\; |\vec{b}_r| \;\equiv\; |\vec{c}_r| \;\equiv\; |\vec{d}_r| \\ \equiv\; |\vec{\bar{a}}_r| \;\equiv\; |\vec{\bar{b}}_r| \;\equiv\; |\vec{\bar{c}}_r| \;\equiv\; |\vec{\bar{d}}_r|, \end{aligned} \tag{6}$$

the *cubic vertex triplex phase constraints*

$$\begin{aligned} \langle \vec{a}_r \rangle \;&\equiv\; \langle \vec{b}_r \rangle - \tfrac{\pi}{2} \;\equiv\; \langle \vec{c}_r \rangle \pm \pi \;\equiv\; \langle \vec{d}_r \rangle - \tfrac{3\pi}{2} \\ \langle \vec{\bar{a}}_r \rangle \;&\equiv\; \langle \vec{\bar{b}}_r \rangle - \tfrac{\pi}{2} \;\equiv\; \langle \vec{\bar{c}}_r \rangle \pm \pi \;\equiv\; \langle \vec{\bar{d}}_r \rangle - \tfrac{3\pi}{2} \end{aligned} \tag{7}$$

such that

$$\begin{aligned} \langle \vec{a}_r \rangle \;&\equiv\; \langle \vec{\bar{a}}_r \rangle \pm \pi \\ \langle \vec{b}_r \rangle \;&\equiv\; \langle \vec{\bar{b}}_r \rangle \pm \pi \\ \langle \vec{c}_r \rangle \;&\equiv\; \langle \vec{\bar{c}}_r \rangle \pm \pi \\ \langle \vec{d}_r \rangle \;&\equiv\; \langle \vec{\bar{d}}_r \rangle \pm \pi, \end{aligned} \tag{8}$$

and the *cubic vertex triplex inclination constraints*

$$\begin{aligned} [\vec{a}_r] \;&\equiv\; [\vec{b}_r] \;\equiv\; [\vec{c}_r] \;\equiv\; [\vec{d}_r] \\ [\vec{\bar{a}}_r] \;&\equiv\; [\vec{\bar{b}}_r] \;\equiv\; [\vec{\bar{c}}_r] \;\equiv\; [\vec{\bar{d}}_r] \end{aligned} \tag{9}$$

such that

$$\begin{aligned} [\vec{a}_r] \;&\equiv\; [\vec{\bar{a}}_r] \pm \pi \\ [\vec{b}_r] \;&\equiv\; [\vec{\bar{b}}_r] \pm \pi \\ [\vec{c}_r] \;&\equiv\; [\vec{\bar{c}}_r] \pm \pi \\ [\vec{d}_r] \;&\equiv\; [\vec{\bar{d}}_r] \pm \pi, \end{aligned} \tag{10}$$

to establish the *cubic vertex triplex antisymmetric constraints*

$$\begin{aligned} \vec{a}_r \;&\equiv\; -\vec{\bar{a}}_r \\ \vec{b}_r \;&\equiv\; -\vec{\bar{b}}_r \\ \vec{c}_r \;&\equiv\; -\vec{\bar{c}}_r \\ \vec{d}_r \;&\equiv\; -\vec{\bar{d}}_r. \end{aligned} \tag{11}$$

Therefore, the cube built from the 8 triplex vertices comprising $V_{T_r^2}$ of eq. (3)—which satisfy eqs. (6–11) and are confined to T_r^2—is depicted in Figures 1 and 2.

