Clifford Fourier Transform on Multivector Fields and Uncertainty Principles for Dimensions $n = 2 \pmod 4$ and $n = 3 \pmod 4$

Eckhard M. S. Hitzer, Bahri Mawardi

ABSTRACT First, the basic concepts of the multivector functions, vector differential and vector derivative in geometric algebra are introduced. Second, we define a generalized real Fourier transform on Clifford multivector-valued functions ($f : \mathbb{R}^n \to Cl_{n,0},\ n = 2, 3 \pmod 4$). Third, we show a set of important properties of the Clifford Fourier transform on $Cl_{n,0},\ n = 2, 3 \pmod 4$ such as differentiation properties, and the Plancherel theorem, independent of special commutation properties. Fourth, we develop and utilize commutation properties for giving explicit formulas for $f\boldsymbol{x}^m$, $f\nabla^m$ and for the Clifford convolution. Finally, we apply Clifford Fourier transform properties for proving an uncertainty principle for $Cl_{n,0},\ n = 2, 3 \pmod 4$ multivector functions.
Keywords: Vector derivative, multivector-valued function, Clifford (geometric) algebra, Clifford Fourier transform, uncertainty principle.

1 Introduction

In applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for image processing and signal analysis where the image or signal from the original domain is transformed to the (spectral or) frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in (real) geometric algebra.

Brackx et al. [1] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied

AMS Subject Classification: 15A66, 43A32.

approach for hypercomplex Clifford Fourier Transformations in $Cl_{0,n}$ was followed by Bülow et. al. [2]. In [3], Li et. al. extended the Fourier Transform holomorphically to a function of m complex variables.

In this paper we adopt and expand[1] to $\mathcal{G}_n,\ n = 2,3\,(\mathrm{mod}\,4)$ the generalization of the Fourier transform in Clifford geometric algebra $\mathcal{G}_3$ recently suggested by Ebling and Scheuermann [6], based on [11]. To avoid ambiguities we recall that

$$\begin{aligned} n = 2\,(\mathrm{mod}\,4) \;\Leftrightarrow\; n = 2+4k,\ k \in \mathbb{N}, \\ n = 3\,(\mathrm{mod}\,4) \;\Leftrightarrow\; n = 3+4l,\ l \in \mathbb{N}. \end{aligned} \tag{1.1}$$

We explicitly show detailed properties of the real[2] Clifford geometric algebra Fourier transform (CFT). As an application we subsequently use some of these properties to define and prove the uncertainty principle for $\mathcal{G}_n$ multivector functions.

2 Clifford's Geometric Algebra $\mathcal{G}_n$ of $\mathbb{R}^n$

Let us consider now and in the following an orthonormal vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \ \boldsymbol{e}_n\}$ of the real n-dimensional Euclidean vector space $\mathbb{R}^n$ with $n = 2,3\,(\mathrm{mod}\,4)$. The geometric algebra over $\mathbb{R}^n$ denoted by $\mathcal{G}_n$ then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \ldots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_n\}. \tag{2.1}$$

Remark 2.1. The fact that we begin by introducing orthonormal bases for both the vector space $\mathbb{R}^n$ and for its associated geometric algebra $\mathcal{G}_n$ is only because we assume readers to be familiar with these concepts. As is well-known, the definitions of vector spaces and geometric algebras are generically basis independent [7]. The definition of the vector derivative of section 3 is basis independent, too. Only when we introduce the infinitesimal scalar volume element for integration over $\mathbb{R}^n$ in section 4 and in the proof of the last theorem 5.5 in [4] do we use a basis explicitly. In the latter case it may well be possible to formulate a basis independent proof. All results derived in this paper are therefore *manifestly invariant* (independent of the use of coordinate systems), apart from the proof of theorem 5.5.

---

[1] In the following we mean with $n = 2,3\,(\mathrm{mod}\,4)$ that $n = 2\,(\mathrm{mod}\,4)$ or $n = 3\,(\mathrm{mod}\,4)$, i.e. with (1.1) that $n \in \{2,3,6,7,10,11,\ldots\}$. For further details and proofs in the case of $n = 3$ compare [4]. In the geometric algebra literature [7] instead of the mathematical notation $Cl_{p,q}$ the notation $\mathcal{G}_{p,q}$ is widely in use. It is convention to abbreviate $\mathcal{G}_{n,0}$ to $\mathcal{G}_n$.

[2] The meaning of *real* in this context is, that we use the oriented n-dimensional unit volume element i_n of the geometric algebra $\mathcal{G}_n$ over the field of the reals $\mathbb{R}$ to construct the kernel of the Clifford Fourier transformation of definition 4.1. This i_n has a clear geometric interpretation, e.g. as n-dimensional hypercube with side length one, and in an orthonormal basis of $\mathbb{R}^n$ it can be factorized as $i_n = \boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_n$.

The squares of vectors are positive definite scalars (Euclidean metric) and so are all even powers of vectors

$$\boldsymbol{x}^2 \geq 0, \quad \boldsymbol{x}^m \geq 0 \quad \text{for} \quad m = 2m', m' \in \mathbb{N}. \tag{2.2}$$

Therefore given a multivector $M \in \mathcal{G}_n$

$$\boldsymbol{x}^m M = M\, \boldsymbol{x}^m, \;\; m = 2m', m' \in \mathbb{N}. \tag{2.3}$$

Note that for $n = 2, 3 \,(\mathrm{mod}\, 4)$

$$i_n^2 = -1, \quad i_n^{-1} = -i_n, \quad i_n^m = (-1)^{\frac{m}{2}} \quad \text{for} \quad m = 2m', m' \in \mathbb{Z}, \tag{2.4}$$

similar to the complex imaginary unit.

The *grade selector* is defined as $\langle M \rangle_k$ for the k-vector part of M, especially $\langle M \rangle = \langle M \rangle_0$. Then M can be expressed as the sum of all its grade parts

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n. \tag{2.5}$$

The *reverse* of M is defined by the anti-automorphism

$$\widetilde{M} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle M \rangle_k. \tag{2.6}$$

The *square norm* of M is defined by

$$\|M\|^2 = \langle M\widetilde{M} \rangle, \tag{2.7}$$

where

$$\langle M\widetilde{N} \rangle = M * \widetilde{N} \tag{2.8}$$

is a real valued (inner and) *scalar product* for any M, N in $\mathcal{G}_n$. As a consequence we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N} \rangle|^2 \;\leq\; \|M\|^2 \,\|N\|^2 \quad \forall\, M, N \in \mathcal{G}_n. \tag{2.9}$$

3 Multivector Functions, Vector Differential and Vector Derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathcal{G}_n$. For an arbitrary vector $\boldsymbol{a} \in \mathbb{R}^n$ we define[3] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{3.1}$$

provided this limit exists and is well defined.

[3]Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_{p,q}$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 3.6).

Remark 3.1. $\boldsymbol{a}\cdot\nabla$ is a scalar operator, therefore the left and right vector differentials[4] agree, i.e.

$$\boldsymbol{a}\cdot\dot{\nabla}\dot{f}(\boldsymbol{x}) = \dot{f}(\boldsymbol{x})\ \boldsymbol{a}\cdot\dot{\nabla}. \tag{3.2}$$

The basis independent *vector derivative* ∇ is defined in [7, 8] to have the algebraic properties of a grade one vector in $\mathbb{R}^n$ and to obey equation (3.1) for all vectors $\boldsymbol{a}\in\mathbb{R}^n$. This allows the following explicit representation.

Remark 3.2. The vector derivative ∇ can be expanded in a basis of $\mathbb{R}^n$ as

$$\nabla = \sum_{k=1}^{n} \boldsymbol{e}_k\partial_k \quad \text{with} \quad \partial_k = \frac{\partial}{\partial x_k},\ \ 1\le k\le n. \tag{3.3}$$

Example 3.3. Here we give a set of multivector functions $f:\mathbb{R}^6\to\mathcal{G}_6$, their vector differentials and vector derivatives [8]. We assume that $\boldsymbol{e}_{1256} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_5\boldsymbol{e}_6$, constant $\boldsymbol{x}_0\in\mathbb{R}^6$, $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}_0$, $r=\|\boldsymbol{r}\|$, and $\boldsymbol{r}^{-1}=\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|^2}$.

$$f_1=\boldsymbol{x},\quad \boldsymbol{a}\cdot\nabla f_1=\boldsymbol{a},\quad \nabla f_1 = 6, \tag{3.4}$$

$$f_2=\boldsymbol{x}^2,\quad \boldsymbol{a}\cdot\nabla f_2=2\boldsymbol{a}\cdot\boldsymbol{x},\quad \nabla f_2 = 2\boldsymbol{x}, \tag{3.5}$$

$$f_3=\|\boldsymbol{x}\|,\quad \boldsymbol{a}\cdot\nabla f_3=\boldsymbol{a}\cdot\boldsymbol{x}/\|\boldsymbol{x}\|,\quad \nabla f_3 = \boldsymbol{x}/\|\boldsymbol{x}\|, \tag{3.6}$$

$$f_4=\boldsymbol{x}\cdot\boldsymbol{e}_{1256},\quad \boldsymbol{a}\cdot\nabla f_4=\boldsymbol{a}\cdot\boldsymbol{e}_{1256},\quad \nabla f_4 = 4\boldsymbol{e}_{1256}, \tag{3.7}$$

$$f_5=\log r,\quad \boldsymbol{a}\cdot\nabla f_5=\boldsymbol{a}\cdot\boldsymbol{r}^{-1},\quad \nabla f_5 = \boldsymbol{r}^{-1}. \tag{3.8}$$

