Some issues on Goldbach Conjecture

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Summary

This paper presents a deterministic process of finding all pairs (p, q) of odd numbers (composites and primes) of natural numbers ≥ 3 whose sum ($p + q$) is equal to a given even natural number $2n \geq 6$. Subsequently, based on the above procedure and also relying on the distribution of primes in the set $\mathbb{N}$ of natural numbers, we propose a closed analytical formula, which estimates the number of primes which satisfy Goldbach's conjecture for positive integers ≥ 6.

1. Introduction

As is known, on June 7, 1742, Christian Goldbach in a letter to Leonhard Euler [1] argued, inter alia that "every even natural number > 4 can be written as a sum of two primes", namely:

$$2n = p + q \quad \text{where } n>2, \text{ and } p, q \text{ are prime numbers.} \tag{1}$$

To save space we do not mention the precise wording of that old time, but we focus our attention on a modified formula (1) which is inextricably linked with the newest definition of all primes $\mathbb{P} = \{2,3,5,7,11,\ldots\}$, so as to exclude the unit.

On August 8, 1900, David Hilbert gave a famous speech during the second International Congress of Mathematics in Paris, in which he proposed 23 problems for mathematicians of the 20th century, including the conjecture of Goldbach [2]. Later, in 1912, Landau sorted four main problems for the first few numbers including the conjecture Goldbach [3,4]. The first scientific work on the Goldbach conjecture was made in the 1920's. Note that in 1921, Hardy said that the Goldbach conjecture is not only the most famous and difficult problem in number theory, but the whole of mathematics.

It is known that the most difficult so-called *strong Goldbach conjecture* was preceded by important work in the so-called *weak Goldbach conjecture*. The weak conjecture Goldbach, which is known as the *odd Goldbach conjecture* or *ternary Goldbach problem* or *3-primes problem*, stated that: any number greater than 7 can be expressed as a sum of three primes (one prime number can be used more than once in the same sum). The above assumption is called "weak" because if the strong conjecture Goldbach (which concerns sums of two primes) is proved, then the weak will be true.

The ***weak formulation*** of the conjecture has not been yet proven, but there have been some useful although somewhat failed attempts. The first of these works was in 1923 when, using the 'circle method' and assuming the validity of the hypothesis of a generalized Riemann, Hardy and Littlewood [5] proved that every sufficiently large odd integer is sum of three odd primes and almost all the even number is the sum of two primes. In 1919, Brun [6], using the method of his sieve proved that every large even number is the sum of two numbers each of whom has at least nine factors of primes. Then in 1930, using the Brun's method along with his own idea of "density" of a sequence of integers, Schnirelman [7] proved that every sufficiently large integer is the sum of maximum c primes for a given number c. Then in 1937, Vinogradov [8], using the circle method and his own method to estimate the exponential sum in a variable prime number, was able to overcome the dependence of the great Riemann hypothesis and thus provide the evidence of the findings of Hardy and Littlewood now without conditions. In other words, he directly proved (theorem of Vinogradov's theorem) that all sufficiently large odd

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number can be expressed as the sum of three primes. The original proof of Vinogradov, based on inefficient theorem of Siegel-Walfisz, did not put a limit for the term “sufficiently large”, while his student K. Borozdkin [9] showed in 1956 that ότι $n_0 = 3^{3^{15}} = 3^{14348907}$ is sufficiently large (has 6,846,169 digits). Later, after improvements in the method of Brun, in 1966 Chen Jing-Run [10] managed to prove that every large integer is the sum of a prime and a product of at most two primes. In 2002, Liu and Wang [11] lowered the threshold around $n > e^{3100} \approx 2\times 10^{1346}$. The exponent is too large to allow control of all smaller numbers with the assistance of a digital computer. According to Internet reports [12,13], the computer assisted search arrived for the strong Goldbach conjecture up to order 10^{18} (http://www.ieeta.pt/~tos/goldbach.html) and, for the weak Goldbach conjecture not much more. In 1997, Deshouillers et al. [14] showed that the generalized Riemann hypothesis implies the weak Goldbach's conjecture for all numbers. Also, Kaniecki [15] showed that every odd number is the sum of at most five primes, provided the validity of Riemann Hypothesis.

Most of these classic works have been included in a collective volume by Wang [16]. Specifically, in this volume the first section includes the representation of an odd number as a sum of three primes with six papers (Hardy and Littlewood; Vinogradov; Linnik; Pan; Vaughan; Deshouillers, Effinger, Riele & Zinoviev), the second section includes the representation of an even number as a sum of two nearly primes in six other works of (Brun; Buchstab; Kuhn; Selberg; Wang; Selberg) and finally the third section includes the representation of an even number as a sum of a prime and an almost prime in nine works (Renyi; Wang; Pan; Barban Til; Buchstab; Vinogradov; Bombieri; Chen Jing-Run; Pan). Finally, apart from the individual reports of certain articles, the collective volume includes 234 additional citations arranged by author, referring to the period 1901-2001.

The ***strong formulation*** of Goldbach conjecture, which is the subject of this paper, is much more difficult than the above weak one. Using the above method of Vinogradov [8], in separate works Chudakov [17], van der Corput [18] and Estermann [19] showed that almost all even number can be written as a sum of two primes (in the sense that the fraction of even number tends to the unit). As mentioned above, in 1930, Schnirelman [7] showed that every even number $n \geq 4$ can be written as a sum of at most 20 primes. This result in turn enriched by other authors; the most well-known result due to Ramaré [20] who in 1995 showed that every even number $n \geq 4$ is indeed a maximum sum of 6 primes. Indeed, resolving the weak Goldbach conjecture will come through that every even number $n \geq 4$ is the sum of at most 4 primes [21]. In 1973, using sieve theory methods (sieve theory) Chen Jing-run showed that every (sufficiently large even number can be written as a sum either of two primes or of one prime and one semiprime (i.e. a product of two primes) [22], e.g. $100 = 23 + 7\cdot 11$. In 1975, Montgomery and Vaughan [23] showed that “most” even number is a sum of two primes. In fact, they showed that there was a positive constants c and C such that for all sufficiently large numbers N, every even number less than N is the sum of two primes with CN^{1-c} exceptions at the most. In particular, all the even integers that are not sum of two primes have zero density. Linnet [24] proved, in 1951, the existence of a constant K such that every sufficiently large even number is the sum of two primes and a maximum of K powers of 2. Heath-Brown and Puchta [25] in 2002 found that the value $K = 13$ works well. The latter improved to $K = 8$ by Pintz and Ruzsa [26] in 2003.