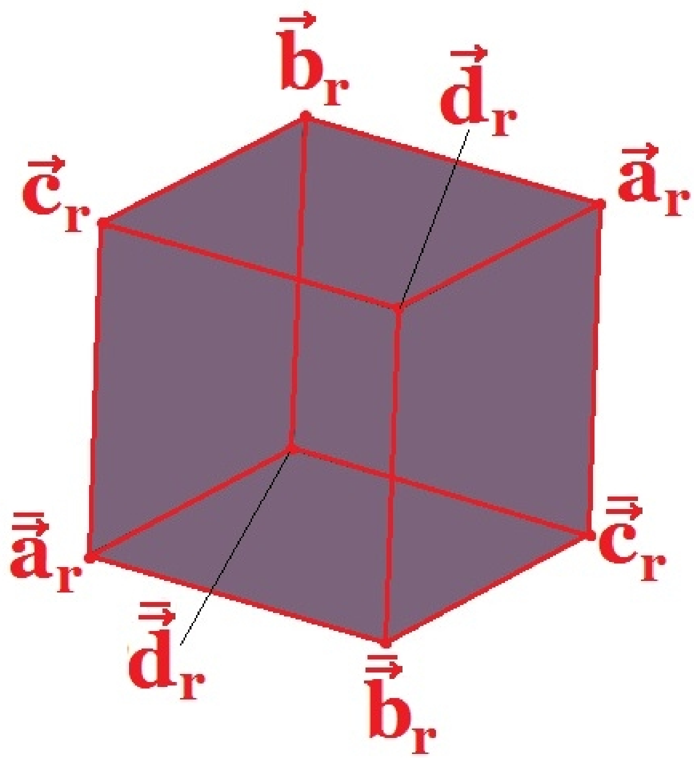

Fig. 1: The 8 triplex vertices of the initial cube comprise the set $\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2}$, which are confined to T_r^2 (not shown).

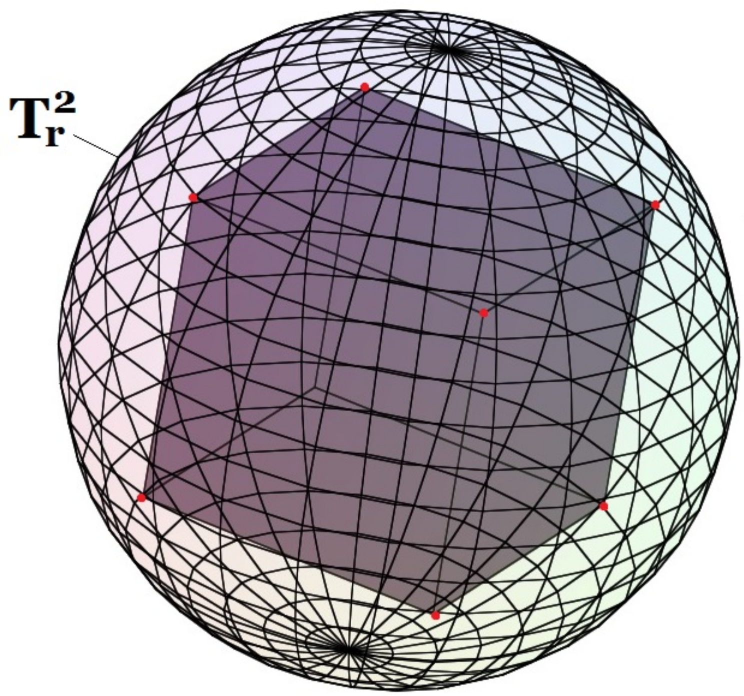

Fig. 2: The 8 triplex vertices of the initial cube comprise the set $\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2}$, which are confined to T_r^2 (shown).

3. Third, in iso-mathematics [1, 2, 3, 4, 5], Santilli proved that the standard multiplicative unit which satisfies the conventional number field axioms [11] is not limited to the number 1, and can thus be replaced with a positive-definite iso-multiplicative iso-unit $\hat{r}_+ > 0$ for iso-numbers. Thus, in accordance to Santilli's methodology [1, 2, 3, 4, 5], we select some $\hat{r}_+$ with the corresponding iso-unit inverse $\hat{r}_- = \frac{1}{\hat{r}_+}$, such that
$$\hat{r}_+ > r > \hat{r}_- > 0. \tag{12}$$