Proposition 3.4 (Left and right linearity)**.**

$$\nabla(f+g)=\nabla f+\nabla g,\quad (f+g)\nabla = f\nabla + g\nabla. \tag{3.9}$$

Proposition 3.5. *For* $f(\boldsymbol{x})=g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x})\in\mathbb{R}$,

$$\boldsymbol{a}\cdot\nabla f = f\ \boldsymbol{a}\cdot\nabla = \{\boldsymbol{a}\cdot\nabla\lambda(\boldsymbol{x})\}\frac{\partial g}{\partial\lambda}. \tag{3.10}$$

Proposition 3.6 (Left and right derivative from differential)**.**

$$\nabla f = \nabla_{\boldsymbol{a}}(\boldsymbol{a}\cdot\nabla f),\quad f\nabla = (\boldsymbol{a}\cdot\nabla f)\nabla_{\boldsymbol{a}}. \tag{3.11}$$

Proposition 3.7 (Left and right product rules)**.**

$$\nabla(fg) = (\dot{\nabla}\dot{f})g + \dot{\nabla}f\dot{g} = (\dot{\nabla}\dot{f})g + \nabla_{\boldsymbol{a}}\, f(\boldsymbol{a}\cdot\nabla g). \tag{3.12}$$

$$(fg)\nabla = f(\dot{g}\dot{\nabla}) + \dot{f}g\dot{\nabla} = f(\dot{g}\dot{\nabla}) + (\boldsymbol{a}\cdot\nabla f)\,g\,\nabla_{\boldsymbol{a}}. \tag{3.13}$$

Note that the multivector functions f *and* g *in (3.12) and (3.13) do not necessarily commute.*

[4]The point symbols specify on which function the vector derivative is supposed to act.

Example 3.8. For two functions $f, g : \mathbb{R}^2 \to \mathcal{G}_2$, $f = \boldsymbol{x}, g = \boldsymbol{x} \cdot \boldsymbol{e}_{12}$ we calculate [8]

$$\begin{aligned}(fg)\nabla &= f(\dot{g}\dot{\nabla}) + (\boldsymbol{a}\cdot\nabla f)\, g\, \nabla_{\boldsymbol{a}} \\ &= \boldsymbol{x}[(\boldsymbol{x}\cdot\boldsymbol{e}_{12})\nabla] + (\boldsymbol{a}\cdot\nabla\boldsymbol{x})(\boldsymbol{x}\cdot\boldsymbol{e}_{12})\nabla_{\boldsymbol{a}} = \boldsymbol{x}(-2\boldsymbol{e}_{12}) + \boldsymbol{a}(\boldsymbol{x}\cdot\boldsymbol{e}_{12})\nabla_{\boldsymbol{a}} \\ &= -2\boldsymbol{x}\boldsymbol{e}_{12} + \boldsymbol{e}_1(\boldsymbol{x}\cdot\boldsymbol{e}_{12})\boldsymbol{e}_1 + \boldsymbol{e}_2(\boldsymbol{x}\cdot\boldsymbol{e}_{12})\boldsymbol{e}_2 = -2\boldsymbol{x}\boldsymbol{e}_{12}. \end{aligned} \tag{3.14}$$

Differentiating twice with the vector derivative, we get the differential Laplacian operator ∇^2. We can write $\nabla^2 = \nabla\cdot\nabla + \nabla\wedge\nabla$. But for integrable functions $\nabla\wedge\nabla = 0$. In this case we have $\nabla^2 = \nabla\cdot\nabla$. Because ∇^2 is a scalar operator, the left and right Laplace derivatives agree, i.e. $\nabla^2 f = f\,\nabla^2$. More generally all even powers of the left and right vector derivative agree

$$\nabla^m f = f\,\nabla^m \quad \text{for} \quad m = 2m', m' \in \mathbb{N}. \tag{3.15}$$

Proposition 3.9 (Integration of parts)**.**

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})[\boldsymbol{a}\cdot\nabla g(\boldsymbol{x})]\, d^n\boldsymbol{x} = \\ \left[\int_{\mathbb{R}^{n-1}} f(\boldsymbol{x})g(\boldsymbol{x})\, d^{n-1}\boldsymbol{x}\right]_{a\cdot x=-\infty}^{a\cdot x=\infty} - \int_{\mathbb{R}^n} [\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})]g(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{3.16}$$

Remark 3.10. Proposition 3.9 reduces to the familiar coordinate form, if we insert for $\boldsymbol{a}$ the grade 1 basis vectors $\boldsymbol{e}_k, 1 \le k \le n$ of (2.1), because

$$\boldsymbol{e}_k\cdot\nabla = \partial_k \quad \text{and} \quad \boldsymbol{e}_k\cdot\boldsymbol{x} = x_k. \tag{3.17}$$

We also note that because of (2.3) even powers of vectors commute with multivector valued functions $f \in L^2(\mathbb{R}^n, \mathcal{G}_n)$

$$\boldsymbol{x}^m f = f\,\boldsymbol{x}^m \quad \text{for} \quad m = 2m', m' \in \mathbb{N}. \tag{3.18}$$

Theorem 3.11. *For all geometric algebras* $\mathcal{G}_n$, $n \in \mathbb{N}$ *we have for* $f : \mathbb{R}^n \to \mathcal{G}_n)$, $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$, *and* $\lambda = \pm\boldsymbol{\omega}\cdot\boldsymbol{x}$

$$\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}} f(\boldsymbol{x})(\pm i_n^{-1})e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} = \boldsymbol{a}\cdot\boldsymbol{x}\, f(\boldsymbol{x})e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{3.19}$$

For $f(\boldsymbol{x}) = 1$ *we get*

$$\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}(\pm i_n^{-1})\, e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} = \boldsymbol{a}\cdot\boldsymbol{x}\; e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{3.20}$$

Proof.

$$\begin{aligned}\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}} f(\boldsymbol{x})(\pm i_n^{-1})e^{\pm i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} &= f(\boldsymbol{x})(\pm i_n^{-1})\; \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}} e^{\pm i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} \\ = f(\boldsymbol{x})(\pm i_n^{-1})\; \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}(\pm\boldsymbol{\omega}\cdot\boldsymbol{x})\, \frac{\partial e^{i_n\lambda}}{\partial\lambda} &= f(\boldsymbol{x})\, i_n^{-1}\; \boldsymbol{a}\cdot\boldsymbol{x}\, i_n\, e^{i_n\lambda} \\ = \boldsymbol{a}\cdot\boldsymbol{x} f(\boldsymbol{x})\, i_n^{-1}\, i_n\, e^{i_n\lambda} &= \boldsymbol{a}\cdot\boldsymbol{x} f(\boldsymbol{x})\, e^{\pm i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\ . \end{aligned} \tag{3.21}$$

For the second equality we used proposition 3.5, and for the third equality proposition 21 of [8]. ☐

Example 3.12. Functions $f : \mathbb{R}^6 \to \mathcal{G}_6$, like $\sin(\boldsymbol{x})$ and $e^{i_6 \boldsymbol{\omega} \cdot \boldsymbol{x}}$ with $i_6 = \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_3 \boldsymbol{e}_4 \boldsymbol{e}_5 \boldsymbol{e}_6$, $i_6^{-1} = \tilde{i}_6 = -i_6$ can be defined by power series. An example for (3.19) is therefore

$$\boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}} \sin(\boldsymbol{x})(-i_6)e^{i_6 \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{x} \sin(\boldsymbol{x})e^{i_6 \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{3.22}$$

By exchanging the roles of $\boldsymbol{x}$ and $\boldsymbol{\omega}$ in (3.20) we obtain

Corollary 3.13.

$$\boldsymbol{a} \cdot \nabla(\pm i_n^{-1})\, e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{\omega}\; e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{3.23}$$

Applying proposition 3.6 to corollary 3.13 and multiplying both sides with $\pm i_n$ from the right we get

Corollary 3.14.

$$\nabla e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{\omega}\, (\pm i_n)\, e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}, \tag{3.24}$$

and its reverse

Corollary 3.15.

$$e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, \nabla = e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, (\pm i_n)\, \boldsymbol{\omega}. \tag{3.25}$$

Theorem 3.16. *For all geometric algebras* $\mathcal{G}_n$, $n \in \mathbb{N}$ *we have for* $f : \mathbb{R}^n \to \mathcal{G}_n)$, $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$, *and* $\lambda = \pm \boldsymbol{\omega} \cdot \boldsymbol{x}$

$$\nabla_{\boldsymbol{\omega}} f(\boldsymbol{x})(\pm i_n^{-1})e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{x} f(\boldsymbol{x})e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{3.26}$$

For $f(\boldsymbol{x}) = 1$ *we get*

$$\nabla_{\boldsymbol{\omega}} (\pm i_n^{-1})e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{x}\; e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{3.27}$$

Proof. We first proof (3.26).

$$\begin{aligned}\nabla_{\boldsymbol{\omega}} f(\boldsymbol{x})(\pm i_n^{-1})e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} &= \nabla_{\boldsymbol{a}}[\boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}} f(\boldsymbol{x})(\pm i_n^{-1})e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}] \\ &= \nabla_{\boldsymbol{a}}(\boldsymbol{a} \cdot \boldsymbol{x}) f(\boldsymbol{x})\, e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} = \boldsymbol{x} f(\boldsymbol{x})\, e^{\pm i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}.\end{aligned} \tag{3.28}$$

For the first equality we used proposition 3.6, for the second equality theorem 3.11, and for the third equality proposition 72 of [8]. □

Example 3.17. Similar to example 3.12 functions $f : \mathbb{R}^7 \to \mathcal{G}_7$, like $\cos(\boldsymbol{x})$ and $e^{i_7 \boldsymbol{\omega} \cdot \boldsymbol{x}}$ with $i_7 = \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_3 \boldsymbol{e}_4 \boldsymbol{e}_5 \boldsymbol{e}_6 \boldsymbol{e}_7$, $i_7^{-1} = \tilde{i}_7 = -i_7$ can be defined by power series. An example for (3.26) is therefore

$$\nabla_{\boldsymbol{\omega}} \cos(\boldsymbol{x})\, i_7\, e^{-i_7 \boldsymbol{\omega} \cdot \boldsymbol{x}} = \boldsymbol{x} \cos(\boldsymbol{x})e^{i_7 \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{3.29}$$

The reverse of (3.27) gives

Corollary 3.18.

$$e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} (\pm i_n^{-1}) \nabla_{\boldsymbol{\omega}} = e^{\pm i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \boldsymbol{x} . \tag{3.30}$$

Convolution is an important operation for smoothing images and for edge detection in image processing. The *Clifford Convolution* of multivector valued functions is defined for arbitrary n.