It is noteworthy that in 2000 the relation (1) was verified using computers for even numbers up to 4×10^{16} [27], and the attempt was repeated by T. Oliveira e Silva with the help of distributed computing network to $n \leq 1.609\times 10^{18}$ and in selected areas up to 4×10^{18} [13]. However, mathematically these checks do not constitute conclusive evidence of validity of (1), and the effort continues today [28].

It is noteworthy that in addition to the above papers, the interested reader can consult internet sources [29-32]. Finally, the object of the Goldbach conjecture has been the subject of statistical approach [33], education [34], narrative storytelling and popular books [35-37].

In this paper we present a theoretical framework that provides an estimate of the number of prime numbers satisfying the relation (1).

2. A deterministic procedure for the decomposition of an even number as sum of two odds

From the definition of prime numbers, $\mathbb{P}=\{2,3,5,7,11,\ldots\}$, in which they are divisible only by themselves and the unit, it follows immediately that "the only even prime number is 2". If for a moment we assume that p = 2, then the only case satisfying the relation (1) is when q = 2, otherwise the resulting sum would be equal to an odd number. Because the resulting even number 4 (i.e. n = 2) is outside the scope of interest because we care only for n> 2, it is obvious that the relation (1) makes sense only for the *odd* primes, i.e. for all primes greater than 2, comprising the set $\mathbb{P}-\{2\}$.

Here in general, we quote the successive steps that led us to develop this work and then develop a series of theorems which lead directly to the desired result.

2.1 Equivalence classes

As known, an equivalence relation on the set A, such as e.g. K(α) = {x/x∈A and xRα} divides the set A into subsets, named *equivalence classes*, which are disjoint to each other and their union gives A.

Suppose $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z}^2$ is its Cartesian (tensor) product and R is the binary relation on the set $\mathbb{Z}^2$, which is defined as follows:

$$R = \{(\alpha,\beta)/(\alpha,\beta) \in \mathbb{Z}^2 \text{ and } (\alpha-\beta) \text{ is divisible by } n\in\mathbb{N}\}, \tag{2}$$

that is $(\alpha-\beta)=\lambda n$ or $\alpha=\lambda n+\beta$ (the identity of division) and $\mathbb{N}$ is the set of natural numbers.

R is an equivalence relation because, as can be proven, it is *reflexive*, *symmetric* and *transitive*. The natural number n is that which determines the number of equivalence classes.

If we take the set of natural numbers $\mathbb{N}$ and define n = 6, then $\mathbb{N}$ is divided into six equivalence classes, those of elements 0, 1, 2, 3, 4 and 5, namely:

$$K(0)=\{x/x=6\lambda+0 \quad \lambda\in\mathbb{N}\} \tag{3a}$$
$$K(1)=\{x/x=6\lambda+1,\ \lambda\in\mathbb{N}\} \tag{3b}$$
$$K(2)=\{x/x=6\lambda+2,\ \lambda\in\mathbb{N}\} \tag{3c}$$
$$K(3)=\{x/x=6\lambda+3,\ \lambda\in\mathbb{N}\} \tag{3d}$$
$$K(4)=\{x/x=6\lambda+4,\ \lambda\in\mathbb{N}\} \tag{3e}$$
$$K(5)=\{x/x=6\lambda+5,\ \lambda\in\mathbb{N}\} \tag{3f}$$

The above classes are disjoint each other and their union gives the set $\mathbb{N}$:

$$K(0)\cup K(1)\cup K(2)\cup K(3)\cup K(4)\cup K(5)=\mathbb{N} \tag{4}$$

At the same time, the equivalence class of any other element except of 0, 1, 2, 3, 4, 5, coincides with one of the above formulas.

The set of natural numbers corresponding to the equivalence classes K(2) and K(4), which can be written as K(2) = 2(3n+1) and K(4) = 2(3n+2) are divisible by 2; K(3) which can be written as K(3) = 3(2n+1) is divisible by 3, and K(0) = 6n is divisible by both 2 and 3.

Therefore the set of natural numbers contained in the equivalence classes K(0), K(2), K(3) and K(4) are *composite numbers* as multiples of 2 and 3. Therefore, the remaining equivalence classes K(1) and K(5) contain *all primes* (except of 2 and 3) and the *multiples of primes* being > 3.

It is understood the class K(0), which equals 6n+0, is followed (higher by 1) by the class K(1) and follows (lower by 1) the class K(5), so both of these equivalence classes can be combined in the formula:

$$\boxed{6\lambda \pm 1 = \text{Primes} + \text{Multiples of Primes} > 3} \tag{5}$$

In this representation of *odd numbers* we managed to condense the prime numbers in the set of natural numbers by a factor of 3 and make more likely the coincidence of two primes in the sums we will generate in the sequence, when we decompose an even number in two odd ones.

For the sake of uniformity, which will be useful in the sequence, the multiples of 3 will be denoted, as appropriate, either as a $6\lambda–3$ for $\lambda = 1,2,3\ldots N$, or like $6\lambda+3$ for $\lambda = 0,1,2\ldots N$:

$$6\lambda \pm 3 = \text{Multiples of 3} \tag{5a}$$

2.2 Even numbers

The even numbers are usually denoted by 2n. From the number theory we know that in any *three consecutive even numbers* one of them is a multiple of 3, so this even number can be written as $3\cdot 2n = 6n$. The former number even of this can be written as 6n-2, while the next as 6n+2.

The triple of numbers (6n-2, 6n, 6n+2) are consecutive even numbers, since they differ by 2 units. This implies that the subsets of even numbers $\mathbb{N} = \{x/x \ \ 6n\text{-}2, 6n, 6n+2\}$ for n = 1,2,3 ... and $n \in \mathbb{N}$ are disjoint and their union is the set $\mathbb{N}_2$ of all even numbers ≥ 4.

So, the set of even numbers ≥ 4 can be denoted with:

$$\mathbb{N}_2 = \{x/x \ \ 6n\text{-}2, 6n, 6n+2\} \ \text{ where } \ n \geq 1. \tag{6}$$

The above triple of even numbers will be the *basic cell* to create the even numbers, for $n \geq 1$, in the sequence of our work.