4. Fourth, we engage $\hat{r}_+$ to iso-topically lift [1, 2, 3, 4, 5] T_r^2 to the *exterior iso-2-sphere IHR* $T_{\hat{r}_+}^2$ via the transition
$$f(T_r^2, \hat{r}_+): \quad T_r^2 \rightarrow T_{\hat{r}_+}^2 \tag{13}$$
and its corresponding inverse
$$f^{-1}(T_{\hat{r}_+}^2, \hat{r}_+): \quad T_{\hat{r}_+}^2 \rightarrow T_r^2, \tag{14}$$
such that the iso-unit $\hat{r}_+$ is the *exterior iso-radius* of $T_{\hat{r}_+}^2$, which is "outside" of T_r^2 because eq. (2) becomes
$$T_{\hat{r}_+}^2 \equiv \{\vec{y}_{\hat{r}_+} \in Y : |\vec{y}_{\hat{r}_+}| = r \times \hat{r}_+\} \tag{15}$$
for
$$\vec{y}_{\hat{r}_+} \equiv \vec{y} \times \hat{r}_+, \ \forall \vec{y} \in T_r^2 \rightarrow \forall \vec{y}_{\hat{r}_+} \in T_{\hat{r}_+}^2, \tag{16}$$
where T_r^2 and $T_{\hat{r}_+}^2$ are *locally iso-morphic* [6, 10]. Therefore, given that $V_{T_r^2} \subset T_r^2$, the iso-topic lifting of eqs. (13–16) indicates
$$\begin{array}{rcl}
\vec{a}_{\hat{r}_+} & \equiv & \vec{a}_r \times \hat{r}_+ \\
\vec{b}_{\hat{r}_+} & \equiv & \vec{b}_r \times \hat{r}_+ \\
\vec{c}_{\hat{r}_+} & \equiv & \vec{c}_r \times \hat{r}_+ \\
\vec{d}_{\hat{r}_+} & \equiv & \vec{d}_r \times \hat{r}_+ \\
\bar{\vec{a}}_{\hat{r}_+} & \equiv & \bar{\vec{a}}_r \times \hat{r}_+ \\
\bar{\vec{b}}_{\hat{r}_+} & \equiv & \bar{\vec{b}}_r \times \hat{r}_+ \\
\bar{\vec{c}}_{\hat{r}_+} & \equiv & \bar{\vec{c}}_r \times \hat{r}_+ \\
\bar{\vec{d}}_{\hat{r}_+} & \equiv & \bar{\vec{d}}_r \times \hat{r}_+,
\end{array} \tag{17}$$

enabling us to rewrite eq. (6) to establish the *exterior cubic iso-vertex iso-triplex amplitude-radius constraints*

$$\begin{aligned} \hat{r}_+ &\equiv |\vec{a}_{\hat{r}_+}| \equiv |\vec{b}_{\hat{r}_+}| \equiv |\vec{c}_{\hat{r}_+}| \equiv |\vec{d}_{\hat{r}_+}| \\ &\equiv |\vec{\bar{a}}_{\hat{r}_+}| \equiv |\vec{\bar{b}}_{\hat{r}_+}| \equiv |\vec{\bar{c}}_{\hat{r}_+}| \equiv |\vec{\bar{d}}_{\hat{r}_+}| \end{aligned} \tag{18}$$

with the *exterior iso-vertex* directional-preservations

$$\begin{array}{ccccccc} \langle\vec{a}_{\hat{r}_+}\rangle & \equiv & \langle\vec{a}_r\rangle & | & [\vec{a}_{\hat{r}_+}] & \equiv & [\vec{a}_r] \\ \langle\vec{b}_{\hat{r}_+}\rangle & \equiv & \langle\vec{b}_r\rangle & | & [\vec{b}_{\hat{r}_+}] & \equiv & [\vec{b}_r] \\ \langle\vec{c}_{\hat{r}_+}\rangle & \equiv & \langle\vec{c}_r\rangle & | & [\vec{c}_{\hat{r}_+}] & \equiv & [\vec{c}_r] \\ \langle\vec{d}_{\hat{r}_+}\rangle & \equiv & \langle\vec{d}_r\rangle & | & [\vec{d}_{\hat{r}_+}] & \equiv & [\vec{d}_r] \\ \langle\vec{\bar{a}}_{\hat{r}_+}\rangle & \equiv & \langle\vec{\bar{a}}_r\rangle & | & [\vec{\bar{a}}_{\hat{r}_+}] & \equiv & [\vec{\bar{a}}_r] \\ \langle\vec{\bar{b}}_{\hat{r}_+}\rangle & \equiv & \langle\vec{\bar{b}}_r\rangle & | & [\vec{\bar{b}}_{\hat{r}_+}] & \equiv & [\vec{\bar{b}}_r] \\ \langle\vec{\bar{c}}_{\hat{r}_+}\rangle & \equiv & \langle\vec{\bar{c}}_r\rangle & | & [\vec{\bar{c}}_{\hat{r}_+}] & \equiv & [\vec{\bar{c}}_r] \\ \langle\vec{\bar{d}}_{\hat{r}_+}\rangle & \equiv & \langle\vec{\bar{d}}_r\rangle & | & [\vec{\bar{d}}_{\hat{r}_+}] & \equiv & [\vec{\bar{d}}_r] \end{array} \tag{19}$$

