A continued fraction expression for Carnot efficiency

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Abstract. . A new and interesting continued fraction expression is derived for Carnot efficiency. The derivation is based on a series combination of a set of Carnot heat engines, wherein heat rejected by a member of the series is absorbed by the following member of the series. Our analysis of this combination of Carnot heat engines shows that mathematical consistency is maintained only if the efficiency of Carnot heat engine is zero. This calls the attention of researchers to look back at the puzzling definition of Carnot efficiency that says the efficiency of ideal heat engine (Carnot heat engine) is less than one, in spite of the fact that each of the steps involved in the cycle of operation is hundred percent efficient.

Keywords: Carnot efficiency, Thermal efficiency, Continued fraction expression

1. Introduction:

Carnot heat engine (CHE) is a device, which transforms heat into mechanical energy. It is an ideal engine and operates in a cycle consisting of four reversible steps that are described below.

1. An isothermal expansion step AB, during which the system absorbs Q_1 units of heat from a heat reservoir (HR) at temperature T_1 K and delivers work of W_1 (= Q_1) units. 2. An adiabatic expansion step BC, during which no heat interaction occurs between the system and the surroundings, but the system delivers W_2 units of work expending its internal energy. 3. An isothermal compression step CD, during which the system absorbs work of W_3 (= Q_2) units and rejects Q_2 units of heat to a HR at temperature T_2 K ($T_2 < T_1$). 4. An adiabatic compression step DA, during which no heat interaction occurs between the system and the surroundings, but the system absorbs W_4 (= W_2) units work to replenish its internal energy that was expended in step 2.

At the end of step 4, the system completes a cycle and returns to its original state suffering no change. The only changes left at the end of the cycle are those suffered by the surroundings - the changes of heat suffered by the two HRs and the upward displacement of a standard mass in the gravitational field, which represents the net work, $W = (W_1 - W_3)$. According to the first law of thermodynamics, $(W_1 - W_3) = (Q_1 - Q_2)$. Standard books that deal with thermodynamics [1-3] give a description of this well known Carnot cycle.

2. Carnot efficiency

The thermal efficiency of Carnot heat engine (Carnot efficiency) η, is defined as the ratio of work delivered W, to the quantity of heat Q_1, absorbed from the high temperature HR. Thus,

$$\eta = \left(\frac{W}{Q_1}\right) = \left[\frac{Q_1 - Q_2}{Q_1}\right] < 1 \qquad (1)$$

Carnot engines can be combined in series and parallel to form networks. Such combinations are used in many contexts in thermodynamics. For example, such combinations are used to explain Kelvin's thermodynamic heating of buildings [4], the development of the absolute scale or the thermodynamic scale of temperature [5], to prove Clausius theorem [1].

3. Derivation of the continued fraction expression for Carnot efficiency

Consider a set of Carnot heat engines connected in series such that the heat rejected by one is absorbed by the next in the series and each engine produces the same amount of work per cycle.

Using subscripts i, i+1 for an arbitrary Carnot engine in this series that interacts with heat reservoirs at absolute temperatures T_i, T_{i+1} ($T_i > T_{i+1}$) we can write,

$$\eta_{i,i+1} = \left[\frac{Q_i - Q_{i+1}}{Q_i}\right] = \left[1 - \frac{Q_{i+1}}{Q_i}\right] < 1 \tag{2}$$

Applying it for the series combination of engines producing equal works, we get,

$$\eta_{i,i+1} = \left[1 - \frac{Q_{i+1}}{Q_i}\right] = \left[1 - \frac{Q_{i+1}}{Q_{i+1} + W}\right] = \left[1 - \frac{1}{1 + (W/Q_{i+1})}\right] \tag{3}$$

But $(W/Q_{i+1}) = \eta_{i+1,i+2}$ $\therefore$ we get,

$$\eta_{i,i+1} = \left[1 - \frac{1}{1 + (\eta_{i+1,i+2})}\right] \tag{4}$$

$$\eta_{i,i+1} = \left[1 - \frac{1}{1 + \left\{1 - \frac{1}{1 + (\eta_{i+2,i+3})}\right\}}\right] \tag{5}$$

$$= \left[1 - \frac{1}{2 - \frac{1}{1 + (\eta_{i+2,i+3})}}\right] \tag{6}$$

$$= 1 - \frac{1}{2 - \frac{1}{2 - \frac{1}{1 + \eta_{i+3,i+4}}}} \tag{7}$$

$$= 1 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots \, 2 - \cfrac{1}{1 + \eta_{j,(j+1)}}}}}} \tag{8}$$