If we replace n in the formula (6) with two *natural numbers* λ_i and λ_j where $(\lambda_i, \lambda_j) \in \mathbb{N}$, such as:

$$n = \lambda_i + \lambda_j \,, \tag{7}$$

the triple of the successive even numbers is transformed to

$$6(\lambda_i + \lambda_j) - 2 \tag{8a}$$
$$6(\lambda_i + \lambda_j) \tag{8b}$$
$$6(\lambda_i + \lambda_j) + 2. \tag{8c}$$

Relations (8) may be further transformed as follows:

$$6(\lambda_i + \lambda_j) - 2 = (6\lambda_i - 1) + (6\lambda_j - 1), \tag{9a}$$
$$6(\lambda_i + \lambda_j) = (6\lambda_i - 1) + (6\lambda_j + 1), \tag{9b}$$
$$6(\lambda_i + \lambda_j) = (6\lambda_i + 1) + (6\lambda_j - 1), \tag{9b′}$$
$$6(\lambda_i + \lambda_j) + 2 = (6\lambda_i + 1) + (6\lambda_j + 1). \tag{9c}$$

Thus symbolizing the even numbers, we managed to transform all the even numbers into a sum of *2 odd numbers of the form* $6\lambda_i \pm 1$, which, as proved, *either are primes or multiples of primes* ≥ 5.

Of course the even numbers can be created also as a sum of two odd numbers of which either one or both be *multiples of 3*. In this case however the even numbers of the form $6n$-2 are generated *solely* as a sum:

$$6n - 2 = (6\lambda_i + 1) + (6\lambda_j - 3) = 6(\lambda_i + \lambda_j) - 2 \text{ where } \lambda_i + \lambda_j = n \quad (10a)$$

The even numbers of the form $6n$ can be created only as a sum of two odd multiples of 3:

$$6n = (6\lambda_i + 3) + (6\lambda_j - 3) = 6(\lambda_i + \lambda_j) \text{ where } \lambda_i + \lambda_j = n \quad (10b)$$

Finally, the even numbers of the form $6n$+2 are generated solely as a sum

$$6n + 2 = (6\lambda_i - 1) + (6\lambda_j + 3) = 6(\lambda_i + \lambda_j) + 2 \text{ where } \lambda_i + \lambda_j = n \quad (10c)$$

Equations (10) imply that for each even number the pairs of odd numbers, which are created on the basis of these equations, can give either *zero* or *at most only one pair prime-to-prime*.

More specifically, given that the set of odd numbers being multiples of 3, ($6\lambda \pm 3$), from which the even numbers of the form $6n$ are formed, only the number 3 is prime, this implies that under (10b), the only *pair* which verifies the Goldbach conjecture, is the “3+3=6”.

Based on (10a), the even numbers of the form $6n$-2 verify the Goldbach conjecture only if ($6\lambda_j$–3)=3 while ($6\lambda_i$+1) = prime.

Finally, the even numbers of the form $6n$+2 verify the Goldbach conjecture, by virtue of (10c), only if ($6\lambda_j$–3)=3 while ($6\lambda_i$–1) = prime.

2.3 Three basic theorems

In the following we describe a deterministic procedure in which every even natural number can be decomposed into all possible sums of odd integers, primes or composites.

Theorem-1: Every even natural number $2n$ (independently of the specific form $6n$–2, $6n$, $6n$+2, it has) can be decomposed into a sum of two odd natural numbers (primes or composites) in so many different ways, n_s, as the integer part *(floor)* of the rational number $(n-1)/2$, that is $n_s = \lfloor (n-1)/2 \rfloor$. The index ‘*s*’ results from the word ‘sample’, thus referring to sample of n_s odd numbers from which will be later choose the prime numbers.

PROOF

We distinguish two cases.

1) When n is odd, we form the sets:

A = {3, 5, …, n} and B = {2n-3, 2n-5, …, n}. Since the order of items is not important in the sets, in order to maintain the desired sequence (in the form of rows or columns) we form the vectors $\vec{a} = [3, 5, \ldots, n]$ και $\vec{b} = [2n-3, 2n-5, \ldots, n]$. It is obvious that all elements of the vector $\vec{c} = \vec{a} + \vec{b}$ are strictly defined and are equal to $2n$ as opposed to *probabilistic pairs* that can be derived from the sets A and B. Also, it is evident that any enhancement of the vector $\vec{a}$ will give terms contained in the vector $\vec{b}$, displayed from right to left, so it makes no sense. Finally, it is obvious that the cardinality of two sets is the same, ie *cardA* = *cardB* = (n-1)/2.

2) When n is even, we form the sets:

A = {3, 5, …, n-1} and B = {2n-3, 2n-5, …, n+1}. As previously, we consider the new vectors $\vec{a}' = [3, 5, \ldots, n-1]$ and $\vec{b}' = [2n-3, 2n-5, \ldots, n+1]$. It is obvious that all elements of the vector $\vec{c}' = \vec{a}' + \vec{b}'$ are again equal to $2n$. As previously, any enhancement of the vector $\vec{a}$ will give terms included into the vector $\vec{b}$, displayed from right to left. Finally, it is obvious that the cardinality of two sets is the same, i.e. *cardA* = *cardB* = *n*/2-1.

Summarizing the results of the two above cases, it is easily concluded that:

$$cardA = cardB = \left\lfloor \frac{(n-1)}{2} \right\rfloor \quad (11) ■$$

Conclusion-1: The higher an even number 2*n* is, the higher the number of pairs of odd numbers n_s .

Theorem-2: If the n_s pairs of odd numbers (p_s, q_s) of Theorem-1, whose sum is equal to the even number $2n$, are plotted in orthocanonical system of axes $p_s\, q_s$, they belong to a straight line which forms 45 degrees to both axes p_s and q_s.

PROOF

From Analytical Geometry we know that in a *x*-*y* system, every straight line intersecting the *x* axis at the point A(α,0) and the *y* axis at the point B(0,β) satisfies the equation: $x/\alpha + y/\beta = 1$. In our specific case, if we select the *x* axis to represent the term $p_s \in \mathbb{N}_0$, while *y* axis to represent the term $q_s \in \mathbb{N}_0$, then obviously it holds $\alpha = \beta = 2n$ (see **Figure 1**). It is also obvious that the (2*n*+1) integers of the interval [0,α] and the corresponding (2*n*+1) integers of the interval [0,β], of which sum equals to 2*n*, correspond to (2*n*+1) discrete points along the straight segment AB. Given that we are interested in only the odd numbers $p_s, q_s > 3$ which satisfy the relation (1), without necessarily being prime numbers, we must leave out the three pairs being closest to the *x* axis: (2*n*,0), (2*n*-1,1), (2*n*-2,2), as well as the three pairs closest to the *y* axis: (0,2*n*), (1,2*n*-1), (2,2*n*-2). These six points are denoted into Figure 1 by the symbol (×). Therefore, the number of candidate points for further examination is $n_s = \lfloor (n-1)/2 \rfloor$, two of which coincide with the ends C and D (note that the set of *all* discrete points/pairs that correspond to even and odd integers is 2*n*-5).