that continue to satisfy the generalized constraints of eqs. (7–11) to establish

$$\{\vec{a}_{\hat{r}_+}, \vec{b}_{\hat{r}_+}, \vec{c}_{\hat{r}_+}, \vec{d}_{\hat{r}_+}, \vec{\bar{a}}_{\hat{r}_+}, \vec{\bar{b}}_{\hat{r}_+}, \vec{\bar{c}}_{\hat{r}_+}, \vec{\bar{d}}_{\hat{r}_+}\} \equiv V_{T^2_{\hat{r}_+}} \subset T^2_{\hat{r}_+} \subset Y_+ \tag{20}$$

for the implied exterior vertex iso-topic lifting $V_{T^2_r} \to V_{T^2_{\hat{r}_+}}$, where $V_{T^2_{\hat{r}_+}}$ is the exterior set of 8 iso-triplex iso-vertices that are confined to $T^2_{\hat{r}_+}$ and form the *exterior cube* of the tesseract for the *exterior dynamical system* of the macro sub-space 3-brane Y_+.

5. Fifth, given eqs. (13–16), the relation $\hat{r}_- = \frac{1}{\hat{r}_+}$ is the foundation of the exterior and interior IHR iso-duality of [10], where the iso-unit inverse $\hat{r}_-$ is the *interior iso-radius* of the *interior iso-2-sphere IHR* $T^2_{\hat{r}_-}$ that is "inside" of T^2_r, such that T^2_r is simultaneously iso-topically lifted to $T^2_{\hat{r}_-}$ via the transition

$$f(T^2_r, \hat{r}_-): \quad T^2_r \to T^2_{\hat{r}_-} \tag{21}$$

and its corresponding inverse

$$f^{-1}(T^2_{\hat{r}_-}, \hat{r}_-): \quad T^2_{\hat{r}_-} \to T^2_r, \tag{22}$$

because eq. (2) becomes

$$T^2_{\hat{r}_-} \equiv \{\hat{\vec{y}} \in Y : |\hat{\vec{y}}| = r \times \hat{r}_-\} \tag{23}$$

for

$$\hat{\vec{y}} \equiv \vec{y} \times \hat{r}_-, \ \forall \vec{y} \in T^2_r \to \forall \hat{\vec{y}} \in T^2_{\hat{r}_-}, \tag{24}$$

where T^2_r and $T^2_{\hat{r}_-}$ are *locally iso-morphic* [6, 10]. Thus, the $T^2_{\hat{r}_+}$ of eqs. (13–16) is *iso-dual* to the $T^2_{\hat{r}_-}$ of eqs. (21–24) with respect to T^2_r in accordance to the exterior and interior IHR iso-duality of [10]. Therefore, given that $V_{T^2_r} \subset T^2_r$, the iso-topic lifting of eqs. (21–24) indicates

$$\begin{array}{rcl}
\vec{a}_{\hat{r}_-} & \equiv & \vec{a}_r \times \hat{r}_- \\
\vec{b}_{\hat{r}_-} & \equiv & \vec{b}_r \times \hat{r}_- \\
\vec{c}_{\hat{r}_-} & \equiv & \vec{c}_r \times \hat{r}_- \\
\vec{d}_{\hat{r}_-} & \equiv & \vec{d}_r \times \hat{r}_- \\
\bar{\vec{a}}_{\hat{r}_-} & \equiv & \bar{\vec{a}}_r \times \hat{r}_- \\
\bar{\vec{b}}_{\hat{r}_-} & \equiv & \bar{\vec{b}}_r \times \hat{r}_- \\
\bar{\vec{c}}_{\hat{r}_-} & \equiv & \bar{\vec{c}}_r \times \hat{r}_- \\
\bar{\vec{d}}_{\hat{r}_-} & \equiv & \bar{\vec{d}}_r \times \hat{r}_-,
\end{array} \tag{25}$$

enabling us to rewrite eq. (6) to establish the *interior cubic iso-vertex iso-triplex amplitude-radius constraints*

$$\begin{array}{rcccccccc}
\hat{r}_- & \equiv & |\vec{a}_{\hat{r}_-}| & \equiv & |\vec{b}_{\hat{r}_-}| & \equiv & |\vec{c}_{\hat{r}_-}| & \equiv & |\vec{d}_{\hat{r}_-}| \\
 & \equiv & |\bar{\vec{a}}_{\hat{r}_-}| & \equiv & |\bar{\vec{b}}_{\hat{r}_-}| & \equiv & |\bar{\vec{c}}_{\hat{r}_-}| & \equiv & |\bar{\vec{d}}_{\hat{r}_-}|
\end{array} \tag{26}$$

