Matrix Determinant as a verifier of a Path (cycle) in the Directed Hamiltonian Cycle Problem under two special conditions: a formal proof

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Abstract

In earlier work, the author conjectured that under two special conditions relating to theorems on the determinant of a matrix: the absence of a zero row (column) and the absence of similar rows (columns), a non-zero determinant value certifies the existence of a Directed Hamiltonian Path in an arbitrary adjacency matrix. Here, a formal proof is provided by means of deductive logic to establish that in an arbitrary adjacency matrix of size *n* (*n* rows and *n* columns), a non-zero determinant value verifies the existence of a Directed Hamiltonian Path in the adjacency matrix.

Keywords: P vs NP, Matrix Determinant, Directed Hamiltonian Path, proof, deductive logic, axioms.

1. INTRODUCTION

The decision version of the directed Hamiltonian cycle problem asks, "Given a graph G, does G have a Hamiltonian Cycle (Path)?" A directed graph G consists of a finite, non-empty set of vertices and edges – which are ordered pairs of vertices [1]. Graph G is said to have a Hamiltonian cycle or path if there exists a sequence of one-way edges across all vertices. This problem is known to be Non-Polynomial complete (NP-complete) [2] and as such not likely to be in class P – the class of problems with feasible algorithms i.e. polynomial-time algorithms [2]. An adjacency matrix – a square matrix of 0's and 1's, denoting the absence (0) or presence (1) of edges between vertices of a graph G is a common representation of a graph G [1]. By this definition it is deducible that an adjacency matrix is a representation of combination of edges of a directed graph across the *n* rows and *n* columns of an adjacency matrix of size *n*.

The determinant of a square matrix A is a special scalar denoted by |A|. In an arbitrary square matrix of size *n* (*n* rows and *n* columns), under two special conditions relating to theorems on determinants with proofs given in [3]: the absence of a zero row (column) and the absence of similar rows (columns), the determinant of a matrix is non-zero. In previous work [3], the author conjectured that under these two conditions the non-zero determinant value of an adjacency matrix certifies the existence of a Directed Hamiltonian path in an arbitrary adjacency matrix – a representation of a directed graph G. Here, the author gives a formal proof by means of deductive logic that given an arbitrary adjacency matrix of size n, the absence of a zero row (column) and the absence of similar rows (columns) i.e. a non-zero determinant value certifies the existence of a directed Hamiltonian path.

2. Proof

A proof is provided by means of deductive logic where four preceding axioms are linked to reach a conclusion that is logically valid.

(1) An adjacency matrix is a square matrix consisting of combinations of edges of a directed graph in the n rows and n columns of adjacency matrix of size n.

(2). A directed Hamiltonian path is a sequence of one-way edges across the n rows (columns) of an adjacency matrix of size n. That is, a directed Hamiltonian path is a unique permutation of edges.

(3). To satisfy the dual conditions of the absence of a zero row (column) and the absence of similar rows (columns) across the n rows (columns) of adjacency matrix, different combinations of edges are applied in row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n of adjacency matrix size n.

(4). Different combinations of edges applied in row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n of an adjacency matrix size n implies a permutation of edges across the n rows (columns) of adjacency matrix size n, because a permutation is an **ordered** combination.

(5). Therefore, drawing from axiom 2, for an arbitrary adjacency matrix of size n satisfying the dual conditions of the absence of a zero row (column) and the absence of similar rows (columns) – implying a non-zero determinant value for the adjacency matrix, there exists a directed Hamiltonian Path.

Proved

3. Discussion and Conclusion

This paper gives a formal proof using the axioms of deductive logic to establish that under two special conditions: the absence of a zero row (column) and the absence of similar rows (columns) – implying a non-zero determinant value, an arbitrary adjacency matrix, which is a common representation of a directed graph, encodes for a directed Hamiltonian path or cycle.

Matrix determinants can be computed in polynomial-time (matrices are an aspect of linear algebra), therefore given adjacency matrices satisfying the dual conditions above, a non-zero determinant value gives an efficient way of verifying the existence of a directed Hamiltonian path or cycle.

As this technique does not verify the existence of a directed Hamiltonian path for adjacency matrices that do not satisfy the two stated conditions, this solution does not imply that the decision version of the directed Hamiltonian path (cycle) problem is in class P. It does prove however that the decision version of the directed Hamiltonian path (cycle) problem which is an NP-complete problem hence in class NP, when expressed as an adjacency matrix, can be solved in polynomial-time when the distribution on its inputs satisfies the two stated conditions. This draws from Levin's theory of average-case completeness [4, 5], where it is relevant to ask whether every NP problem with a reasonable probability distribution on its inputs can be solved in polynomial-time on average.

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