A better representation, especially for various small numbers is detailed in **Table 1**. In full agreement with the immediately above, we observe that the middle M appears to be among the candidate pairs only when the number *n* is odd, i.e. for the even numbers: 6, 10, 14, ..., 50, and so on. But if we consider the middle M of segment CD, each pair P(p_s, q_s) has a corresponding equivalent pair being represented by the symmetric point P'(q_s, p_s) of P with respect to M. If we want to exclude the repetition of a pair (p_s, q_s), then we can restrict our attention only to half of CD, e.g. the segment DM. Concerning the point M, it satisfies the relation $p_s + q_s = 2n$ only when the number *n* is odd. ■

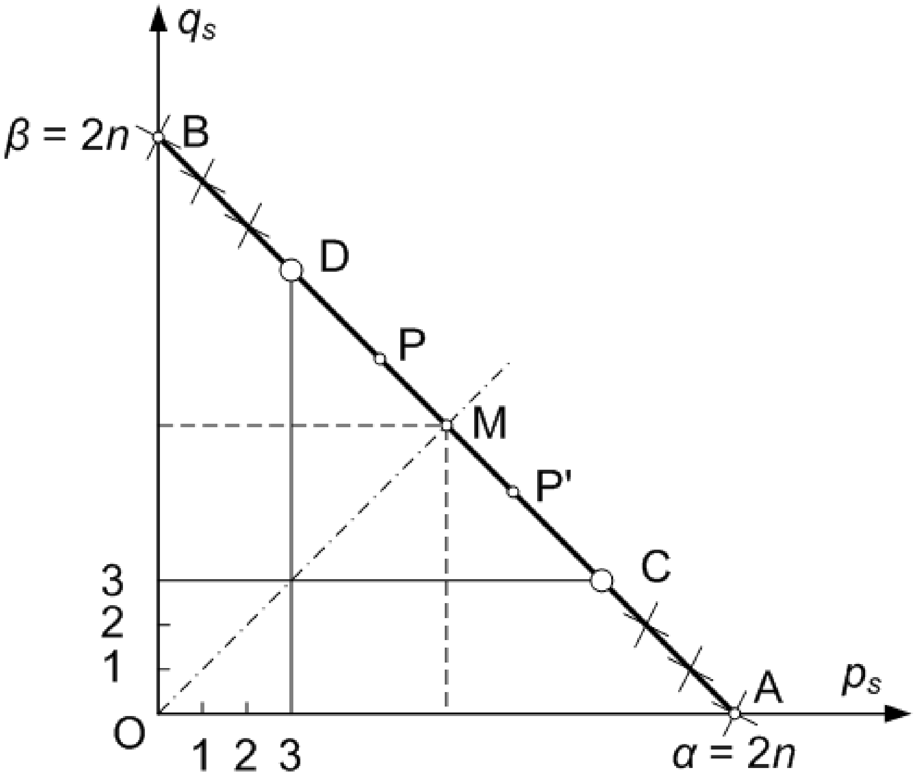


Figure 1: Diagram showing the interval CD on which lie the arranged pairs (p_s, q_s) which satisfy the relationship $p_s + q_s = 2n,\ n > 2$. To avoid repeated pairs, we work only on the part DM, where M denotes the common mid-point of the straight line segments AB and CD.

Table 1: Representative way of decomposing even numbers in a sum of two odd ones. Every even integer has been put in the form $2n \equiv 2(v+2)$, and is produced by the sum of red odd numbers $(2v+1)$ of the horizontal axis and the corresponding red values in the vertical axis. The colors displayed in the order green, blue and magenta correspond to the even numbers of the form $6n$, $6n+2$ and $6n-2$, respectively ($n \in \mathbb{N}$).

2v+1	**v**													
25	**12**	28	30	32	34	36	38	40	42	44	46	48	50	
23	**11**	26	28	30	32	34	36	38	40	42	44	46	48	
21	**10**	24	26	28	30	32	34	36	38	40	42	44	46	
19	**9**	22	24	26	28	30	32	34	36	38	40	42	44	
17	**8**	20	22	24	26	28	30	32	34	36	38	40	42	
15	**7**	18	20	22	24	26	28	30	32	34	36	38	40	
13	**6**	16	18	20	22	24	26	28	30	32	34	36	38	
11	**5**	14	16	18	20	22	24	26	28	30	32	34	36	
9	**4**	12	14	16	18	20	22	24	26	28	30	32	34	
7	**3**	10	12	14	16	18	20	22	24	26	28	30	32	
5	**2**	8	10	12	14	16	18	20	22	24	26	28	30	
3	**1**	6	8	10	12	14	16	18	20	22	24	26	28	
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**v**
		3	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**2v+1**

Theorem-3: Suppose the point P in Figure 1 starts from point D and moves towards M traversing all distinct pairs (p_s, q_s) on the straight segment DM with $p_s \le q_s$. In this movement, two sets A and B of odd natural numbers are created, of which the first (A) is formed by the values of p_s while the second (B) from the values of q_s. Both these two sets have the same cardinality: $n_s = \lfloor (n-1)/2 \rfloor$, which equals the number of distinct pairs that satisfy the relationship $p_s + q_s = 2n$. Assuming that the number of prime numbers in sets A and B are α_1 and β_1, respectively, and that the distribution of primes along the arithmetic line is completely *random*, show that the number of pairs of primes (p, q) that satisfy the relation (1), is approximated by:

$$n_p = (\alpha_1 \beta_1)/n_s \qquad (12).$$

PROOF

According to the theory of probability, in each pair (p_s,q_s) selected from the set of $n_s = \lfloor (n-1)/2 \rfloor$ elements, there are four possible events as shown in **Table 2**.

Table 2: All possible combinations for the formation of a pair (p_s, q_s) from the columns A and B.