with the *interior iso-vertex* directional-preservations

$$\begin{array}{ccccccccccc}
\langle\vec{a}_{\hat{r}_-}\rangle & \equiv & \langle\vec{a}_r\rangle & \equiv & \langle\vec{a}_{\hat{r}_+}\rangle & | & [\vec{a}_{\hat{r}_-}] & \equiv & [\vec{a}_r] & \equiv & [\vec{a}_{\hat{r}_+}] \\
\langle\vec{b}_{\hat{r}_-}\rangle & \equiv & \langle\vec{b}_r\rangle & \equiv & \langle\vec{b}_{\hat{r}_+}\rangle & | & [\vec{b}_{\hat{r}_-}] & \equiv & [\vec{b}_r] & \equiv & [\vec{b}_{\hat{r}_+}] \\
\langle\vec{c}_{\hat{r}_-}\rangle & \equiv & \langle\vec{c}_r\rangle & \equiv & \langle\vec{c}_{\hat{r}_+}\rangle & | & [\vec{c}_{\hat{r}_-}] & \equiv & [\vec{c}_r] & \equiv & [\vec{c}_{\hat{r}_+}] \\
\langle\vec{d}_{\hat{r}_-}\rangle & \equiv & \langle\vec{d}_r\rangle & \equiv & \langle\vec{a}_{\hat{r}_+}\rangle & | & [\vec{d}_{\hat{r}_-}] & \equiv & [\vec{d}_r] & \equiv & [\vec{d}_{\hat{r}_+}] \\
\langle\bar{\vec{a}}_{\hat{r}_-}\rangle & \equiv & \langle\bar{\vec{a}}_r\rangle & \equiv & \langle\bar{\vec{a}}_{\hat{r}_+}\rangle & | & [\bar{\vec{a}}_{\hat{r}_-}] & \equiv & [\bar{\vec{a}}_r] & \equiv & [\bar{\vec{a}}_{\hat{r}_+}] \\
\langle\bar{\vec{b}}_{\hat{r}_-}\rangle & \equiv & \langle\bar{\vec{b}}_r\rangle & \equiv & \langle\bar{\vec{b}}_{\hat{r}_+}\rangle & | & [\bar{\vec{b}}_{\hat{r}_-}] & \equiv & [\bar{\vec{b}}_r] & \equiv & [\bar{\vec{b}}_{\hat{r}_+}] \\
\langle\bar{\vec{c}}_{\hat{r}_-}\rangle & \equiv & \langle\bar{\vec{c}}_r\rangle & \equiv & \langle\bar{\vec{c}}_{\hat{r}_+}\rangle & | & [\bar{\vec{c}}_{\hat{r}_-}] & \equiv & [\bar{\vec{c}}_r] & \equiv & [\bar{\vec{c}}_{\hat{r}_+}] \\
\langle\bar{\vec{d}}_{\hat{r}_-}\rangle & \equiv & \langle\bar{\vec{d}}_r\rangle & \equiv & \langle\bar{\vec{d}}_{\hat{r}_+}\rangle & | & [\bar{\vec{d}}_{\hat{r}_-}] & \equiv & [\bar{\vec{d}}_r] & \equiv & [\bar{\vec{d}}_{\hat{r}_+}]
\end{array} \tag{27}$$

that incorporate eq. (19) and continue to satisfy the generalized constraints of eqs. (7–11) to establish

$$\{\vec{a}_{\hat{r}_-}, \vec{b}_{\hat{r}_-}, \vec{c}_{\hat{r}_-}, \vec{d}_{\hat{r}_-}, \vec{\bar{a}}_{\hat{r}_-}, \vec{\bar{b}}_{\hat{r}_-}, \vec{\bar{c}}_{\hat{r}_-}, \vec{\bar{d}}_{\hat{r}_-}\} \equiv V_{T^2_{\hat{r}_-}} \subset T^2_{\hat{r}_-} \subset Y_- \tag{28}$$

for the implied interior vertex iso-topic lifting $V_{T^2_r} \rightarrow V_{T^2_{\hat{r}_-}}$, where $V_{T^2_{\hat{r}_-}}$ is the interior set of 8 iso-triplex iso-vertices that are confined to $T^2_{\hat{r}_-}$ and form the *interior cube* of the tesseract for the *interior dynamical system* of the micro sub-space 3-brane Y_-.

