The Lonely Runner Conjecture - The Proven Case of Correlated Factors Extended to *n* Arbitrary Integer Values.

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Abstract

In number theory, and especially the study of the diophantine approximation, the *Lonely Runner Conjecture* is a conjecture with important and widespread applications in mathematics.

A previous paper by this author proved the *lonely runner conjecture* for any *n,* for the special case of integers with particularly correlated prime factors. In this paper we attempt to extend this work to the general case of *n* arbitrary integers. The paper demonstrates that any set integers with correctly correlated prime factors, such that they satisfy the conjecture, can approximate any set of arbitrary integers with infinite precision.

Background

In a previous paper it was shown that the *lonely runner conjecture* is true in the case of speeds with specially correlated prime factors. The nature of the correlation was such that each of the *n* runners speeds was *constructed* from a distinct set of (*n*-1) different factors from a set of n arbitrary prime factors.

The conclusions from that paper were as follows:

(1) When the speeds of any (*n*-1) runners share a common prime factor p, they will coincide at the origin O, at a time T= 1/p, where T < 1.

(2) When the speeds of any (*n*-1) runners share a common factor p, the excluded runner (whose speed does not share this prime factor) can be shown to become *lonely* at some time t = M x T, where M is an integer and T = 1/p.

(3) When the prime factors of the speeds of *n* runners are correlated such that each possesses a different set of (*n*-1) prime factors, selected from a set of *n*, [described above], then each runner becomes *lonely* at some time.

(4) If the speeds of any *n* runners are such that, relative to each other, their speeds are related by the inverse proportions of different arbitrary prime numbers, then every runner will become *lonely* at some time.

The original paper noted the following:

"the special correlation which satisfies the l*onely runner conjecture* for any *n*, may include the powers of those primes. That is to say, as long as the factors are correlated correctly, it does not appear to matter whether any or all are raised to any arbitrary positive integer powers"

A cursory review of the proof shows that this statement is correct. However, we would advise that the paper be reviewed in its entirety to assure oneself of this fact. The paper in question is added *here* as Appendix (1) and it should be read before continuing.

Consequently, the *lonely runner conjecture* would be correct for any *n* integer speeds *constructed* from any arbitrary set of prime factors (where p_1 to p_n are a set of prime factors). For example the following set of values are correctly correlated and would therefore, satisfy the lonely runner conjecture for any n:

The table below indicates the requirements for the correct correlation of prime factors and for which the *lonely runner conjecture* holds true. That is, *in this case,* zero's powers along the leading diagonal and integer positive powers elsewhere.

	p_1	p_2	p_3	p_4	p_5			$p_{(n-1)}$	p_n
S_1	**0**	2	4	3	3			3	2
S_2	3	**0**	3	2	1			4	5
S_3	1	2	**0**	3	4			4	3
S_4	1	1	1	**0**	3			1	1
S_5	2	3	1	3	**0**			2	1
$S_{(n-1)}$	2	3	1	3	3			**0**	6
S_n	6	4	1	2	2			7	**0**

Prime factors along the top, speeds on the left, powers of factors in the body of the table.

(Table 1)

The nature of the correlation can be easily explained - each runners speed excludes a different factor from the set {p_1 to p_n}. For example, the 1st runners' speed excludes p_1 as a factor. Under these circumstances, the lonely runner conjecture is true.

Extending the Special Case to the General Case by Approximation

We now attempt to make the argument that any set of arbitrary integer values (*speeds*) can be can be *constructed* in this manner with unlimited precision. That is to say, given a set of arbitrary integer values - $I_1, I_2, I_3, I_4, \ldots. I_n$, a set of values can be satisfactorily *constructed* using correlated prime factors such that:

$S_1 = C\,(I_1 \pm \Delta_1)$

$S_2 = C\,(I_2 \pm \Delta_2)$

$S_3 = C\,(I_3 \pm \Delta_3)$

$S_4 = C\,(I_4 \pm \Delta_4)$

$S_5 = C\,(I_5 \pm \Delta_5)$

……

$S_n = C\,(I_n \pm \Delta_n)$

Such that $\Delta i \rightarrow 0$. Through the use of a simple algorithm it is possible to find a Δi which is below any predefined limit, such that Δi can be as small as we choose. C is a scaling factor, such that the condition for *loneliness* is now that the separation from the nearest runner is C/n instead of 1/n.