Definition 3.19 (Clifford Convolution)**.** The Clifford Convolution of $f, g \in L^2(\mathbb{R}^n, \mathcal{G}_n)$ is defined as

$$(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) d^n \boldsymbol{y} . \tag{3.31}$$

Example 3.20. As an example let us compute the convolution of two exponential functions $f, g : \mathbb{R}^2 \to \mathcal{G}_2$, $f(\boldsymbol{x}) = \boldsymbol{e}_2 \exp(-i_2 \boldsymbol{\omega} \cdot \boldsymbol{x})$, $g(\boldsymbol{x}) = 3 \boldsymbol{e}_1 \exp(i_2 \boldsymbol{\omega}' \cdot \boldsymbol{x})$ with $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^2$, $i_2 = \boldsymbol{e}_1 \boldsymbol{e}_2$, $i_2 \boldsymbol{e}_{1,2} = -\boldsymbol{e}_{1,2} i_2$,

$$\begin{aligned}(f \star g)(\boldsymbol{x}) &= \int_{\mathbb{R}^2} \boldsymbol{e}_2 \mathrm{e}^{-i_2 \boldsymbol{\omega} \cdot \boldsymbol{y}} 3 \boldsymbol{e}_1 \mathrm{e}^{i_2 \boldsymbol{\omega}' \cdot (\boldsymbol{x} - \boldsymbol{y})} d^2 \boldsymbol{y} \\ &= 3 \boldsymbol{e}_2 \boldsymbol{e}_1 \mathrm{e}^{i_2 \boldsymbol{\omega}' \cdot \boldsymbol{x}} \int_{\mathbb{R}^2} \mathrm{e}^{i_2 (\boldsymbol{\omega} - \boldsymbol{\omega}') \cdot \boldsymbol{y}} d^2 \boldsymbol{y} \\ &= -3 (2\pi)^2 i_2 \mathrm{e}^{i_2 \boldsymbol{\omega}' \cdot \boldsymbol{x}} \delta(\boldsymbol{\omega} - \boldsymbol{\omega}'). \end{aligned} \tag{3.32}$$

Exchanging the order of the functions we get

$$(g \star f)(\boldsymbol{x}) = 3 (2\pi)^2 i_2 \mathrm{e}^{-i_2 \boldsymbol{\omega}' \cdot \boldsymbol{x}} \delta(\boldsymbol{\omega} - \boldsymbol{\omega}'), \tag{3.33}$$

illustrating the general non-commutativity $(f \star g) \neq (g \star f)$ due to the geometric product.

Note that the following identity, which follows from the substitution of variables ($\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$), is valid for all dimensions n. Let $f, g \in L^2(\mathbb{R}^n, \mathcal{G}_n)$ then

$$\int_{\mathbb{R}^n} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) d^n \boldsymbol{y} = \int_{\mathbb{R}^n} f(\boldsymbol{z}) g(\boldsymbol{x} - \boldsymbol{z}) d^n \boldsymbol{z} . \tag{3.34}$$

Ebling and Scheuermann [6] distinguish between right and left convolution. They are right that products of multivector valued functions do not commute (compare example 3.20), so after e.g. a linear and shift-invariant (LSI) multivector filter is chosen it matters if a multivector image function is multiplied with the filter from the right or from the left. But because of (3.34) we only define one kind of convolution and leave it up to particular applications which factor is taken as multivector filter and which for the multivector image function, etc. The CFT formulas of the convolution which we derive for $n = 2, 3 \pmod 4$ are valid for whatever choice is made in applications.

4 Clifford Fourier Transform (CFT)

Definition 4.1. The Clifford Fourier transform[5] of $f(\boldsymbol{x})$ is the function $\mathcal{F}\{f\}\colon \mathbb{R}^n \to \mathcal{G}_n$, $n = 2, 3 \pmod 4$ given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}\,, \tag{4.1}$$

with $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$.

Note that

$$d^n\boldsymbol{x} = d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge \ldots \wedge d\boldsymbol{x_n}\; i_n^{-1} \tag{4.2}$$

is scalar valued ($d\boldsymbol{x_k} = dx_k\, \boldsymbol{e_k} \in \mathbb{R}^n$, $k = 1, 2, \ldots, n$, no summation). Because for $n = 3 \pmod 4$ i_n commutes with every element of $\mathcal{G}_n$, the Clifford Fourier kernel $e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ will also commute with every element of $\mathcal{G}_n$. This is not the case for $n = 2 \pmod 4$.

Example 4.2. We give an example for an integrable function $f : \mathbb{R}^n \to \mathcal{G}_n$, the n-dimensional $\mathrm{rect}(\boldsymbol{x})$ function, which can be given in terms of the real scalar $\mathrm{rect}(x)$ function with $x \in \mathbb{R}$ as

$$\mathrm{rect}(\boldsymbol{x}) = \prod_{k=1}^{n} \mathrm{rect}(x_k)\, \boldsymbol{e}_k = i_n \prod_{k=1}^{n} \mathrm{rect}(x_k). \tag{4.3}$$

The CFT of $\mathrm{rect}(\boldsymbol{x})$ gives

$$\begin{aligned}\mathcal{F}\{\mathrm{rect}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \mathrm{rect}(\boldsymbol{x}) e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\ = i_n \int_{\mathbb{R}^n} \prod_{k=1}^{n} \mathrm{rect}(x_k) e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} &= i_n \prod_{k=1}^{n} \mathrm{sinc}(\frac{\omega_k}{2\pi}).\end{aligned} \tag{4.4}$$

Theorem 4.3. *The Clifford Fourier transform* $\mathcal{F}\{f\}$ *of* $f \in L^2(\mathbb{R}^n, \mathcal{G}_n)$, $n = 2, 3 \pmod 4$, $\int_{\mathbb{R}^n} \|f\|^2 d^n\boldsymbol{x} < \infty$ *is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega}\,. \tag{4.5}$$

For a full proof of theorem 4.3 in dimension $n = 3$ that can be generalised straight forwardly to dimensions $n = 2, 3 \pmod 4$ see e.g. [4]. Though definition 4.1 and theorem 4.3 are the same for the dimensions $n = 2 \pmod 4$ and $n = 3 \pmod 4$, care has to be taken of the general non-commutativity of i_n for $n = 2 \pmod 4$. However it turns out, that many properties of the

[5] Compare e.g. [14], article 160 for the precise conditions on the existence of Fourier integrals.

CFT for $n = 2,3 \pmod 4$ can be expressed independent of the commutation properties of i_n, if sufficient care is taken to avoid commuting i_n with other multivectors (except scalars and powers of i_n itself). Exceptions are the CFT of the Clifford convolution, $f\boldsymbol{x}^m$ and $f\nabla^m$, which need to be studied dimension dependent.

We therefore continue with a general section on investigating properties of the CFT for $n = 2 \pmod 4$ and $n = 3 \pmod 4$ aiming at expressions that do not depend on the commutation properties of i_n. This will be followed by a section on the properties of the CFT of the Clifford convolution, $f\boldsymbol{x}^m$ and $f\nabla^m$. This second section will also include one table each for $n = 2 \pmod 4$ and $n = 3 \pmod 4$ that summarize all the properties of the CFT studied in this paper and fully utilize the commutation properties of i_n.

4.1 Properties of the CFT for $n = 2,3 \pmod 4$ expressed independent of i_n commutations

The properties of the CFT we will treat now are linearity, scaling, delay, shift, transformations of powers of the vector differential, of left and right powers of the vector derivative, of the vector variable $\boldsymbol{x} \in \mathbb{R}^n$, and finally the Plancherel and Parseval theorems. The unique feature of our study is the independence of theorem formulations and proofs on the commutation properties of the pseudoscalars i_n in dimensions $n = 2,3 \pmod 4$. If not otherwise stated, n is assumed to be $n = 2,3 \pmod 4$ in the remainder of this section.

Theorem 4.4 (Left linearity)**.** *For $f(\boldsymbol{x}) = \alpha f_1(\boldsymbol{x}) + \beta f_2(\boldsymbol{x})$ with constants $\alpha, \beta \in \mathcal{G}_n$, and functions $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathcal{G}_n$ we have*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \alpha\mathcal{F}\{f_1\}(\boldsymbol{\omega}) + \beta\mathcal{F}\{f_2\}(\boldsymbol{\omega}). \tag{4.6}$$

Proof. Follows from the linearity of the geometric product and the integration involved in the definition 4.1 of the CFT. □

Remark 4.5. Restricting the constants in theorem 4.4 to $\alpha, \beta \in \mathbb{R}$ we get both left and right linearity of the CFT.