6. Sixth, given the 8 exterior iso-triplex iso-vertices of $T^2_{\hat{r}_+}$ in eq. (20) and the 8 interior iso-triplex iso-vertices of $T^2_{\hat{r}_-}$ in eq. (28), we identify the 16 iso-triplex iso-vertices of the iso-dual tesseract as

$$V_{T^2_{\hat{r}_\pm}} \equiv V_{T^2_{\hat{r}_+}} \cup V_{T^2_{\hat{r}_-}}, \tag{29}$$

where 8 additional edges are inserted to inter-link the iso-vertex pairs in a pairwise fashion to inter-connect Santilli's exterior and interior dynamical systems for Y_+ and Y_-, respectively. See Figure 3 for a depiction of the iso-dual tesseract.

7. Seventh, it is straightforward to assign triplex order parameters [6, 7, 9] to the iso-vertices of eq. (29) to topologically deform the tesseract. For example, suppose that one layer of triplex order parameters [6, 7, 9] is assigned to the 8 vertices of $V_{T^2_r}$ as

$$\{\vec{\psi}(\vec{a}_r), \vec{\psi}(\vec{b}_r), \vec{\psi}(\vec{c}_r), \vec{\psi}(\vec{d}_r), \vec{\psi}(\vec{\bar{a}}_r), \vec{\psi}(\vec{\bar{b}}_r), \vec{\psi}(\vec{\bar{c}}_r), \vec{\psi}(\vec{\bar{d}}_r)\} \equiv \vec{\psi}_{T^2_r} \tag{30}$$

to encode topological deformations that comply with the antisymmetric constraints

$$\begin{array}{rcl} \vec{\psi}(\vec{a}_r) & \equiv & -\vec{\psi}(\vec{\bar{a}}_r) \\ \vec{\psi}(\vec{b}_r) & \equiv & -\vec{\psi}(\vec{\bar{b}}_r) \\ \vec{\psi}(\vec{c}_r) & \equiv & -\vec{\psi}(\vec{\bar{c}}_r) \\ \vec{\psi}(\vec{d}_r) & \equiv & -\vec{\psi}(\vec{\bar{d}}_r) \end{array} \tag{31}$$

that are depicted in Figure 4.