We now attempt to demonstrate a method of approximating any arbitrary integer values A, B, C, D, E to any required level of precision using correctly correlated prime factors.

Procedure

Select the first *n* primes (in this case 5) - 2, 3, 5, 7, 11 and correlate as follows:

	2	**3**	**5**	**7**	**11**
S_1	**0**	1	1	1	1
S_2	1	**0**	1	1	1
S_3	1	1	**0**	1	1
S_4	1	1	1	**0**	1
S_5	1	1	1	1	**0**

(Table 2)

Our base speeds are then:

$S_1 = 3 \times 5 \times 7 \times 11$

$S_2 = 5 \times 7 \times 11 \times 2$

$S_3 = 7 \times 11 \times 2 \times 3$

$S_4 = 11 \times 2 \times 3 \times 5$

$S_5 = 2 \times 3 \times 5 \times 7$

For now these values (specifically the ratio of the speeds) do not match our required ratios. However, we can adjust the powers of the factors appropriately to match the actual integers (subject to scaling) to any required accuracy.

Our aim is to raise each of these prime factors to an appropriate integer power such that the ratios are approximately (to any required precision) A:B, B:C, C:D, D:E.

We describe below an appropriate algorithm:

Our base position is:

	2	3	5	7	11	Actual Ratio	Required Ratio
S_1	0	1	1	1	1		
S_2	1	0	1	1	1	3/2	A/B
S_3	1	1	0	1	1	5/3	B/C
S_4	1	1	1	0	1	7/5	C/D
S_5	1	1	1	1	0	11/7	D/E

(Table 3)

Method:

We want to adjust the ratios from one to the next, from the current ratios to the required ratios. We start with the first pair. We use only the factors which differ between each base speed. In this case 2 and 3. The other factors between S_1 and S_2 are the same.

The existing ratio of S_1/S_2 is 3/2, we want A/B. Therefore need to find values for Q and P such that $3^Q/2^P = A/B$ (where Q and P will replace the existing powers).

Thus:

$2^P = (B/A)(3^Q)$

$P \, Log2 = [Log(B/A) + Q \, Log3]$

Yielding:-

$\boldsymbol{P = [Log(B/A)+Q \, Log3]/[Log2]}$ **Equation 1**

$\boldsymbol{\Delta \, p = [Log(B/A)+Q \, Log3]/[Log2] \; (Mod \; 1) < \Delta \; ; \; Or > (1- \Delta)};$ **Equation 2**

In more general terms:

$P_i = [Log(I_{(i+1)}/I_i)+P_{(i+1)} \, Log \, p_{(i+1)}]/[Log \, p_i]$ **Equation 3**

$\Delta \, p_i = [Log(I_{(i+1)}/I_i)+P_{(i+1)} \, Log \, p_{(i+1)}]/[Log \, p_i] \; (Mod \; 1) < \Delta \; ; \; Or > (1- \Delta);$

Equation 4

Where:

P_i	*The power of the ith prime*
p_i	*The ith Prime*
I_i	*The ith Integer*
$\Delta \, p_i$	*The ith delta term*

Equation 2 forms the basis of our algorithm.

We use *Equation 2* to find an appropriate value for ***Q*** and then *Equation 1* to find a corresponding value for ***P***.

What we want to find is an integer value for Q, where Δ ***is*** *our predetermined tolerance, such that:*

$\Delta \, p = [Log(B/A)+\boldsymbol{Q} \, Log3]/[Log2] \; (Mod \; 1) < \Delta \; ; \; Or > (1- \Delta);$

Note - this is a modular arithmetic operation on the integer multiples of irrational numbers. We we will never get a value for $\Delta \, p = 0$*. However we will get a* $\Delta \, p \rightarrow 0$ *and it can be as small as we like, provided that we look for long enough.*

This process is repeated for $Q = 1$ *to* m*, until* $\Delta \, p < \Delta$

We do this and establish an appropriate integer $\boldsymbol{Q} = m$ *.*

Then using this value of $\boldsymbol{Q}$ *we find P*

$P = [Log(B/A)+\boldsymbol{Q}\ Log3]/[Log2]$

Doing this will yield a value of *P* as sufficiently close as we like (depending on Δ) to an integer such that $P \rightarrow \boldsymbol{P}$ (an Integer) and $3^{\wedge}\boldsymbol{Q}/2^{\wedge}\boldsymbol{P} \rightarrow A/B$