Column A	Column B	Number of pairs	
Prime	Prime	$n_{pp} = (\alpha_1 \beta_1)/n_s$	(13a)
Prime	Composite	$n_{pc} = [\alpha_1 (n_s - \beta_1)]/n_s$	(13b)
Composite	Prime	$n_{cp} = [(n_s - \alpha_1)\beta_1]/n_s$	(13c)
Composite	Composite	$n_{cc} = [(n_s - \alpha_1)(n_s - \beta_1)]/n_s$	(13d)
Sum of pairs		n_s	

If we select a *random pair* (p_s, q_s), where $p_s \in A \wedge q_s \in B$, the probability p_s be prime is $P_A = \alpha_1/n_s$, while the probability q_s be prime is $P_B = \beta_1/n_s$. Since the first event (p_s=prime) is entirely independent on the second event (q_s=prime), the probability P_{prime} of the intersection of these two independent events equals to the product of their probabilities, that is:

$$P_{prime} = P_A \cdot P_B = (\alpha_1/n_s)\cdot(\beta_1/n_s) = (\alpha_1 \beta_1)/n_s^2 \qquad (14)$$

Since the probability P_{prime} on the sample of n_s pairs is known, the number of pairs of prime numbers (p, q) will be also known and will be approximated by the relationship:

$$n_p = P_{prime} \cdot n_s = \left[(\alpha_1 \beta_1)/n_s^2\right] \cdot n_s = (\alpha_1 \beta_1)/n_s \qquad (15)$$

The relationship (15) completes the proof of Theorem-3. ■

It is noted that generally, probabilistic analyzes are conducted under conditions of uncertainty. When talking about probability, we refer to the realization of one event in relation to other possible events. Obviously, the possibility to verify the Goldbach conjecture on the number of n_p pairs, belongs to the first category (13a).

Remark

The distribution of prime numbers on the numerical line is not accidental but strictly predetermined. It is therefore a purely deterministic phenomenon. The prime numbers are in a predetermined position, waiting to be discovered.

Today, we know the number and distribution of prime numbers for a very large number of natural numbers. Therefore, we know the number of primes, both in column A (α_1) and in column B (β_1), arranged in (n_s) pairs in which the even numbers are decomposed on the basis of equations (9) and (10). So, we are able to rigorously examine whether the predetermined distribution of prime numbers is such as to ensure the validity of probabilistic relationship (13a), “prime-to-prime” in a representative sample (e.g. the 12,000 numbers) the delicate area of small even numbers, where the number of pairs n_s is small.

As an example, **Table 3** represents the way in which we decompose a triple (triad) of even numbers (6*n*-2, 6*n*, 6*n*+2), where *n*=15, in the sample of 12,000.

To facilitate discrimination of pairs “prime-to-prime” we encode the composite numbers with *gray* color, the odd (prime numbers) of the form 6λ-1 with *turquoise* color, the odd (prime numbers) of the form 6λ+1 with *tile* color, and finally the first number 3 with *yellow* color.

Since even numbers of the form 6*n* are formed on the basis of equation (9b) and (9b′), while the even numbers of both the form 6*n*-2 and the form 6*n*+2 are formed only by an equation [(9a) and (9c), respectively], an *asymmetry* appears in the number of ordered pairs that verify (13a), as shown in **Table 3**.

Table 3: Example for the decomposition of a triad of numbers (6*n*-2, 6*n*, 6*n*+2), for *n*=15, in sum of two odd numbers.

6*n*-2 = 88

	COLUMN A	COLUMN B
1	3	85
2	5	83
3	7	81
4	9	79
5	11	77
6	13	75
7	15	73
8	17	71
9	19	69
10	21	67
11	23	65
12	25	63
13	27	61
14	29	59
15	31	57
16	33	55
17	35	53
18	37	51
19	39	49
20	41	47
21	43	45

Sum of pairs that fulfill Goldbach conjecture: **4**

Sum of Pairs: 21 (Equation 9a)

6*n* = 90

	COLUMN A	COLUMN B
1	3	87
2	5	85
3	7	83
4	9	81
5	11	79
6	13	77
7	15	75
8	17	73
9	19	71
10	21	69
11	23	67
12	25	65
13	27	63
14	29	61
15	31	59
16	33	57
17	35	55
18	37	53
19	39	51
20	41	49
21	43	47
22	45	45

Sum of pairs that fulfill Goldbach conjecture: **9**

Sum of Pairs: 22 (Equations 9b and 9b´)

6*n*+2 = 92

	COLUMN A	COLUMN B
1	3	89
2	5	87
3	7	85
4	9	83
5	11	81
6	13	79
7	15	77
8	17	75
9	19	73
10	21	71
11	23	69
12	25	67
13	27	65
14	29	63
15	31	61
16	33	59
17	35	57
18	37	55
19	39	53
20	41	51
21	43	49
22	45	47

Sum of pairs that fulfill Goldbach conjecture: **4**

Sum of Pairs: 22 (Equation 9c)

Composite number
Odd prime in the form (6λ-1)
Odd prime in the form (6λ+1)
Prime number 3

(Sum of pairs fulfilling Golbach conjecture for the entire triad: 4+9+4 = 17)

Figure 2 shows, with a remarkable correlation, the expected 'coincidence' of the curve which represents the number of cases that verify the Goldbach conjecture, only for the even numbers of the form 6*n* (on one hand) and the sum of cases for the even numbers in the form 6*n*-2 and 6*n*+2 (on the other hand).