Theorem 4.6 (Scaling)**.** *Let $a \in \mathbb{R}, a \neq 0$ be a scalar constant, then the Clifford Fourier transform of the function $f_a(\boldsymbol{x}) = f(a\boldsymbol{x})$ becomes*

$$\mathcal{F}\{f_a\}(\boldsymbol{\omega}) = \frac{1}{|a|^n}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}). \tag{4.7}$$

Proof. Follows from variable substitution $\boldsymbol{u} = a\boldsymbol{x}$. □

Theorem 4.7 (Shift in space domain, delay)**.** *If the argument of $f(\boldsymbol{x}) \in \mathcal{G}_n$ is offset by a constant vector $\boldsymbol{a} \in \mathbb{R}^n$, i.e. $f_d(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{a})$, then*

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{a}}. \tag{4.8}$$

Proof. Definition 4.1 gives

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x} - \boldsymbol{a}) e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \, d^n \boldsymbol{x}.$$

We substitute $\boldsymbol{t}$ for $\boldsymbol{x} - \boldsymbol{a}$ in the above expression, and get with $d^n \boldsymbol{x} = d^n \boldsymbol{t}$

$$\begin{aligned} \mathcal{F}\{f_d\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} f(\boldsymbol{t}) \, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{t}} e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{a}} \, d^n \boldsymbol{t} \\ &= \mathcal{F}\{f\}(\boldsymbol{\omega}) e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{a}}. \end{aligned} \tag{4.9}$$

This proves (4.8). □

Example 4.8. Using example 4.2 we can calculate the CFT of a shifted n-dimensional $\operatorname{rect}(\boldsymbol{x})$ function with center at $\boldsymbol{a} = 3\boldsymbol{e}_2$, $\omega_2 = \boldsymbol{\omega} \cdot \boldsymbol{e}_2$ as

$$\begin{aligned} \mathcal{F}\{\operatorname{rect}_d\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \operatorname{rect}(\boldsymbol{x} - 3\boldsymbol{e}_2) e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \, d^n \boldsymbol{x} \\ &= i_n \, e^{-3 i_n \omega_2} \prod_{k=1}^{n} \operatorname{sinc}(\frac{\omega_k}{2\pi}). \end{aligned} \tag{4.10}$$

Theorem 4.9 (Shift in frequency domain)**.** *If* $\boldsymbol{\omega}_0 \in \mathbb{R}^n$ *and* $f_0(\boldsymbol{x}) = f(\boldsymbol{x}) \, e^{i_n \boldsymbol{\omega}_0 \cdot \boldsymbol{x}}$ *, then*

$$\mathcal{F}\{f_0\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0) \tag{4.11}$$

Proof. Using definition 4.1 and simplifying it we obtain

$$\mathcal{F}\{f_0\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i_n (\boldsymbol{\omega} - \boldsymbol{\omega}_0) \cdot \boldsymbol{x}} \, d^n \boldsymbol{x} = \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0). \tag{4.12}$$

□

The CFT $\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$ is centered on the point $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ in the frequency domain.

Theorem 4.10 (Powers of $\boldsymbol{x} \in \mathbb{R}^n$ from left)**.**

$$\mathcal{F}\{\boldsymbol{x}^m f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \nabla_{\boldsymbol{\omega}}^m \, \mathcal{F}\{f\}(\boldsymbol{\omega}) \, i_n^m, \quad m \in \mathbb{N}. \tag{4.13}$$

Proof. We first proof theorem 4.10 for $m = 1$. Direct calculation leads to

$$\begin{aligned} \mathcal{F}\{\boldsymbol{x} f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \boldsymbol{x} f(\boldsymbol{x}) \, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \, d^n \boldsymbol{x} \\ = \int_{\mathbb{R}^n} \nabla_{\boldsymbol{\omega}} f(\boldsymbol{x}) \, i_n \, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \, d^n \boldsymbol{x} &= \nabla_{\boldsymbol{\omega}} \int_{\mathbb{R}^n} f(\boldsymbol{x}) \, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} \, d^n \boldsymbol{x} \; i_n \\ &= \nabla_{\boldsymbol{\omega}} \, \mathcal{F}\{f\}(\boldsymbol{\omega}) \, i_n \, , \end{aligned} \tag{4.14}$$

where we have used definition 4.1 and (3.26) of theorem 3.16. We therefore have

$$\mathcal{F}\{\boldsymbol{x}\, f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \nabla_{\boldsymbol{\omega}}\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n\,. \tag{4.15}$$

Repeating this process $m-1$ times we get

$$\mathcal{F}\{\boldsymbol{x}^m f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \nabla_{\boldsymbol{\omega}}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m, \quad m \in \mathbb{N}. \tag{4.16}$$

□

Example 4.11. The CFT of a Gaussian function $f(\boldsymbol{x}) = \exp(-\boldsymbol{x}^2)$, $\boldsymbol{x} \in \mathbb{R}^n$ is again a Gaussian function

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} e^{-\boldsymbol{x}^2} e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} = \pi^{\frac{n}{2}} e^{-\frac{\boldsymbol{\omega}^2}{4}} \tag{4.17}$$

The CFT of its first moment is therefore according to theorem 4.10 and propositions 3.5 and 3.6

$$\begin{aligned} \mathcal{F}\{\boldsymbol{x} f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \boldsymbol{x} e^{-\boldsymbol{x}^2} e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x} \\ &= \pi^{\frac{n}{2}} \nabla_{\boldsymbol{\omega}} e^{-\frac{\boldsymbol{\omega}^2}{4}} i_n = -\frac{\pi^{\frac{n}{2}}}{2}\, \boldsymbol{\omega}\; i_n\; e^{-\frac{\boldsymbol{\omega}^2}{4}}. \end{aligned} \tag{4.18}$$

Theorem 4.12 ($\boldsymbol{x}^m$ from right)**.**

$$\mathcal{F}\{f(\boldsymbol{x})\; \boldsymbol{x}^m\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \nabla_{\boldsymbol{\omega}}^m\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}\;\, i_n^m, \quad m \in \mathbb{N}. \tag{4.19}$$

Proof. Direct calculation leads to

$$\begin{aligned} \mathcal{F}\{f(\boldsymbol{x})\,\boldsymbol{x}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\,\boldsymbol{x}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\ = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \nabla_{\boldsymbol{\omega}} i_n\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \nabla_{\boldsymbol{\omega}}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}\; i_n \end{aligned} \tag{4.20}$$

where we have used definition 4.1 and (3.27) of theorem 3.16. Replacing f $m-1$ times by $f\boldsymbol{x}$ and converting the additional right factor $\boldsymbol{x}$ each time into a derivative of $e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ leads to the full proof of the theorem. □

Remark 4.13. In the next section we will use theorem 4.12 and the dimension dependent commutation properties of i_n to derive final formulas for the CFTs of $f(\boldsymbol{x})\,\boldsymbol{x}^m$, $m \in \mathbb{N}$.

Theorem 4.14.

$$\mathcal{F}\{(\boldsymbol{a}\cdot\boldsymbol{x})^m f(\boldsymbol{x})\}(\boldsymbol{\omega}) = (\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}})^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m, \quad m \in \mathbb{N}. \tag{4.21}$$

Proof. We first proof theorem 4.14 for $m = 1$.

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{a}\cdot\boldsymbol{x}\, f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x}\, f(\boldsymbol{x})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \boldsymbol{a}\cdot\boldsymbol{x}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\
&\overset{\text{Theor. 3.11}}{=} \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\, i_n\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\
&= \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}} \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}\, i_n \\
&= \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n. \qquad (4.22)
\end{aligned}$$

Repeatedly inserting $\boldsymbol{a}\cdot\boldsymbol{x} f$ for f in (4.22) we obtain theorem 4.14 for every $m \in \mathbb{N}$. □

Inserting $\boldsymbol{b}\cdot\boldsymbol{x} f$ with $\boldsymbol{b} \in \mathbb{R}^n$ for f in (4.22) we obtain the following corollary.

