The complete classification of self-similar solutions of the Navier-Stokes equations for incompressible flow

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A new classification of self-similar solutions of the Navier-Stokes system of equations is presented here. We consider equations of motion for incompressible flow (of Newtonian fluids) in the curl rotating co-ordinate system. Then the equation of momentum should be split into the sub-system of 2 equations: an irrotational (*curl-free*) one, and a solenoidal (*divergence-free*) one.

The irrotational (*curl-free*) equation used for obtaining of the components of pressure gradient ∇p. As a term of such an equation, we used the irrotational (*curl-free*) *vector field of flow velocity*, which is given by the proper potential (*besides, the continuity equation determines such a potential as a harmonic function*).

As for solenoidal (*divergence-free*) equation, the transition from Cartesian to curl rotating co-ordinate system transforms equation of motion to the *Helmholtz* vector differential equation for time-dependent self-similar solutions. The *Helmholtz* differential equation can be solved by separation of variables in only 11 coordinate systems, so it forms a complete set of all possible cases of self-similar solutions for Navier-Stokes system of equations.

Keywords: Navier-Stokes equations, self-similar solutions, incompressible flow.

1. Introduction, the Navier-Stokes system of equations.

In accordance with [1-2], the Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below:

$$\nabla \cdot \vec{u} = 0 , \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F} , \tag{1.2}$$

- where ***u*** is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and ***F*** represents body forces (*per unit of mass in a volume*) acting on the fluid and ∇ is the del (nabla) operator. Let us also choose the Oz axis coincides to the main direction of flow propagation.

The system of Navier-Stokes equations is known to be the system of the mixed parabolic and hyperbolic type [3].

2. The curl rotating co-ordinate system.

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$, and then using the curl of the curl identity $\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$, we could present the equation (1.2) in the case of incompressible flow of Newtonian fluids as below:

$$\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{w} + \nu \cdot \nabla^2 \vec{u} - \left(\frac{1}{2}\nabla(u^2) + \frac{\nabla p}{\rho} - \vec{F} \right) \tag{2.1}$$

- here we denote *the curl field* ***w***, a pseudovector field (*time-dependent*).

Let us consider equation (2.1) in the *curl* rotating co-ordinate system by adding of the proper *Coriolis force* to the equation of motion (2.1) as below

$$\frac{\partial \vec{u}}{\partial t} - 2\vec{\Omega}\times\vec{u} = \vec{u}\times\vec{w} + \nu\cdot\nabla^2\vec{u} - \left(\frac{1}{2}\nabla(u^2) + \frac{\nabla p}{\rho} - \vec{F}\right) \qquad (2.2)$$

- where $\mathbf{\Omega}$ - is the angular velocity of curl rotation in the vicinity of initial Cartesian system of co-ordinates. Thermodynamic variables and the net viscous stress are independent of the reference frame; velocity $\boldsymbol{u}$ in the *curl* rotating co-ordinate system coincides with the velocity of flow in previous co-ordinates system (*if vicinity of vortex rotation is negligible*).

Besides, the equality below is valid for the angular velocity of curl rotation in the case of Newtonian fluids [2]:

$$\mathbf{\Omega} = (\nabla\times \boldsymbol{u})/2$$

So, from the equation (2.2) we obtain

$$\frac{\partial \vec{u}}{\partial t} = \nu\cdot\nabla^2\vec{u} - \left(\frac{1}{2}\nabla(u^2) + \frac{\nabla p}{\rho} - \vec{F}\right) \qquad (2.3)$$

Let us denote as below (*according to the Helmholtz fundamental theorem of vector calculus*):

$$\nabla\times\vec{u} \equiv \vec{w}, \qquad \vec{u} \equiv \vec{u}_p + \vec{u}_w,$$

$$\nabla\cdot\vec{u}_w \equiv 0, \qquad \nabla\times(\vec{u}_p) \equiv 0,$$

- where $\boldsymbol{u}_p$ is *an irrotational* (*curl-free*) field of flow velocity, and $\boldsymbol{u}_w$ - is *a solenoidal* (*divergence-free*) field of flow velocity which generates a curl field $\boldsymbol{w}$:

$$\vec{u}_p \equiv \nabla \varphi , \qquad \vec{u}_w \equiv \nabla \times \vec{A} ,$$

- here φ - is the proper scalar potential, $\boldsymbol{A}$ – is the appropriate vector potential.
Thus, we could obtain from the equation (1.1) the equality below

$$\nabla \cdot (\nabla \varphi + \nabla \times \vec{A}) = 0 , \quad \Rightarrow \quad \Delta \varphi = 0 , \tag{2.4}$$

- it means that φ - is the proper *harmonic function* [3].