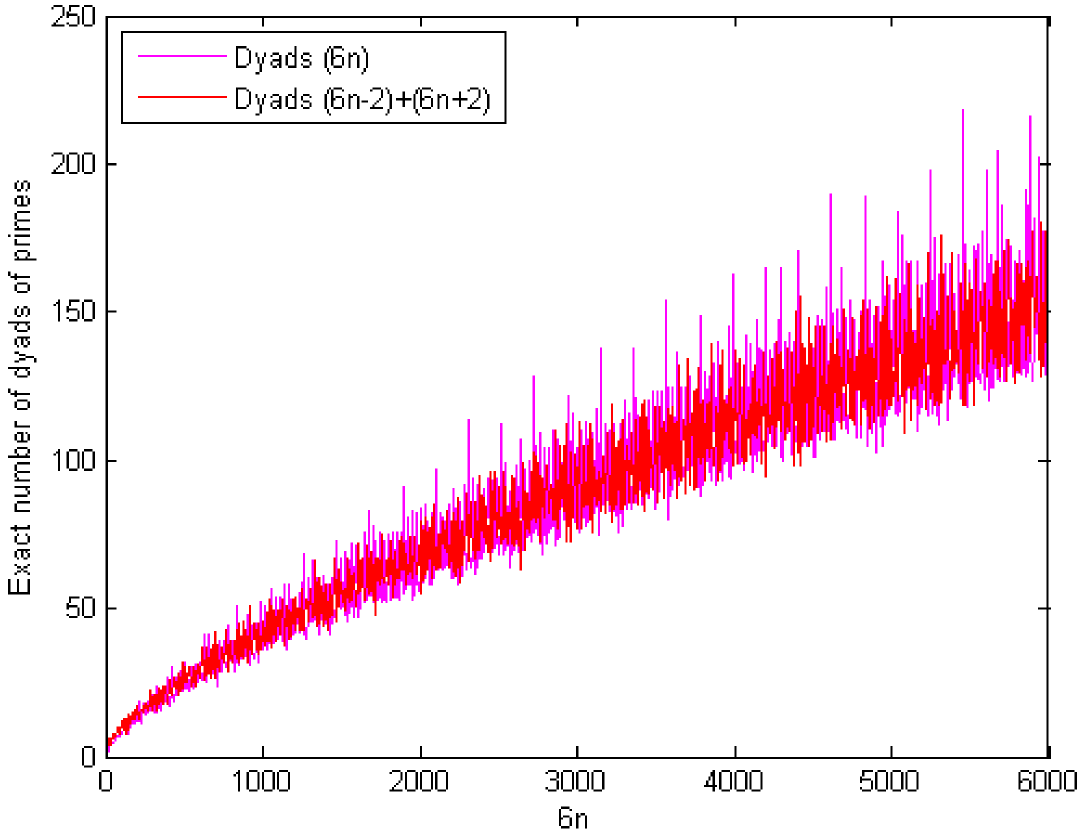


Figure 2: Number of pairs (dyads) of prime numbers that fulfill Goldbach Conjecture for the lowest 6000 numbers, in two characteristic categories [magenta line: dyads 6*n*, red line: dyads (6*n*-2) and (6*n*+2)].

Figure 3 shows, separately for each form of the triad of even numbers (6*n*, 6*n*-2, 6*n*+2), the number of pairs (dyads) of prime numbers that verify the Goldbach conjecture. As previously observed, even in this case the anticipated 'coincidence' by equations (9a) and (9c) occurs.

To remove the above asymmetry, due to the four relations (9), which in turn reflect the three types of even numbers, we transform (13a) by introducing three weights whose sum is equal to 3 (the weight of 6*n* is twice the others), and therefore adapt in more detail as follows:

For 6*n*-2 : $$n_{pp}=3/4(\alpha_1\beta_1/n_s) \quad (14a)$$

For 6*n* : $$n_{pp}=3/2(\alpha_1\beta_1/n_s) \quad (14b)$$

For 6*n*+2 : $$n_{pp}=3/4(\alpha_1\beta_1/n_s) \quad (14c)$$

Therefore, if we assume that the Goldbach conjecture is not verified for some even numbers, these cases should be essentially searched within the even numbers of the form 6*n*-2 and 6*n*+2, which have the smallest values of verification, (14a) and (14c).

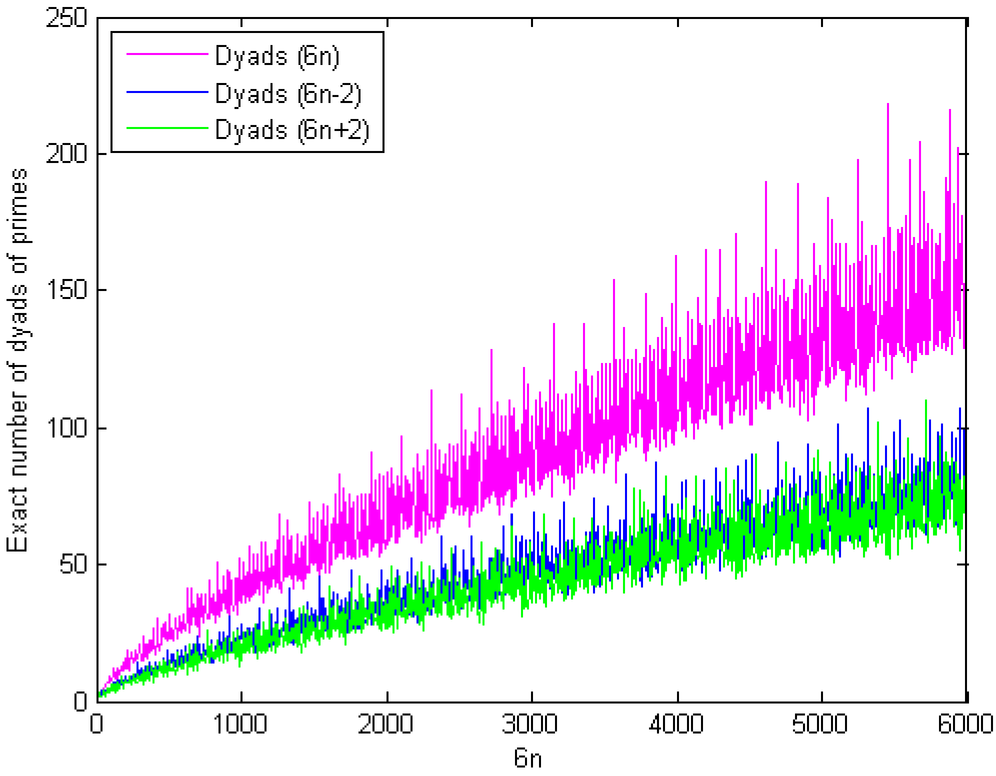


Figure 3: Number of dyads of prime numbers that fulfill Goldbach Conjecture for the smallest 6000 numbers, in three characteristic categories according to the cell (6*n*-2, 6*n*, 6*n*+2).

3. The distribution of prime numbers

It is known, by virtue of the Prime Number Theorem (PNT) [see Appendix A] that, as we move to larger numbers N, the density of primes $1/\log N$ gradually decreases. In the following, the symbol $\log x$ [or $\log(x)$] is equivalent to the symbol $\ln x$ [or $\ln(x)$] corresponding to the Neperian (natural) logarithm of base e.

Here we should clarify that, despite the gradual decrease in the density of the primes (see percentage in the fifth column of **Table 4**), the odd numbers starting from the number 3 (from which we start the column A) until the even number of $6n+2 = 92$ (see right column of Table 3) is 44. Within these 44 odd numbers, 23 of primes are contained. Thus the percentage of primes in the ordered pairs of odd number in which the number 92 is analyzed will be $23\times44/100 = 52.3\%$. This implies that *in any distribution of prime and composite numbers*, in the ordered in pairs in which the even numbers between 6 and 92 are analyzed, the value of any *prime-to-prime* event will be larger than 1. This means that the Goldbach Conjecture is a priori verified for the even numbers with $n \leq 15$.