We now amend our base position as follows and note from the table below that as well as adjusting the values of S_1 and S_2 we have also adjusted the values of the remaining speeds by increasing the powers of 2 by **P**. This has the advantage of leaving the remaining original ratios i.e. 5/3, 7/5, 11/7, unchanged for now. However the ratio of S_1/S_2 is now adjusted to the required value, within the limits of our chosen Δ, such that $3^{\wedge}Q/2^{\wedge}P \rightarrow A/B$

	2	3	5	7	11	Actual Ratio	Required Ratio
S_1	0	Q	1	1	1		
S_2	P	0	1	1	1	$[3^{\wedge}Q/2^{\wedge}P] \rightarrow A/B$	A/B
S_3	P	1	0	1	1	5/3	B/C
S_4	P	1	1	0	1	7/5	C/D
S_5	P	1	1	1	0	11/7	D/E

(Table 4)

In the same manner we repeat this process to find values for ***R*** *and* ***S*** *using:*

$\Delta_r = [Log(C/B)+\boldsymbol{S}\ Log5]/[Log3]\ (Mod\ 1) < \Delta$; *Or* $> (1- \Delta)$; *To find* ***S***

And** R = [Log(C/B)+**S** Log5]/[Log3], To find **R.

After our second iteration of the process we have the following. We note that again not only do we adjust the second and third terms we must also adjust the others appropriately to maintain the existing ratios. We note that the ratio of S_1/S_2 is still $3^{\wedge}\boldsymbol{Q}/2^{\wedge}\boldsymbol{P}$ as we now adjust backwards and forwards to maintain existing ratios following our adjustments to S_2 and S_3.

	2	**3**	**5**	**7**	**11**	**Actual Ratio**	**Required Ratio**
S_1	0	Q	S	1	1		
S_2	P	0	S	1	1	[3^Q/2^P] → A/B	A/B
S_3	P	R	0	1	1	[5^S/3^R] → B/C	B/C
S_4	P	R	1	0	1	7/5	C/D
S_5	P	R	1	1	0	11/7	D/E

(Table 5)

*We can repeat this process again to find a value for **T** and **U***

	2	**3**	**5**	**7**	**11**	**Actual Ratio**	**Required Ratio**
S_1	0	Q	S	U	1		
S_2	P	0	S	U	1	[3^Q/2^P] → A/B	A/B
S_3	P	R	0	U	1	[5^S/3^R] → B/C	B/C
S_4	P	R	T	0	1	[7^U/5^T] → C/D	C/D
S_5	P	R	T	1	0	11/7	D/E

(Table 6)

*And finally we can find **V** and **W**.*

	2	3	5	7	11	Actual Ratio	Required Ratio
S_1	0	Q	S	U	W		
S_2	P	0	S	U	W	[3^Q/2^P] → A/B	A/B
S_3	P	R	0	U	W	[5^S/3^R] → B/C	B/C
S_4	P	R	T	0	W	[7^U/5^T] → C/D	C/D
S_5	P	R	T	V	0	[11^W/7^V] → D/E	D/E

(Table 7)

Results

We have thus succeeded in the task of finding the appropriate powers of our correlated factors such that they are arbitrarily close to the ratios of the actual integers required . This is done by way of a straightforward algorithm which has been explained. In this manner we have created a set of correlated factors that can be approximated as closely as we choose to any set of integer values and for which the *lonely runner conjecture* is true. There will, in this scenario, be an issue of scaling, but this is of no importance so long as the *relative* values are the same, from one to the other, as the original integers.

A Specific Example:

To complete this work we will show how the integers 2, 3, 4, 5, 6 can be approximated sufficiently closely using the powers of properly correlated factors.

We use the factors 2, 3, 5, 7, 11.