This continued fraction expression is new, and is an interesting expression for Carnot efficiency. Adding 1 to both sides of equation (8) gives,

$$(1 + \eta_{i,i+1}) = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots \, 2 - \cfrac{1}{(1 + \eta_{j,(j+1)})}}}}} \tag{9}$$

We can write the following continued fraction expression for zero and compare it with equation (8) to see that $\eta_{i,\,i+1} = 0$.

$$0 = 1 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots \, 2 - \cfrac{1}{1 + 0}}}}} \tag{10}$$

Similarly, we can write the following continued fraction expression for 1and compare it with equation (9) to see again that $\eta_{i,\,i+1} = 0$.

$$1 = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots \, 2 - \frac{1}{\ddots}}}}} \qquad (11)$$

Thus, it is clear that η must be zero.

4. Discussion

Since η is the efficiency of an arbitrarily chosen member of the series, it follows that every Carnot engine in the series must have zero efficiency. Again since any given Carnot heat engine cycle can be considered as a series combination of Carnot heat engines each producing equal amounts of work per cycle, it follows that any given Carnot heat engine must have zero efficiency, in order that the definition of efficiency is mathematically consistent. Zero efficiency in turn demands that the temperatures of the two HRs with which the Carnot heat engine interacts must be equal. The cycle then becomes an isothermal reversible cycle that produces zero work out put as is to be expected. Thus, if η is less than one it must be zero. The only other alternative is η = 1. This calls the attention of researchers to look back at the puzzling definition of Carnot efficiency that says the efficiency of an ideal heat engine (Carnot heat engine) is less than one, in spite of the fact that each of the steps involved in the cycle of operation is hundred percent efficient.

Prof. R Vittal Rao of Mathematics department of Indian Institute of Science, Bangalore, India, who reviewed the manuscript, gave a simple deduction of the result, given in the appendix.

6. Appendix

Let X be the infinite continued fraction

$$X = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots}}}} \qquad (A1)$$

Let $X_1, X_2, \ldots\ldots, X_n, \ldots$ be the convergents defined as

$$X_1 = 2 \qquad (A2)$$

$$X_2 = 2 - \frac{1}{2} = 2 - \frac{1}{X_1} \qquad (A3)$$

$$X_3 = 2 - \frac{1}{2-\frac{1}{2}} \qquad = 2 - \frac{1}{X_2} \tag{A4}$$

Continuing this we get $$X_n = 2 - \frac{1}{X_{n-1}} \tag{A5}$$

We easily see that $$X_1 = 2 = \frac{2}{1} \tag{A6}$$

$$X_2 = 2 - \frac{1}{X_1} = 2 - \frac{1}{2} = \frac{3}{2} \tag{A7}$$

Using the fact that $$X_n = 2 - \frac{1}{X_{n-1}} \tag{A8}$$

We get by mathematical induction that $$X_n = \frac{n+1}{n} \tag{A9}$$

Hence $$X = \lim_{n\to\infty} X_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \tag{A10}$$

Thus the infinite continued fraction X has the value 1. We can also see this intuitively as follows:

We can write $$X_2 = 2 - \frac{1}{X} \tag{A11}$$

Hence we get $$X^2 - 2X + 1 = (X-1)^2 = 0 \tag{A12}$$

Therefore, $X = 1$.

Consequently we get

$$1 - \frac{1}{2-\frac{1}{2-\cdots}} = -1 + \left\{2 - \frac{1}{2-\frac{1}{2-\cdots}}\right\} \tag{A13}$$

$$= -1 + 1 = 0 \tag{A14}$$

Confirming that η is equal to zero.

5. Acknowledgement

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7. References

[1] M. W. Zemansky and H. C. Van Ness, Basic engineering thermodynamics McGraw-Hill, New York, 1966) Ch. 8, p. 9.

[2] G. W. Castellan, Physical chemistry, 3rd edn., (Narosa, New Delhi 1985) p. 153.

[3] E. Fermi, Thermodynamics, (Dover, New York, 1936) p. 32.

[4] F. H. Crawford, Heat, thermodynamics and statistical Physics (Harcourt, Brace & World Inc., New York, 1963) p. 217.

[5] Jaime Wisniak, J. Chem. Educ., **77,** 518 (2000).