Based on the default deterministic creation of ordered pairs, to ensure the sum of all dyads be $6n$-2, and given the distribution of prime numbers in between 3 and $6\times15-2 = 88$, the number n_{pp} of verifications is illustrated in **Table 4**, where we present the data α_1, β_1 and n_s, involved in (14a) to calculate n_{pp}, for even numbers of the form $6n$-2 with n varying from 2 to 15.

A first observation arising from the data of **Table 4** is that the density and distribution of primes in that interval is such as to ensure in a great approach, the coincidence of the values derived from (14a) with actual checks.

The density and distribution of prime numbers also ensure that α_1 and β_1 increase in such a way that the product ($\alpha_1\beta_1$) to increase at a faster rate than the rate at which increases the number of ordered pairs n_s. Direct result of the distribution of prime numbers is that as we move to larger even numbers, the verification of conjecture Goldbach, is *constantly increasing*, and is increasingly removed from the value 1 required by the conjecture.

Table 4: Estimation of the number of verifications $\boldsymbol{n_{pp}}$ concerning Goldbach Conjecture based on Eq(14a) compared with reality

n	Number (6n-2)	α_1	β_1	Percentage %	n_s	Number of verifications $\boldsymbol{n_{pp}}$ of Goldbach Conjecture Equation (14a)	Real $\boldsymbol{n_{pp}}$
2	10	2	2	100	2	1.5	2
3	16	3	2	80	3	1.5	2
4	22	4	4	80	5	2.4	3
5	28	5	3	67	6	1.9	2
6	34	6	5	69	8	2.8	4
7	40	7	4	61	9	2.3	3
8	46	8	6	64	11	3.3	4
9	52	8	6	58	12	3.0	3
10	58	9	7	57	14	3.4	4
11	64	10	7	57	15	3.5	5
12	70	10	8	53	17	3.5	5
13	76	11	9	55	18	4.1	5
14	82	12	10	55	20	4.5	5
15	88	13	9	52	21	4.2	4

Therefore, besides the percentage (%) which, *in the test sample*, ensures a priori verification of Goldbach conjecture, it is also the distribution of prime numbers, and more *especially the distribution of the primes*, which ensures the appropriate values in α_1 and β_1 so that the results derived from (14) to be in a much closed agreement with the actual verification. This fact urges us to investigate whether the distribution of prime numbers, in a larger and more representative sample of even numbers, where the percentage (%) of the primes with its continuous reduction, goes down below 50%, shall ensure, by itself, the values for α_1 and β_1 which are also in agreement with the actual verifications.

Due to the reduced space, in Table 5 the following quantities are randomly recorded;

-the values of α_1 and β_1,

-the number of columns n_s

-and the verifications n_{pp} based on equations (14),

for several values of the even numbers in the form 6n-2, 6n and 6n+2 for n=20 until n=2000, which correspond to the even numbers 118, 120, 122 until 11998, 12000 and 12002.

The even number 12002 includes 6001 odd numbers and 1437 primes. Therefore, the percentage (in %) of the primes included in the 6001 odd numbers by which we create the ordered pairs, will be 1437/6001×100=23.9% Obviously, this percentage does not ensure the a-priori verification of Goldbach Conjecture.

Table 5: Estimation of the number of verifications n_{pp} of Goldbach Conjecture based on relations (14) for several values of even numbers in the form 6*n*-2, 6*n*, and 6*n*+2