Corollary 4.15.

$$\mathcal{F}\{\boldsymbol{a}\cdot\boldsymbol{x}\;\; \boldsymbol{b}\cdot\boldsymbol{x}\, f(\boldsymbol{x})\}(\boldsymbol{\omega}) = -\,\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\;\; \boldsymbol{b}\cdot\nabla_{\boldsymbol{\omega}}\, \mathcal{F}\{f\}(\boldsymbol{\omega}). \qquad (4.23)$$

Theorem 4.16 (Vector differential)**.** *The Clifford Fourier transform of the* m^{th} *power vector differential of* $f(\boldsymbol{x})$ *is*

$$\mathcal{F}\{(\boldsymbol{a}\cdot\nabla)^m f\}(\boldsymbol{\omega}) = (\boldsymbol{a}\cdot\boldsymbol{\omega})^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m, \quad m \in \mathbb{N}. \qquad (4.24)$$

Proof. We first proof theorem 4.16 for $m = 1$.

$$\begin{aligned}
\boldsymbol{a}\cdot\nabla f(\boldsymbol{x}) &= \boldsymbol{a}\cdot\nabla \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, \boldsymbol{a}\cdot\nabla\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega} \\
&\overset{\text{Cor. 3.13}}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, \boldsymbol{a}\cdot\boldsymbol{\omega} i_n\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega} \\
&= \mathcal{F}^{-1}[\boldsymbol{a}\cdot\boldsymbol{\omega}\, \mathcal{F}\{f\}\, i_n](\boldsymbol{x}). \qquad (4.25)
\end{aligned}$$

Application of the inverse CFT theorem 4.3 proves theorem 4.16 for $m = 1$

$$\mathcal{F}\{\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\} = \boldsymbol{a}\cdot\boldsymbol{\omega}\, \mathcal{F}\{f\}\, i_n. \qquad (4.26)$$

By repeatedly replacing f with $\boldsymbol{a}\cdot\nabla f$ in (4.26) we obtain theorem 4.16 for all $m \in \mathbb{N}$. □

Theorem 4.17 (Left vector derivative)**.** *The Clifford Fourier transform of the* m^{th} *power vector derivative of* $f(\boldsymbol{x})$ *is*

$$\mathcal{F}\{\nabla^m f\}(\boldsymbol{\omega}) = \boldsymbol{\omega}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m, \quad m \in \mathbb{N}. \qquad (4.27)$$

Proof. We first proof theorem 4.17 for $m = 1$. According to proposition 3.6 we can calculate the derivative from the differential of theorem 4.16

$$\begin{aligned} \mathcal{F}\{\nabla f(\boldsymbol{x})\} &= \mathcal{F}\{\nabla_{\boldsymbol{a}}[\boldsymbol{a} \cdot \nabla f(\boldsymbol{x})]\} = \nabla_{\boldsymbol{a}} \mathcal{F}\{\boldsymbol{a} \cdot \nabla f(\boldsymbol{x})\} \\ &= \nabla_{\boldsymbol{a}}(\boldsymbol{a} \cdot \boldsymbol{\omega})\, \mathcal{F}\{f\}\, i_n = \boldsymbol{\omega}\, \mathcal{F}\{f\}\, i_n \end{aligned} \tag{4.28}$$

By repeatedly replacing f with ∇f in (4.28) we obtain theorem 4.17 for all $m \in \mathbb{N}$. □

Example 4.18. Using the CFT of a Gaussian function $f(\boldsymbol{x}) = \exp(-\boldsymbol{x}^2)$ of (4.17) we can calculate the CFT of its third vector derivative with (4.27) as

$$\mathcal{F}\{\nabla^3 f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} \nabla^3 e^{-\boldsymbol{x}^2} e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^n \boldsymbol{x} = -\pi^{\frac{n}{2}}\, \boldsymbol{\omega}^3\, i_n\, e^{-\frac{\boldsymbol{\omega}^2}{4}}. \tag{4.29}$$

We now prove a theorem, which we will use in the next section together with the dimension dependent commutation properties of i_n to derive final formulas for the CFT of powers of the right vector derivative $f(\boldsymbol{x})\nabla^m$, $m \in \mathbb{N}$.

Theorem 4.19 (∇^m from right)**.**

$$f(\boldsymbol{x})\nabla^m = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m\, e^{i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\ \boldsymbol{\omega}^m\, d^n \boldsymbol{\omega}. \tag{4.30}$$

Proof. We first proof theorem 4.19 for $m = 1$.

$$\begin{aligned} f(\boldsymbol{x})\nabla &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega} \cdot \boldsymbol{x}} d^n \boldsymbol{\omega} \nabla \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, \nabla\, d^n \boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, i_n\, \boldsymbol{\omega}\, d^n \boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n\, e^{i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, \boldsymbol{\omega}\, d^n \boldsymbol{\omega}, \end{aligned} \tag{4.31}$$

where we used corollary 3.15 for the third equality. Repeating the application of the vector derivative ∇ from the right $m - 1$ times to both sides of (4.31) completes the proof. □

Next we will prove a Plancherel theorem and deduce a scalar Parseval theorem, which we need in the last section on the uncertainty principles.

Theorem 4.20 (Plancherel)**.** *Assume that* $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \in \mathcal{G}_n$ *with Clifford Fourier transform* $\mathcal{F}\{f_1\}(\boldsymbol{\omega})$ *and* $\mathcal{F}\{f_2\}(\boldsymbol{\omega})$ *respectively, then*

$$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\, \widetilde{f_2(\boldsymbol{x})}\, d^n \boldsymbol{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega}) \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n \boldsymbol{x}. \tag{4.32}$$

Proof. Direct calculation yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\, \widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{\omega}]\, \widetilde{f_2(\boldsymbol{x})} d^n\boldsymbol{x} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega}) \left[\int_{\mathbb{R}^n} f_2(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}\right]^{\sim} d^n\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega}) \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})} d^n\boldsymbol{\omega}.
\end{aligned} \tag{4.33}$$

□

Note that theorem 4.20 is multivector valued. It holds for each grade k, $0 \leq k \leq n$ of the multivectors on both sides of equation (4.32). We therefore have

Corollary 4.21.

$$\langle \int_{\mathbb{R}^n} f_1(\boldsymbol{x})\, \widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x} \rangle_k = \frac{1}{(2\pi)^n} \langle \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega}) \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n\boldsymbol{x} \rangle_k . \tag{4.34}$$

Note further, that with $f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}) = f(\boldsymbol{x})$, we get the following multivector version of the *Parseval theorem*, i.e.

Theorem 4.22 (Multivector Parseval)**.**

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, \widetilde{f(\boldsymbol{x})}\, d^n\boldsymbol{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, \widetilde{\mathcal{F}\{f\}(\boldsymbol{\omega})}\, d^n\boldsymbol{x}. \tag{4.35}$$

Example 4.23. According to (4.17) and left linearity of theorem 4.4 the CFT of the function $f(\boldsymbol{x}) = (1 + \boldsymbol{e}_1)\exp(-\boldsymbol{x}^2)$, $\boldsymbol{x} \in \mathbb{R}^2$ is

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} (1 + \boldsymbol{e}_1) e^{-\boldsymbol{x}^2} e^{-i_2 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^2\boldsymbol{x} = (1 + \boldsymbol{e}_1)\pi e^{-\frac{\boldsymbol{\omega}^2}{4}} \tag{4.36}$$

Inserting this f into (4.35) gives on the left side

$$\int_{\mathbb{R}^2} f(\boldsymbol{x})\, \widetilde{f(\boldsymbol{x})}\, d^2\boldsymbol{x} = (1 + \boldsymbol{e}_1)^2 \int_{\mathbb{R}^2} e^{-2\boldsymbol{x}^2}\, d^2\boldsymbol{x} = (1 + \boldsymbol{e}_1)\pi. \tag{4.37}$$

We can check (4.35) by inserting (4.36) on the right side. The scalar part of the result is π, the vector part is $\pi \boldsymbol{e}_1$.

The scalar part of theorem 4.22 together with (2.7), gives us the scalar Parseval theorem

Theorem 4.24 (Scalar Parseval)**.**

$$\int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\, d^n\boldsymbol{\omega}. \tag{4.38}$$

4.2 Properties of the CFT for $n = 2, 3 \pmod 4$ dependent on i_n commutations

Now we concentrate on properties of the CFT, which need to make use of the commutation properties of the unit oriented pseudoscalar $i_n \in \mathcal{G}_n$ in order to be fully developed. In this category we have for $m \in \mathbb{N}$ the CFTs of $f\boldsymbol{x}^m$, $f\nabla^m$, and the CFT of the Clifford convolution with distinct expressions for the dimensions of $n = 2 \pmod 4$ and $n = 3 \pmod 4$. Before we proceed, we note that for $n = 3 \pmod 4$ the pseudoscalar i_n commutes with all elements of the algebra. For the case of $n = 2 \pmod 4$ we first establish a theorem and two corollaries.

Theorem 4.25. *Any odd grade multivector* $A_r \in \mathcal{G}_n, r = 2s+1, s \in \mathbb{N}, s < \frac{n}{2}$ *anti-commutes with* i_n *for* $n = 2 \pmod 4$

$$A_r i_n = -i_n A_r. \tag{4.39}$$

Any even grade multivector $A_r \in \mathcal{G}_n, r = 2s, s \in \mathbb{N}, s \leq n/2$ *commutes with* i_n *for* $n = 2 \pmod 4$

$$A_r i_n = +i_n A_r. \tag{4.40}$$

Proof. For the case of $n = 2$ we have [2] for a vector $\boldsymbol{a} \in \mathbb{R}^2$

$$i_2\,\boldsymbol{a} = -\,\boldsymbol{a}\,i_2. \tag{4.41}$$

For the general case of $n = 2 \pmod 4$ we can factorize i_n for any vector $\boldsymbol{a} \in \mathbb{R}^n$ such that

$$i_n = i_{n-2}\,\hat{\boldsymbol{b}}\;\hat{\boldsymbol{a}},\;\; \hat{\boldsymbol{b}} * \hat{\boldsymbol{a}} = \hat{\boldsymbol{b}} \rfloor i_{n-2} = \hat{\boldsymbol{a}} \rfloor i_{n-2} = 0,\;\; \hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{a},\;\; \hat{\boldsymbol{b}}^{\,2} = 1. \tag{4.42}$$

i_{n-2} will therefore be of grade $0 \pmod 4$, represent a subspace[6] of $\mathbb{R}^n$ perpendicular to $\boldsymbol{a}$ and therefore commute with $\boldsymbol{a}$. According to (4.41) and (4.42) the two-blade $\hat{\boldsymbol{b}}\;\hat{\boldsymbol{a}}$ anti commutes with $\boldsymbol{a}$ and hence

$$i_n\,\boldsymbol{a} = i_{n-2}\,\hat{\boldsymbol{b}}\;\hat{\boldsymbol{a}}\;\boldsymbol{a} = -i_{n-2}\,\boldsymbol{a}\;\hat{\boldsymbol{b}}\;\hat{\boldsymbol{a}} = -\,\boldsymbol{a}\,i_{n-2}\,\hat{\boldsymbol{b}}\;\hat{\boldsymbol{a}} = -\,\boldsymbol{a}\,i_n. \tag{4.43}$$

Any odd grade multivector A_{odd} can be written as a sum over homogeneous odd grade parts. These parts can in turn be decomposed into sums of odd grade blades, which can be factorized into products of an odd number of vectors [13, 2]. Since a single vector anti commutes with i_n, a geometric product of an odd number of vectors will also animate with i_n and hence by linearity any odd grade multivector will animate with i_n. This proves (4.39). Similarly any even grade multivector A_{even} can be written as a sum over homogeneous even grade parts. These parts can in turn be

[6] A subspace in the sense of outer product null space.

decomposed into sums of even grade blades, which can be factorized into geometric products of an even number of vectors [13, 2]. The even number of commutations with an even number of vector factors leads via linearity to the total commutation relationship (4.40). □

Based on theorem 4.25 we derive two useful corollaries.