Thus, equation (2.3) could be presented as the system of equations below:

$$\begin{cases} \dfrac{\partial (\nabla \varphi)}{\partial t} = \vec{F} - \dfrac{1}{2} \nabla \{ (\nabla \varphi + \vec{u}_w)^2 \} - \dfrac{\nabla p}{\rho} , \\ \\ \dfrac{\partial \vec{u}_w}{\partial t} = \nu \cdot \nabla^2 \vec{u}_w , \end{cases} \tag{2.5}$$

- so, if we solve the second equation of (2.5) for the components of vector $\boldsymbol{u}_w$, we could substitute it into the 1-st equation of (2.5) for obtaining of a proper expression for vector function ∇p:

$$\frac{\nabla p}{\rho} = \vec{F} - \frac{\partial (\nabla \varphi)}{\partial t} - \frac{1}{2} \nabla \{ (\nabla \varphi + \vec{u}_w)^2 \} , \tag{2.6}$$

- where φ - is the proper *harmonic function*, see Eq. (2.4).

The system of equations (1.1)-(2.5) is equivalent to the Navier-Stokes system of equations for incompressible Newtonian fluids (1.1)-(1.2) in the sense of existence and smoothness of a general solution.

The inverse transformation of exact solutions from the curl rotating system to the Cartesian coordinate system is possible only in case Ω = const.

3. <u>Classification of exact solutions for Navier-Stokes Eq.</u>

For non-stationary solutions $\partial/\partial t \neq 0$, the 2-nd of Eq. (2.5) could be solved analytically only in the cases below:

1) $\partial/\partial t \sim \partial/\partial z$ - it means that the Oz axis represents a preferential direction similar to the time arrow in mechanical processes [4];

2) Time-dependent self-similar case, $\boldsymbol{u}_w = \exp(-\omega t)\cdot\boldsymbol{u}_w\,(x, y, z)$, ω = const > 0 (*frequency-parameter*).

For the time-dependent self-similar case, 2-nd of Eq. (2.5) should be presented as

$$\nabla^2 \vec{u}_w + \left(\frac{\omega}{\nu}\right)\vec{u}_w = 0\,, \qquad (3.1)$$

- which is the proper *Helmholtz* differential equation for vector fields $\boldsymbol{u}_w$ [2].

The *Helmholtz* differential equation can be solved by separation of variables in only 11 coordinate systems, 10 of which (*with the exception of confocal paraboloidal coordinates*) are particular cases of the confocal ellipsoidal system: Cartesian, confocal ellipsoidal, confocal paraboloidal, conical, cylindrical, elliptic cylindrical, oblate spheroidal, paraboloidal, parabolic cylindrical, prolate spheroidal, and spherical coordinates [5-6].

Thus, above 11 classes form a complete set of all possible cases of self-similar solutions for Navier-Stokes system of equations.

4. <u>Conclusion.</u>

A new classification of self-similar solutions of the Navier-Stokes system of equations is presented here. We consider equations of motion for incompressible flow (of Newtonian fluids) in the curl rotating co-ordinate system. Then the equation of momentum should be split into the sub-system of 2 equations: an irrotational (*curl-free*) one, and a solenoidal (*divergence-free*) one.

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5. Discussions.

The main result, which should be outlined, is that the initial system of Navier-Stokes equations could be reduced to the equivalent system of equations where for time-dependent self-similar solutions the key equation should be the *Helmholtz* vector differential Eq. For such a reduction we should change Cartesian to the curl rotating co-ordinate system.

The *Helmholtz* differential equation could be solved by separation of variables in only 11 coordinate systems; so, we have classified all the self-similar solutions.

Also we should note that for *Helmholtz* vector differential equation it was proved the existence and smoothness of a general solution.

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