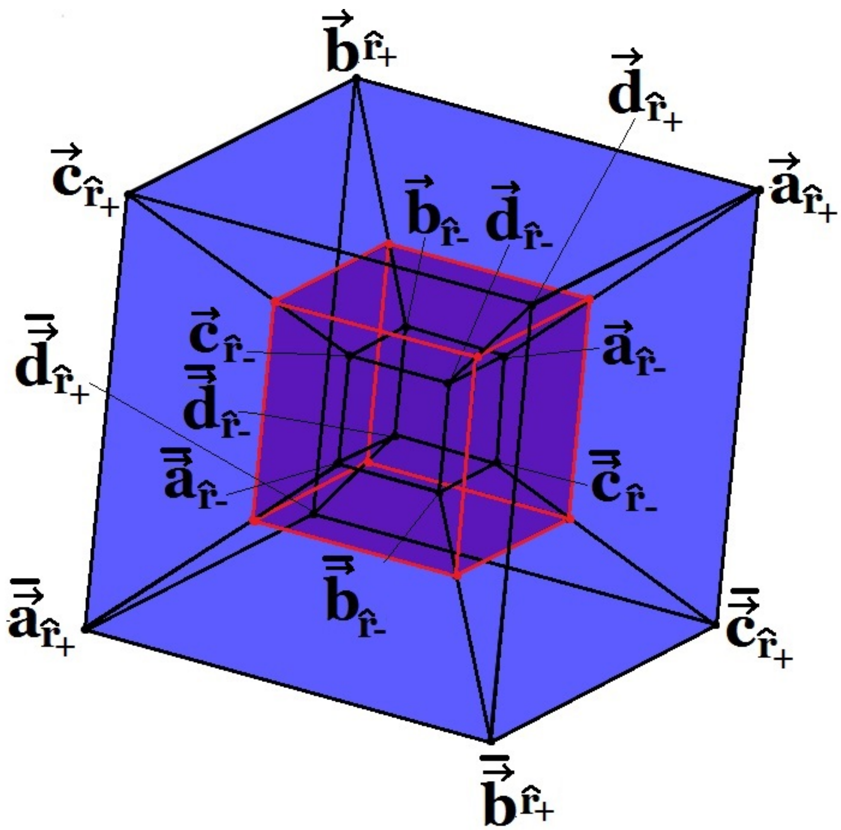

Fig. 3: The 8 triplex vertices of $V_{T_r^2} \subset T_r^2$ are iso-topically lifted via the double-projection iso-dual transition $V_{T^2_{\hat{r}_-}} \subset T^2_{\hat{r}_-} \leftarrow V_{T^2_r} \rightarrow V_{T^2_{\hat{r}_+}} \subset T^2_{\hat{r}_+}$ to generate the 16 iso-triplex iso-vertices of $V_{T^2_{\hat{r}_\pm}}$ for the iso-dual tesseract. Here, the exterior cube's 8 exterior iso-vertices in $V_{T^2_{\hat{r}_+}}$ are confined to the exterior IHR $T^2_{\hat{r}_+} \subset Y_+$ (not shown) in the exterior dynamical system while the interior cube's 8 interior iso-vertices in $V_{T^2_{\hat{r}_-}}$ are confined to the interior IHR $T^2_{\hat{r}_-} \subset Y_-$ (not shown) in the interior dynamical system, which are iso-dual to each other and are both iso-morphic, inter-locking, and synchronized to the initial cube [10].

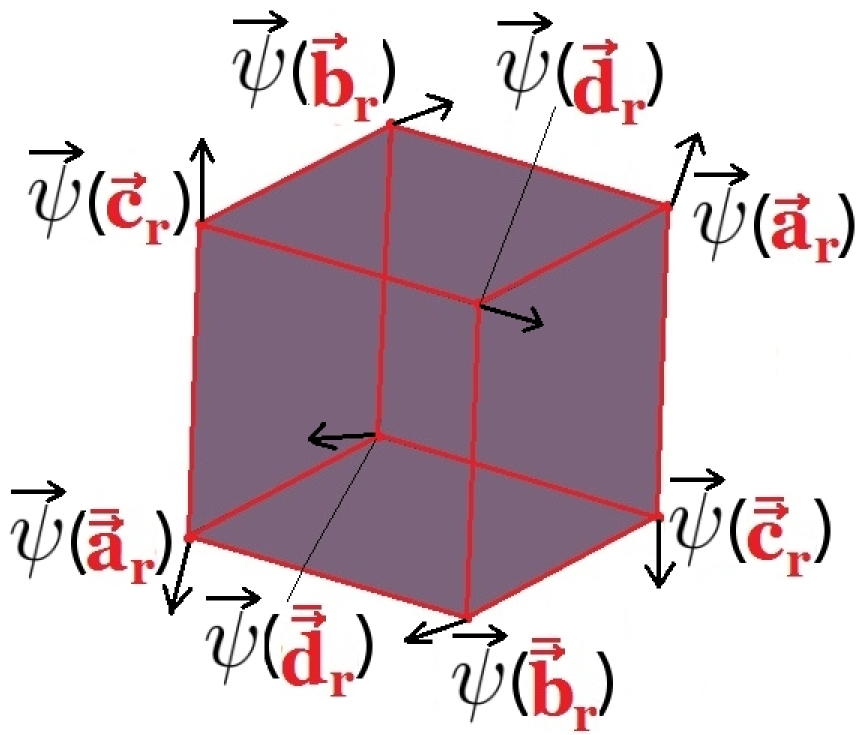

Fig. 4: The 8 triplex vertices of $V_{T_r^2} \subset T_r^2$ are assigned one layer of triplex order parameters [6, 7, 9] to encode topological deformations. These order parameter states can be iso-topically lifted [1, 2, 3, 4, 5, 6] to iso-triplex iso-vertex order parameter states in a double-projective fashion for the iso-dual tesseract.

8. Finally, we can simply select some positive-definite iso-unit with a corresponding inverse (i.e. we can reuse $\hat{r}_+$ and $\hat{r}_-$ or select alternative quantities) and repeat the iso-dual iso-topic lifting of Steps 1–6 for the vertice's triplex order parameters of eqs. (30–31) to define iso-triplex order parameters for the iso-dual tesseract. Thus, if we opt to redeploy $\hat{r}_+$ and $\hat{r}_-$ we define the iso-dual iso-topic liftings

$$
\begin{array}{rcl|rcl}
\vec{\psi}(\vec{a}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{a}_r) \times \hat{r}_+ & \vec{\psi}(\vec{a}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{a}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{b}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{b}_r) \times \hat{r}_+ & \vec{\psi}(\vec{b}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{b}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{c}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{c}_r) \times \hat{r}_+ & \vec{\psi}(\vec{c}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{c}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{d}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{d}_r) \times \hat{r}_+ & \vec{\psi}(\vec{d}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{d}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{\bar{a}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{\bar{a}}_r) \times \hat{r}_+ & \vec{\psi}(\vec{\bar{a}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{\bar{a}}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{\bar{b}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{\bar{b}}_r) \times \hat{r}_+ & \vec{\psi}(\vec{\bar{b}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{\bar{b}}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{\bar{c}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{\bar{c}}_r) \times \hat{r}_+ & \vec{\psi}(\vec{\bar{c}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{\bar{c}}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{\bar{d}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{\bar{d}}_r) \times \hat{r}_+ & \vec{\psi}(\vec{\bar{d}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{\bar{d}}_r) \times \hat{r}_-
\end{array}
\tag{32}
$$