Corollary 4.26. *For* $n = 2 \pmod 4$ *and* $\boldsymbol{a} \in \mathbb{R}$ *we have for even* $m \in \mathbb{N}$

$$(\boldsymbol{a}\, i_n)^m = \boldsymbol{a}^m \tag{4.44}$$

and for odd $m \in \mathbb{N}$

$$(\boldsymbol{a}\, i_n)^m = \boldsymbol{a}^m\, i_n \tag{4.45}$$

Proof. We have

$$(\boldsymbol{a}\, i_n)^2 = \boldsymbol{a}\, i_n \boldsymbol{a}\, i_n = \boldsymbol{a}\, \boldsymbol{a}\, (-i_n i_n) = \boldsymbol{a}^2, \tag{4.46}$$

where we used (4.39) for the second equality. Using (4.46) $m/2$ times [$(m-1)/2$ times] we arrive at equations (4.44) [and (4.45)]. □

Corollary 4.27. *Let the odd grade part of a general multivector* $A \in \mathcal{G}_n$ *be* $A_{odd} = \langle A \rangle_{odd}$ *and the even grade part be* $A_{even} = \langle A \rangle_{even}$ *. Then for* $n = 2 \pmod 4$ *we have*

$$A\, i_n = A_{odd}\, i_n + A_{even}\, i_n = -i_n\, A_{odd} + i_n\, A_{even}, \tag{4.47}$$

and for $\lambda \in \mathbb{R}$

$$A e^{i_n \lambda} = e^{-i_n \lambda} A_{odd} + e^{+i_n \lambda} A_{even}, \tag{4.48}$$

and

$$e^{i_n \lambda} A = A_{odd}\, e^{-i_n \lambda} + A_{even}\, e^{+i_n \lambda}. \tag{4.49}$$

Proof. Corollary 4.27 follows from theorem 4.25 and the fact that $e^{i_n \lambda}$ is a power series of i_n . □

Theorem 4.28 (Powers of $\boldsymbol{x} \in \mathbb{R}^n$ from right)**.** *For* $n = 2 \pmod 4$ *we have*

$$\mathcal{F}\{f(\boldsymbol{x})\, \boldsymbol{x}^m\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}((-1)^m \boldsymbol{\omega})\, \nabla_{\boldsymbol{\omega}}^m\, i_n^m\,, \quad m \in \mathbb{N}. \tag{4.50}$$

For $n = 3 \pmod 4$ *we have*

$$\mathcal{F}\{f(\boldsymbol{x})\, \boldsymbol{x}^m\}(\boldsymbol{\omega}) = i_n^m \mathcal{F}\{f\}(\boldsymbol{\omega})\, \nabla_{\boldsymbol{\omega}}^m\,, \quad m \in \mathbb{N}. \tag{4.51}$$

Proof. We first proof theorem 4.28 for $n = 2 \,(\mathrm{mod}\, 4)$. We start with theorem 4.12 and apply corollary 4.27 to commute the vector derivative $\nabla_{\boldsymbol{\omega}}$ to the right of the integral

$$\mathcal{F}\{f(\boldsymbol{x})\,\boldsymbol{x}^m\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \nabla_{\boldsymbol{\omega}}^m \, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} \, d^n\boldsymbol{x} \; i_n^m$$
$$= \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i_n(-1)^m \boldsymbol{\omega}\cdot\boldsymbol{x}} \, d^n\boldsymbol{x} \, \nabla_{\boldsymbol{\omega}}^m \, i_n^m = \mathcal{F}\{f\}((-1)^m \boldsymbol{\omega}) \, \nabla_{\boldsymbol{\omega}}^m \, i_n^m. \tag{4.52}$$

The proof of (4.51) with $n = 3 \,(\mathrm{mod}\, 4)$ is the same, except that the sign of $\boldsymbol{\omega}$ in the exponent does not change and that we can freely commute i_n to the left. □

Theorem 4.29 (Right vector derivative)**.** *The Clifford Fourier transform of the* m^{th} *power vector derivative (applied from the right) of* $f(\boldsymbol{x})$ *is for* $n = 2 \,(\mathrm{mod}\, 4)$

$$\mathcal{F}\{f \, \nabla^m\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}((-1)^m \boldsymbol{\omega}) \; \boldsymbol{\omega}^m \, i_n^m, \quad m \in \mathbb{N}, \tag{4.53}$$

and for $n = 3 \,(\mathrm{mod}\, 4)$

$$\mathcal{F}\{f \, \nabla^m\}(\boldsymbol{\omega}) = i_n^m \mathcal{F}\{f\}(\boldsymbol{\omega}) \; \boldsymbol{\omega}^m, \quad m \in \mathbb{N}. \tag{4.54}$$

Proof. We first proof theorem 4.29 for $n = 2 \,(\mathrm{mod}\, 4)$. We start with theorem 4.19 and apply corollary 4.27 to commute the vector $\boldsymbol{\omega}^m$ with $e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$

$$f(\boldsymbol{x})\nabla^m = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega}) \, i_n^m \, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} \, \boldsymbol{\omega}^m \, d^n\boldsymbol{\omega}$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega}) \, i_n^m \, \boldsymbol{\omega}^m e^{i_n(-1)^m \boldsymbol{\omega}\cdot\boldsymbol{x}} \, d^n\boldsymbol{\omega}$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}((-1)^m \boldsymbol{\omega}) \, i_n^m \, (-\boldsymbol{\omega})^m e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} \, d^n\boldsymbol{\omega}$$
$$= \mathcal{F}^{-1}[\mathcal{F}\{f\}((-1)^m \boldsymbol{\omega}) \, \boldsymbol{\omega}^m \, i_n^m](\boldsymbol{x}), \tag{4.55}$$

where for odd m we substituted $-\boldsymbol{\omega} \to \boldsymbol{\omega}$ for the third equality. For the fourth equality we applied apply corollary 4.27 once more to commute $\boldsymbol{\omega}^m$ and i_n^m. Equation (4.53) is obtained by applying the inverse CFT theorem 4.3 to both sides of (4.55).

Once again the proof of (4.54) with $n = 3 \,(\mathrm{mod}\, 4)$ is the same, except that the sign of $\boldsymbol{\omega}$ in the exponent does not change and that we can freely commute i_n to the left. □

Remark 4.30. Theorem 4.17 and (4.53) show that all signs of the right hand sides of all five lines in the derivative theorem 5.7 in [6] are wrong. Line three there contains further errors.

For even m we get from theorems 4.10 and 4.28, and from (2.4)

Corollary 4.31.

$$\mathcal{F}\{\boldsymbol{x}^{m} f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}\{f(\boldsymbol{x})\, \boldsymbol{x}^{m}\}(\boldsymbol{\omega}) = (-1)^{\frac{m}{2}} \nabla_{\boldsymbol{\omega}}^{m} \mathcal{F}\{f\}(\boldsymbol{\omega}). \quad (4.56)$$

We further get for even m from theorems 4.17 and 4.29, and from (2.4)

Corollary 4.32.

$$\mathcal{F}\{\nabla^{m} f(\boldsymbol{x})\} = \mathcal{F}\{f(\boldsymbol{x})\, \nabla^{m}\} = (-1)^{\frac{m}{2}}\, \boldsymbol{\omega}^{m}\, \mathcal{F}\{f\}(\boldsymbol{\omega}). \quad (4.57)$$

Theorem 4.33 (CFT of Clifford Convolution)**.** *For* $n = 2\,(\mathrm{mod}\, 4)$, $f, g \in L^2(\mathbb{R}^n, \mathcal{G}_n)$, *and* g_{odd} (g_{even}) *the odd (even) grade part of* g *we have*

$$\mathcal{F}\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(-\boldsymbol{\omega})\, \mathcal{F}\{g_{odd}\}(\boldsymbol{\omega}) + \mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathcal{F}\{g_{even}\}(\boldsymbol{\omega}). \quad (4.58)$$

For $n = 3\,(\mathrm{mod}\, 4)$ *we have*

$$\mathcal{F}\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathcal{F}\{g\}(\boldsymbol{\omega}). \quad (4.59)$$