The base correlations are:

3, 5, 7, 11

2, 5, 7, 11

2, 3, 7, 11

2, 3, 5, 7

We now find the powers using our algorithm selecting an appropriate $\Delta < 1/5{,}000$ *and find the following values for P, Q, R, S, T, U, V (in this particular case we use the algorithm to target the actual integer values from below the actual integer value):*

P=1,053; Q=664; R =11,994; S =8,187; T =8,067; U =6,672; V =3,639; W=2,953

These results are set out in the table below:

	2	3	5	7	11	Actual Ratio	Required Ratio
S_1	0	664	8187	6672	2953		
S_2	1053	0	8187	6672	2953	2.0000.../2.9998...	2/3
S_3	1053	11994	0	6672	2953	2.9998.../3.9995...	3/4
S_4	1053	11994	8067	0	2953	3.9995.../4.9992...	4/5
S_5	1053	11994	8067	3639	0	4.9992.../5.9989...	5/6

What does this mean in practice

In this particular case, it means that we can be sure that runners with speeds 2.0000..., 2.9998..., 3.9995..., 4.9992..., 5.9989..., satisfy the lonely runner conjecture. These values are very close to the actual integer values and in fact we could find many other sets of values which can be as close as we choose to the actual integers. Furthermore these sets of values are well defined.

The main thrust of this argument however, is that there is a simple algorithm (*described above*) which allows us to find values arbitrarily close to the required integer values *(approaching those values from either above or below)* and for which the *lonely runner conjecture* is true.

The conclusions can be summarized as follows:

The *lonely runner conjecture* is true for any *n* runners with speeds *$(I_1 \pm \Delta_1)$, $(I_2 \pm \Delta_2)$, $(I_3 \pm \Delta_3)$, ... $(I_n \pm \Delta_n)$, where I_i is any discrete set of arbitrary integers and* Δ_i is infinitesimally small.

APPENDIX (1)

A Special Case of the Lonely Runner Conjecture for any *n*

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Abstract

In number theory, and especially the study of the diophantine approximation, the *Lonely Runner Conjecture* is a conjecture with important and widespread applications in mathematics.

This paper attempts to prove the conjecture for any n runners in the special case of speeds with specially correlated prime factors.

Statement of the Conjecture

Consider n runners on a circular track of unit length. At time t = 0, all runners are at the same position O and start to run; the runners' speeds are pair-wise distinct. A runner is said to be lonely if at a distance of at least 1/n from each other runner. The *Lonely Runner Conjecture* states that every runner gets lonely at some time.

Proof

The track is of unit length, with all runners running at integer speeds. The runners coincide at the starting point O, at all integer units of time. Consequently, the condition that every runner becomes lonely, must occur first for times $t < 1$.

Now we consider if it is possible for any (n-1) runners to coincide at the starting point O, at any time $t < 1$. After careful consideration, it is evident that if each of these runners share a common prime factor *p, then they will coincide at the origin at a time* $T = 1/p$.

This is best explained by example:

Consider that the speeds of the (*n*-1) runners from 1*st* to (n-1)*th* are **A***p*, **B***p*, **C***p* , **D***p* ...., **Ʊ**p, where *p* is the common factor and **A < B < C< ... < Ʊ**.

After a time t = 1/*p all (n-1) runners will have coincided at the origin O, with the 1st runner having completed* ***A*** *cycles, the 2nd* ***B*** *cycles and the (n-1)th.,* **Ʊ** cycles.

We now enquire as to the location of the *n*th runner at this time T=1/p. We imagine for the moment that he is not at *O* and examine the consequences.

If the nth runner is not at O at time T = 1/p then he is distance DELTA from O. DELTA, is either greater than or less than 1/n. If DELTA is greater than 1/n then that runner is lonely. If DELTA is less than 1/n then after a time 2 x *T, the other runners will have coincided at O once again, whereas the nth runner will have increased his separation from O to a distance 2* x *DELTA. (This follows from the simple application of modular arithmetic) .*

This process can be repeated M times until after a time t = M x *T, M* x *DELTA > 1/n and that runner becomes lonely.*

Now we consider what conditions are necessary for the nth runner ***not*** *to be at O at time T = 1/p. Clearly, if the condition for coincidence of the (n-1) runners at a time t < 1 is that they share a common factor p, then the condition for the* ***non-coincidence*** *of the nth runner is that his speed does not share this prime factor.*