for the double-projection iso-morphic transitions

$$
\begin{array}{rcccl}
\vec{\psi}(\vec{a}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{a}_r) & \rightarrow & \vec{\psi}(\vec{a}_{\hat{r}_+}) \\
\vec{\psi}(\vec{b}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{b}_r) & \rightarrow & \vec{\psi}(\vec{b}_{\hat{r}_+}) \\
\vec{\psi}(\vec{c}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{c}_r) & \rightarrow & \vec{\psi}(\vec{c}_{\hat{r}_+}) \\
\vec{\psi}(\vec{d}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{d}_r) & \rightarrow & \vec{\psi}(\vec{d}_{\hat{r}_+}) \\
\vec{\psi}(\vec{\bar{a}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{\bar{a}}_r) & \rightarrow & \vec{\psi}(\vec{\bar{a}}_{\hat{r}_+}) \\
\vec{\psi}(\vec{\bar{b}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{\bar{b}}_r) & \rightarrow & \vec{\psi}(\vec{\bar{b}}_{\hat{r}_+}) \\
\vec{\psi}(\vec{\bar{c}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{\bar{c}}_r) & \rightarrow & \vec{\psi}(\vec{\bar{c}}_{\hat{r}_+}) \\
\vec{\psi}(\vec{\bar{d}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{\bar{d}}_r) & \rightarrow & \vec{\psi}(\vec{\bar{d}}_{\hat{r}_+})
\end{array}
\tag{33}
$$

and the corresponding inverses

$$\begin{array}{ccccc}
\vec{\psi}(\vec{a}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{a}_r) & \leftarrow & \vec{\psi}(\vec{a}_{\hat{r}_+}) \\
\vec{\psi}(\vec{b}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{b}_r) & \leftarrow & \vec{\psi}(\vec{b}_{\hat{r}_+}) \\
\vec{\psi}(\vec{c}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{c}_r) & \leftarrow & \vec{\psi}(\vec{c}_{\hat{r}_+}) \\
\vec{\psi}(\vec{d}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{d}_r) & \leftarrow & \vec{\psi}(\vec{d}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{a}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{a}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{a}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{b}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{b}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{b}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{c}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{c}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{c}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{d}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{d}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{d}}_{\hat{r}_+}).
\end{array} \tag{34}$$

At this point, we've completed the construction of the iso-dual tesseract by generalizing the dual 4D space-time IHR topology of Section 2.1 with the exterior and interior iso-duality [10].

3 Conclusion

In this research investigation, we deployed Santilli's iso-mathematics [1, 2, 3, 4, 5, 6] and Inopin's dual 4D space-time IHR topology [6, 7, 8, 9] as a platform to assemble the iso-dual tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space that fundamentally comply with exterior and interior IHR iso-duality [10]. To prove that such a tesseract can be built from one cube (rather than two distinct cubes), we presented the step-by-step procedure of Section 2 with simple, flexible, topologically-preserving instructions, where the single, initial cube was iso-topically lifted to simultaneously infer the exterior cube and the interior cube via double-projection. Subsequently, the exterior cube and the interior cube were inter-linked together in a point-by-point fashion by inter-linking the 8 iso-vertex pairs with 8 additional edges to superstruct the iso-dual tesseract. In total, the outcomes of this exploration are significant because an original iso-geometrical inter-connection between Santilli's exterior and interior dynamical systems has been established, which advances the application of iso-mathematics [1, 2, 3, 4, 5, 6] in a new direction.

We suggest that the next logical step of this research process should be to assign triplex order parameters [6, 7, 8, 9] to further encode topological deformations and thereby define a complete "iso-dual tesseract wave-function". From there, we may continue to launch from this platform to explore this frontier along various trajectories and assess the application of geno-mathematics and hyper-mathematics [1, 2, 3, 4, 5]. Thus, this developing iso-geometrical framework warrants further development, scrutiny, collaboration, and hard work in order to advance it for future application in the discipline of science.

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