Proof. For $n = 2\,(\mathrm{mod}\, 4)$ we have

$$\begin{aligned}
\mathcal{F}\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x}-\boldsymbol{y}) d^n\boldsymbol{y}] e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{y}) [\int_{\mathbb{R}^n} g(\boldsymbol{x}-\boldsymbol{y}) e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x}] d^n\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{y}) [\int_{\mathbb{R}^n} g(\boldsymbol{z}) e^{-i_n \boldsymbol{\omega}\cdot(\boldsymbol{y}+\boldsymbol{z})} d^n\boldsymbol{z}] d^n\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{y}) \int_{\mathbb{R}^n} [e^{+i_n \boldsymbol{\omega}\cdot\boldsymbol{y}} g_{odd}(\boldsymbol{z}) + e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{y}} g_{even}(\boldsymbol{z})] e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{z}} d^n\boldsymbol{z} d^n\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{y}) e^{-i_n (-\boldsymbol{\omega})\cdot\boldsymbol{y}} d^n\boldsymbol{y} \mathcal{F}\{g_{odd}\}(\boldsymbol{\omega}) + \int_{\mathbb{R}^n} f(\boldsymbol{y}) e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{y}} d^n\boldsymbol{y} \mathcal{F}\{g_{even}\}(\boldsymbol{\omega}) \\
&= \mathcal{F}\{f\}(-\boldsymbol{\omega})\, \mathcal{F}\{g_{odd}\}(\boldsymbol{\omega}) + \mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathcal{F}\{g_{even}\}(\boldsymbol{\omega}).
\end{aligned}$$

For the third equality we used the variable substitution $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ and for the fourth equality we used corollary 4.27.

The corresponding proof in [4] for $n = 3$, can be generalized straight forwardly to $n = 3\,(\mathrm{mod}\, 4)$. This part of the proof is also strictly invariant. □

Remark 4.34. The above proof of theorem 4.33 for $n = 2\,(\mathrm{mod}\, 4)$ depends on the i_n commutation relationships. But on the other hand, it has the advantage of being manifestly invariant, since no coordinate system needed to be introduced.

TABLE 1.1. Properties of the Clifford Fourier transform (CFT) with $n = 3 \pmod 4$. Multivector functions (Multiv. Funct.) $f, g, f_1, f_2 \in L^2(\mathbb{R}^n, \mathcal{G}_n)$, the constants are $\alpha, \beta \in \mathcal{G}_n$, $0 \neq a \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{\omega}_0 \in \mathbb{R}^n$ and $m \in \mathbb{N}$.

Property	**Multiv. Funct.**	**CFT**
Left lin.	$\alpha f(\boldsymbol{x}) + \beta\, g(\boldsymbol{x})$	$\alpha \mathcal{F}\{f\}(\boldsymbol{\omega}) + \beta \mathcal{F}\{g\}(\boldsymbol{\omega})$
$\boldsymbol{x}$-Shift	$f(\boldsymbol{x} - \boldsymbol{a})$	$e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{a}} \mathcal{F}\{f\}(\boldsymbol{\omega})$
$\boldsymbol{\omega}$-Shift	$e^{i_n \boldsymbol{\omega}_0 \cdot \boldsymbol{x}} f(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{\lvert a\rvert^n} \mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f \star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathcal{F}\{g\}(\boldsymbol{\omega})$
Vec. diff.	$(\boldsymbol{a} \cdot \nabla)^m f(\boldsymbol{x})$	$i_n^m\, (\boldsymbol{a} \cdot \boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$(\boldsymbol{a} \cdot \boldsymbol{x})^m\, f(\boldsymbol{x})$	$i_n^m\, (\boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}})^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})$
Powers of $\boldsymbol{x}$	$\boldsymbol{x}^m f(\boldsymbol{x})$	$i_n^m\, \nabla_{\boldsymbol{\omega}}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$f(\boldsymbol{x})\boldsymbol{x}^m$	$i_n^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, \nabla_{\boldsymbol{\omega}}^m$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$i_n^m\, \boldsymbol{\omega}^m \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$f(\boldsymbol{x})\nabla^m$	$i_n^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, \boldsymbol{\omega}^m$
Plancherel	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega}$
sc. Parseval	$\int_{\mathbb{R}^n} \lVert f(\boldsymbol{x})\rVert^2\, d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \lVert \mathcal{F}\{f\}(\boldsymbol{\omega})\rVert^2\, d^n\boldsymbol{\omega}$

Theorem 4.33 correctly generalizes the results for $n = 2$ in [6] to $n = 2 \pmod 4$. Comparing theorem 4.33 with the convolution theorem 5.6 of [6] in two dimensions, we see that the fourth line of convolution theorem 5.6 in [6] must be wrong. On the right hand side of this formula the dot over the vector filter function $\mathbf{h}$ under the CFT indicating $\dot{\mathbf{h}}(\boldsymbol{x}) = \mathbf{h}(-\boldsymbol{x})$ is incorrect. Because of (3.34) $\mathbf{h}$ should also have no dot, in agreement with the correct dot-free vector filter function $\mathbf{f}$ on the right hand side of line two of theorem 5.6 in [6].

In order to give an overview of the properties of the CFT we list its properties for $n = 3 \pmod 4$ in table 1.1 and for $n = 2 \pmod 4$ in table 1.2. Comparing the tables, the differences caused by the different commutation rules for the pseudoscalars i_n in $n = 3 \pmod 4$ and $n = 2 \pmod 4$ dimensions are obvious. In table 1.1 the positions of i_n and of exponentials $e^{i_n\lambda}$, $\lambda \in \mathbb{R}$ are arbitrary. In table 1.2 the pseudoscalar i_n and its exponentials $e^{i_n\lambda}$, $\lambda \in \mathbb{R}$ cannot be freely commuted.

5 Uncertainty Principle

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing [9]. In quantum physics it states e.g. that conjugate properties like particle momentum and position cannot be be simultaneously measured accurately. In Fourier analysis such conjugate entities correspond to a func-

TABLE 1.2. Properties of the Clifford Fourier transform (CFT) with $n = 2 \,(\mathrm{mod}\, 4)$. Multivector functions (Multiv. Funct.) $f, g, f_1, f_2 \in L^2(\mathbb{R}^n, \mathcal{G}_n)$, the constants are $\alpha, \beta \in \mathcal{G}_n$, $0 \neq a \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{\omega}_0 \in \mathbb{R}^n$ and $m \in \mathbb{N}$.

Property	**Multiv. Funct.**	**CFT**
Left lin.	$\alpha f(\boldsymbol{x}) + \beta\, g(\boldsymbol{x})$	$\alpha \mathcal{F}\{f\}(\boldsymbol{\omega}) + \beta \mathcal{F}\{g\}(\boldsymbol{\omega})$
$\boldsymbol{x}$-Shift	$f(\boldsymbol{x} - \boldsymbol{a})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{a}}$
$\boldsymbol{\omega}$-Shift	$f(\boldsymbol{x})\, e^{i_n \boldsymbol{\omega}_0 \cdot \boldsymbol{x}}$	$\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{\lvert a\rvert^n} \mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f \star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(-\boldsymbol{\omega})\, \mathcal{F}\{g_{odd}\}(\boldsymbol{\omega})$ $+\mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathcal{F}\{g_{even}\}(\boldsymbol{\omega})$
Vec. diff.	$(\boldsymbol{a} \cdot \nabla)^m f(\boldsymbol{x})$	$(\boldsymbol{a} \cdot \boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
	$(\boldsymbol{a} \cdot \boldsymbol{x})^m\, f(\boldsymbol{x})$	$(\boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}})^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
Powers of $\boldsymbol{x}$	$\boldsymbol{x}^m f(\boldsymbol{x})$	$\nabla_{\boldsymbol{\omega}}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
	$f(\boldsymbol{x})\, \boldsymbol{x}^m$	$\mathcal{F}\{f\}((-1)^m \boldsymbol{\omega})\, \nabla_{\boldsymbol{\omega}}^m\, i_n^m$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$\boldsymbol{\omega}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
	$f(\boldsymbol{x}) \nabla^m$	$\mathcal{F}\{f\}((-1)^m \boldsymbol{\omega})\, \boldsymbol{\omega}^m\, i_n^m$
Plancherel	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x}) \widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega}) \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega}$
sc. Parseval	$\int_{\mathbb{R}^n} \lVert f(\boldsymbol{x}) \rVert^2\, d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \lVert \mathcal{F}\{f\}(\boldsymbol{\omega}) \rVert^2\, d^n\boldsymbol{\omega}$

tion and its Fourier transform which cannot both be simultaneously sharply localized. Furthermore much work (e.g. [9, 10]) has been devoted to extending the uncertainty principle to a function and its Fourier transform. Yet Felsberg [11] notes even for two dimensions: *In 2D however, the uncertainty relation is still an open problem. In [12] it is stated that there is no straightforward formulation for the 2D uncertainty relation.*

From the view point of geometric algebra an uncertainty principle gives us information about how the variations of a multivector valued function and its Clifford Fourier transform are related.

The theorems and the corollary below have all been proved with great detail for the case of $n = 3$ in [4]. The key steps of the proofs there involve the CFT of the vector differential of theorem 4.16 and the (scalar) Parseval theorem 4.24, the Cauchy Schwarz inequality (2.9) for multivectors, and finally the coordinate free integration of parts formula of proposition 3.9. Otherwise the proofs are very analogous, and do not involve dimension dependent i_n commutations. Therefore we don't repeat them here, we only list the resulting formulas.