Statement

(1) When the speeds of any (n-1) runners share a common prime factor *p, they will coincide at the origin O, at a time T= 1/p, where T < 1.*

(2) *When the speeds of any (n-1) runners share a common factor p, the excluded runner (whose speed does not share this prime factor) can be shown to become lonely at some time t = M* x *T, where M is an integer and T = 1/p.*

The Case of Specially Correlated Factors

We now give further consideration to the implications of runners speeds sharing common prime factors:

Imagine that there is a set of any *n* prime numbers. From that set of *n* primes, the speeds of *n* runners is constructed by selecting a set of (*n*-1) of those prime factors. Each runner has a speed which is the product of a different set of (*n*-1) of those primes. That is, the speeds of the runners from s1 to sn are as follows:

s1 = p2 x p3 x p3 x p4 .... x pn

s2 = p1 x p3 x p4 x p5 .... x pn

....

s(n-1) = p1 x p2 x p3 x p4 ... x pn

sn = p1 x p2 x p3 x p4 ..... x p(n-1)

There is something special about this arrangement of speeds with highly correlated prime factors:

Every combination of (n-1) runners is such that they share a common factor with each other, whilst not sharing that common factor with the excluded runner. For example, it is evident from the list above, that the *1st to the (n-1)th runners share the common factor* **pn**. *That is to say, in this arrangement, the speed of the* ***Qth*** *runner does not contain the* ***Qth*** *prime factor, whereas the speeds of all other runners do.*

It was demonstrated earlier and summarized in ***Statement 2*** *that:*

When the speeds of any (n-1) runners share a common factor p, the excluded runner can be shown to become lonely at some time t = M x T, where M is an integer and T = 1/p.

It is clear that we have created a situation in which every runner can be considered to be an excluded runner by virtue of some prime factor being absent in its speed, yet present in the other (n-1) runners speed. In this situation it is clear that all n runners become lonely at some point and the Lonely Runner Conjecture is true.

Statement

(3) When the prime factors of the speeds of n runners is correlated such that each possesses a different set of (n-1) prime factors, selected from a set of n, *[described above]* then each runner becomes lonely at some time.

It is possible to simplify this idea further:

Imagine now that we have correlated the prime factors of the runners speeds in the method described above. For this purpose, we have used a set of n arbitrary prime factors. We have clearly shown that, in this special case, each runner becomes lonely for any *n* runners.

Let us call the product of all prime factors in our set of *n* prime factors **P =** p1 x p2 x p3 ....x pn

Hence, it follows from our original argument, that the speeds of the n runners are:

1st **P**/p1;

2nd **P**/p2;

3rd **P**/p3;

......

*n*th **P**/pn;

From this observation the following statement then becomes evident.

Statement

(4) If the speeds of any *n* runners is such that, relative to each other, their speeds are related by the inverse proportions of different arbitrary prime numbers, then every runner will become lonely at some time.

Suggestions for further work

Something not discussed until this point, is that the special correlation which satisfies the lonely runner conjecture for any n, may include the powers of those primes. That is to say, as long as the factors are correlated correctly, it does not appear to matter whether any or all are raised to any arbitrary positive integer powers. This fact is not important for these investigations, but may be helpful for any further enquiries. It might be the case that any n runners with arbitrary, pair-wise distinct, integer speeds, can be sufficiently modeled by reference to specially correlated prime factors as discussed in this paper.

Conclusions

In the specific case of any n runners, having speeds with highly correlated prime factors as described above, it has been shown that the *Lonely Runner Conjecture* is true. Following our investigations the following statements are evident:

(1) When the speeds of any (n-1) runners share a common prime factor *p, they will coincide at the origin O, at a time T= 1/p, where T < 1.*

(2) *When the speeds of any (n-1) runners share a common factor p, the excluded runner (whose speed does not share this prime factor) can be shown to become lonely at some time t = M* x *T, where M is an integer and T = 1/p.*

(3) When the prime factors of the speeds of n runners is correlated such that each possesses a different set of (n-1) prime factors, selected from a set of n, *[described above]* then each runner becomes lonely at some time.

(4) If the speeds of any n runners is such that, relative to each other, their speeds are related by the inverse proportions of different arbitrary prime numbers, then every runner will become lonely at some time.