Theorem 5.1 (Directional uncertainty principle)**.** *Let f be a multivector valued function in $\mathcal{G}_n$, $n = 2, 3 \pmod 4$, which has the Clifford Fourier transform $\mathcal{F}\{f\}(\boldsymbol{\omega})$. Assume $\int_{\mathbb{R}^n} \lVert f(\boldsymbol{x}) \rVert^2\, d^n\boldsymbol{x} = F < \infty$, then the fol-*

lowing inequality holds for arbitrary constant vectors $\boldsymbol{a}$, $\boldsymbol{b}$*:*

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \;\geq\; (\boldsymbol{a}\cdot\boldsymbol{b})^2\, \frac{1}{4}\, F^2. \tag{5.1}$$

Proof. Applying the results stated so far we have[7]

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\overset{\text{tab. 1.1,1.2, line 6}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \|\mathcal{F}\{\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega})\|^2\, d^n\boldsymbol{\omega} \\
&\overset{\text{sc. Parseval}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} \int_{\mathbb{R}^n} \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} \\
&\overset{\text{footnote 7}}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x}\, \|f(\boldsymbol{x})\|\, \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|\, d^n\boldsymbol{x}\right)^2 \\
&\overset{(2.9)}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} |\langle \widetilde{f(\boldsymbol{x})}\, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle|\, d^n\boldsymbol{x}\right)^2 \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} \langle \widetilde{f(\boldsymbol{x})}\, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle\, d^n\boldsymbol{x}\right)^2.
\end{aligned}$$

Because of (2.7) and (2.8)

$$(\boldsymbol{b}\cdot\nabla)\|f\|^2 = 2\langle \widetilde{f}\, (\boldsymbol{b}\cdot\nabla) f\rangle, \tag{5.2}$$

we furthermore obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x}\, \frac{1}{2}\, (\boldsymbol{b}\cdot\nabla\|f\|^2)\, d^n\boldsymbol{x}\right)^2 \\
&\overset{\text{Prop. 3.9}}{=} \frac{1}{4}\left(\left[\int_{\mathbb{R}^{n-1}} \boldsymbol{a}\cdot\boldsymbol{x} \|f(\boldsymbol{x})\|^2 d^{n-1}\boldsymbol{x}\right]_{b\cdot x=-\infty}^{b\cdot x=\infty} \right. \\
&\qquad \left. - \int_{\mathbb{R}^n} [(\boldsymbol{b}\cdot\nabla)(\boldsymbol{a}\cdot\boldsymbol{x})]\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\right)^2 \\
&= \frac{1}{4}\left(0 - \boldsymbol{a}\cdot\boldsymbol{b} \int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x})\right)^2 \\
&= (\boldsymbol{a}\cdot\boldsymbol{b})^2\, \frac{1}{4}\, F^2.
\end{aligned}$$

□

[7] $\phi, \psi : \mathbb{R}^n \to \mathbb{C}, \quad \int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq (\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}\, d^n x)^2$

Choosing $\boldsymbol{b} = \pm\boldsymbol{a}$, i.e. $\boldsymbol{b} \parallel \boldsymbol{a}$, with $\boldsymbol{a}^2 = 1$ we get from theorem 5.1 the following

Corollary 5.2 (Uncertainty principle).

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\ \geq\ \frac{1}{4}\,F^2. \tag{5.3}$$

Remark 5.3. In (5.3) equality holds for *Gaussian* multivector valued functions

$$f(\boldsymbol{x}) = C_0\, e^{-k\,\boldsymbol{x}^2}, \tag{5.4}$$

where $C_0 \in \mathcal{G}_n$ is an arbitrary but constant multivector, $0 < k \in \mathbb{R}$. This follows from the observation that we have for the f of (5.4)

$$-2k\ \boldsymbol{a}\cdot\boldsymbol{x}\ f = \boldsymbol{a}\cdot\nabla f. \tag{5.5}$$

Theorem 5.4. *For* $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, *i.e.* $\boldsymbol{b} \perp \boldsymbol{a}$, *we get*

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\ \geq\ 0. \tag{5.6}$$

Theorem 5.5. *Under the same assumptions as in theorem 5.1, we obtain*

$$\int_{\mathbb{R}^n} \boldsymbol{x}^2\ \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \boldsymbol{\omega}^2\ \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\ \geq\ n\,\frac{1}{4}F^2. \tag{5.7}$$

Remark 5.6. To proof theorem 5.5 we first insert $\boldsymbol{x}^2 = \sum_{k=1}^n (\boldsymbol{e}_k\cdot\boldsymbol{x})^2$, $\boldsymbol{\omega}^2 = \sum_{l=1}^n (\boldsymbol{e}_l\cdot\boldsymbol{\omega})^2$. After that we apply (5.3) and (5.6) depending on the relative directions of the vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}_l$.

6 Conclusions

The (real) Clifford Fourier transform extends the traditional Fourier transform on scalar functions to $\mathcal{G}_n$ multivector functions with $n = 2, 3 \pmod 4$ over the vector space domain $\mathbb{R}^n$. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems are demonstrated in a manifestly invariant fashion. We then presented an uncertainty principle in the geometric algebra $\mathcal{G}_n$, which describes how a multivector-valued function and its Clifford Fourier transform relate.

In many fields the Fourier transform is successfully applied to solving physical equations such as heat and wave equations, in optics, in signal and image processing, etc. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving the whole range of k-vector fields $(k = 0, 1, 2, \ldots, n)$ in geometric algebras $\mathcal{G}_n$ with $n = 2, 3 \pmod 4$. The calculations will be real, have clear geometric interpretations and manifestly invariant. The use of coordinate bases is optional.

References

[1] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Vol. 76 of Research Notes in Mathematics, Pitman Advanced Publishing Program, 1982.

[2] T. Bülow, M. Felsberg and G. Sommer, *Non-commutative Hypercomplex Fourier Transforms of Multidimensional Signals*, in G. Sommer (ed.), *Geom. Comp. with Cliff. Alg., Theor. Found. and Appl. in Comp. Vision and Robotics,* Springer, (2001), 187–207.

[3] C. Li, A. McIntosh and T. Qian, *Clifford Algebras, Fourier Transform and Singular Convolution Operators On Lipschitz Surfaces*, Revista Matematica Iberoamericana, **10 (3)**, (1994), 665–695.

[4] B. Mawardi, E. Hitzer, *Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra* $Cl_{3,0}$, Adv. App. Cliff. Alg., **16 (1)**, (2006), 41–61.

[5] E. Hitzer, B. Mawardi, *Uncertainty Principle for Clifford Geometric Algebras* $Cl_{n,0}, n = 3 \,(\mathrm{mod}\, 4)$ *based on Clifford Fourier Transform*, in T. Qian, M.I. Vai, X.Yusheng (eds.), Wavelet Analysis and Applications, Springer (SCI) Book Series Applied and Numerical Harmonic Analysis, (2006), 45–54.

[6] J. Ebling and G. Scheuermann, *Clifford Fourier Transform on Vector Fields*, IEEE Transactions on Visualization and Computer Graphics, **11 (4)**, July/August (2005), 469–479.

[7] D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Kluwer, 1984.

[8] E. Hitzer, *Vector Differential Calculus*, Mem. Fac. Eng. Fukui Univ. **49 (2)**, (2001), 283–298.
http://sinai.mech.fukui-u.ac.jp/gcj/publications/vdercalc/vderc_abs.html
E. Hitzer, *Multivector Differential Calculus*, Adv. App. Cliff. Alg., **12 (2)**, (2002), 135–182.

[9] J. M. Rassias, *On The Heisenberg-Weyl Inequality*, Jour. of Inequalities in Pure and Appl. Math., **6 (1)**, article 11, (2005), 1–18. http://www.primedu.uoa.gr/~jrasssias/

[10] J. G. Christensen, *Uncertainty Principles*, Master Thesis, University of Copenhagen, 2003.

[11] M. Felsberg, *Low-Level Image Processing with the Structure Multivector*, PhD thesis, Kiel, 2002.

[12] G.H. Granlund, H. Knutsson, *Signal Processing for Computer Vision*, Kluwer, Dordrecht, 1995.

[13] E. Hitzer, *The geometric product and derived products*, preprint.
http://sinai.mech.fukui-u.ac.jp/gala2/index.html#GAtopics

[14] K. Ito (ed.), *Encyclopedic Dictionary of Mathematics*, 2nd. ed., MIT Press, 1996.

[15] Psalm 92, verse 5, *New Int. Version of the Bible*, www.biblegateway.com

Acknowledgment

This research was financially supported by the Global Engineering Program for International Students 2004 of the University of Fukui. We thank A. Hayashi for his continuous support and O. Yasukura and his colleagues from the Department of Applied Physics of the University of Fukui for their comments. We thank U. Kähler for his hints and E.J. Bayro-Corrochano, H. Ishi, S. Krausshar and G. Sommer for good discussions. E. Hitzer wants to thank his wife and children for their patient support. *Soli Deo Gloria.*

How great are your works, O LORD, how profound your thoughts! [15]

Eckhard M. S. Hitzer
Department of Applied Physics
University of Fukui
Bunkyo 3-9-1
910-8507 Fukui, Japan
E-mail: hitzer@mech.fukui-u.ac.jp

Bahri Mawardi
Department of Applied Physics
University of Fukui
Bunkyo 3-9-1
910-8507 Fukui, Japan
E-mail: mawardi@quantum.apphy.fukui-u.ac.jp

Submitted: November 18, 2005; Revised: June 9, 2007.