For n=20	α_1	β_1	n_s	n_{pp} [Eq(14)]
6n-2=118	16	14	29	$3/4\alpha_1\beta_1/n_s$ = 3/4*16*14/29 = 5.8
6n=120	16	13	29	$3/2\alpha_1\beta_1/n_s$ = 3/4*16*13/29 = 10.7
6n+2=122	17	13	30	$3/4\alpha_1\beta_1/n_s$ = 3/4*17*13/30 = 5.5
				Sum of triad = 21.9
For n=40	α_1	β_1	n_s	
6n-2=238	29	21	59	$3/4\alpha_1\beta_1/n_s$ = 3/4*29*21/59 = 7.7
6n=240	29	21	59	$3/2\alpha_1\beta_1/n_s$ = 3/2*29*21/59 = 15.5
6n+2=242	29	21	60	$3/4\alpha_1\beta_1/n_s$ = 3/4*29*21/60 = 7.6
				Sum of triad = 30.8
For n=60	α_1	β_1	n_s	
6n-2=358	40	31	89	$3/4\alpha_1\beta_1/n_s$ = 3/4*40*31/89 = 10.4
6n=360	40	30	89	$3/2\alpha_1\beta_1/n_s$ = 3/2*40*30/89 = 20.2
6n+2=362	41	30	90	$3/4\alpha_1\beta_1/n_s$ = 3/4*41*30/90 = 10.2
				Sum of triad = 40.8
For n=80	α_1	β_1	n_s	
6n-2=478	51	40	119	$3/4\alpha_1\beta_1/n_s$ = 3/4*51*40/119 = 12.9
6n=480	51	39	119	$3/2\alpha_1\beta_1/n_s$ = 3/2*51*39/119 = 25.1
6n+2=482	52	39	120	$3/4\alpha_1\beta_1/n_s$ = 3/4*54*41/120 = 12.7
				Sum of triad = 50.7
For n=100	α_1	β_1	n_s	
6n-2=598	61	46	149	$3/4\alpha_1\beta_1/n_s$ =3/4*61*46/149=14.1
6n=600	61	46	149	$3/2\alpha_1\beta_1/n_s$ =3/2*61*46/149=28.2
6n+2=602	61	47	150	$3/4\alpha_1\beta_1/n_s$ =3/4*61*47/150=14.3
				Sum of triad = 56.6
For n=200	α_1	β_1	n_s	
6n-2=1198	108	88	299	$3/4\alpha_1\beta_1/n_s$ = 3/4*108*88/299 = 23.8
6n=1200	108	87	299	$3/2\alpha_1\beta_1/n_s$ = 3/2*108*87/299 = 47.1
6n+2=1202	109	87	300	$3/4\alpha_1\beta_1/n_s$ = 3/4*109*87/300 = 23.7
				Sum of triad = 94.6
For n=300	α_1	β_1	n_s	
6n-2=1798	153	124	449	$3/4\alpha_1\beta_1/n_s$ =3/4*153*124/449=31.7
6n=1800	153	124	449	$3/2\alpha_1\beta_1/n_s$ =3/2*153*124/449=63.4
6n+2=1802	153	124	450	$3/4\alpha_1\beta_1/n_s$ =3/4*153*124/450=31.6
				Sum of triad = 126.7
For n=400	α_1	β_1	n_s	
6n-2=2398	195	160	599	$3/4\alpha_1\beta_1/n_s$ = 3/4*195*160/599 = 39.0
6n=2400	195	160	599	$3/2\alpha_1\beta_1/n_s$ = 3/2*195*160/599 = 78.1
6n+2=2402	196	161	600	$3/4\alpha_1\beta_1/n_s$ = 3/4*196*161/600 = 39.4
				Sum of triad = 156.5
For n=500	α_1	β_1	n_s	
6n-2=2998	238	191	749	$3/4\alpha_1\beta_1/n_s$ = 3/4*238*191/749 = 45.5
6n=3000	238	190	749	$3/2\alpha_1\beta_1/n_s$ = 3/2*238*191/749 = 91.0
6n+2=3002	238	191	750	$3/4\alpha_1\beta_1/n_s$ = 3/4*238*191/450 = 45.4
				Sum of triad = 181.9
For n=750	α_1	β_1	n_s	
6n-2=4498	333	276	1124	$3/4\alpha_1\beta_1/n_s$ = 3/4*333*276/1124 = 61.3
6n=4500	333	276	1124	$3/2\alpha_1\beta_1/n_s$ = 3/2*333*276/1124 = 126.6
6n+2=4502	334	276	1125	$3/4\alpha_1\beta_1/n_s$ = 3/4*334*276/1122 = 61.3
				Sum of triad = 249.2
For n=1000	α_1	β_1	n_s	
6n-2=5998	429	354	1499	$3/4\alpha_1\beta_1/n_s$ = 3/4*429*354/1499 = 76
6n=6000	429	354	1499	$3/2\alpha_1\beta_1/n_s$ = 3/2*429*354/1499 = 152
6n+2=6002	430	354	1500	$3/4\alpha_1\beta_1/n_s$ = 3/4*430*354/1500 = 76
				Sum of triad = 304
For n=1500	α_1	β_1	n_s	
6n-2=8998	609	506	2249	$3/4\alpha_1\beta_1/n_s$ = 3/4*609*506/2249 = 102.8
6n=9000	609	506	2249	$3/2\alpha_1\beta_1/n_s$ = 3/2*609*506/2249 = 205.5
6n+2=9002	609	507	2250	$3/4\alpha_1\beta_1/n_s$ = 3/4*609*507/2250 = 102.7
				Sum of triad = 411
For n=2000	α_1	β_1	n_s	
6n-2=11998	782	655	2999	$3/4\alpha_1\beta_1/n_s$ = 3/4*782*655/2999 = 128.1
6n=12000	782	655	2999	$3/2\alpha_1\beta_1/n_s$ = 3/2*782*655/2999 = 256.2
6n+2=12002	782	655	3000	$3/4\alpha_1\beta_1/n_s$ = 3/4*782*655/3000 = 128.0
				Sum of triad = 512.3

The following **Table 6** includes both the actual verifications of conjecture Goldbach, in red, and the verifications based on the formula (14); the latter are due to the distribution of prime numbers (α_1,β_1), in columns A and B, in which each even number is decomposed according to (9) and (10), in blue (already included in Table 5). From the results in Table 6, we can notice that the distribution of prime numbers is enough to ensure the values of the verification under (14) to be in close accordance with the reality, as clearly shown in **Figure 4** (for these particular results only).

Table 6: Real verifications of Goldbach Conjecture-CG (in red colour) compared with the verifications based on relation (14) (in blue colour).

n	**n=20**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	118	120	122	
Verifications of GC (Facts)	**6**	**12**	**4**	**22**
Verifications using Eq(14)	**5.8**	**10.7**	**5.5**	**21.9**
n	**n=40**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	238	240	242	
Verifications of GC (Facts)	**9**	**18**	**7**	**34**
Verifications using Eq(14)	**7.7**	**15.5**	**7.6**	**30.8**
n	**n=60**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	358	360	362	
Verifications of GC (Facts)	**10**	**22**	**7**	**39**
Verifications using Eq(14)	**10.4**	**20.2**	**10.2**	**40.8**
n	**n=80**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	478	480	482	
Verifications of GC (Facts)	**11**	**29**	**10**	**50**
Verifications using Eq(14)	**12.9**	**25.1**	**12.7**	**50.7**
n	**n=100**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	598	600	602	
Verifications of GC (Facts)	**15**	**32**	**11**	**58**
Verifications using Eq(14)	**14.1**	**28.2**	**14.3**	**56.8**
n	**n=200**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	1198	1200	1202	
Verifications of GC (Facts)	**24**	**54**	**19**	**97**
Verifications using Eq(14)	**23.8**	**47.1**	**23.7**	**94.6**
n	**n=300**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	1798	1800	1802	
Verifications of GC (Facts)	**27**	**74**	**31**	**132**
Verifications using Eq(14)	**31.7**	**63.4**	**31.6**	**126.7**
n	**n=400**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	2398	2400	2402	
Verifications of GC (Facts)	**37**	**90**	**37**	**164**
Verifications using Eq(14)	**39.0**	**78.1**	**39.4**	**156.5**
n	**n=500**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	2998	3000	3002	
Verifications of GC (Facts)	**46**	**103**	**39**	**188**
Verifications using Eq(14)	**45.5**	**91.0**	**45.4**	**181.9**
n	**n=750**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	4498	4500	4502	
Verifications of GC (Facts)	**64**	**138**	**52**	**254**
Verifications using Eq(14)	**61.3**	**126.6**	**61.3**	**249.2**
n	**n=1000**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	5998	6000	6002	
Verifications of GC (Facts)	**72**	**179**	**62**	**313**
Verifications using Eq(14)	**76.0**	**152.0**	**76.0**	**304.0**
n	**n=1500**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	8998	9000	9002	
Verifications of GC (Facts)	**101**	**243**	**110**	**454**
Verifications using Eq(14)	**102.8**	**205.5**	**102.7**	**411.0**
n	**n=2000**			
Type of even number	6n-2	6n	6n+2	**Sum**
Even number	11998	12000	12002	
Verifications of GC (Facts)	**144**	**303**	**115**	**562**
Verifications using Eq(14)	**128.1**	**256.2**	**128.0**	**512.3**

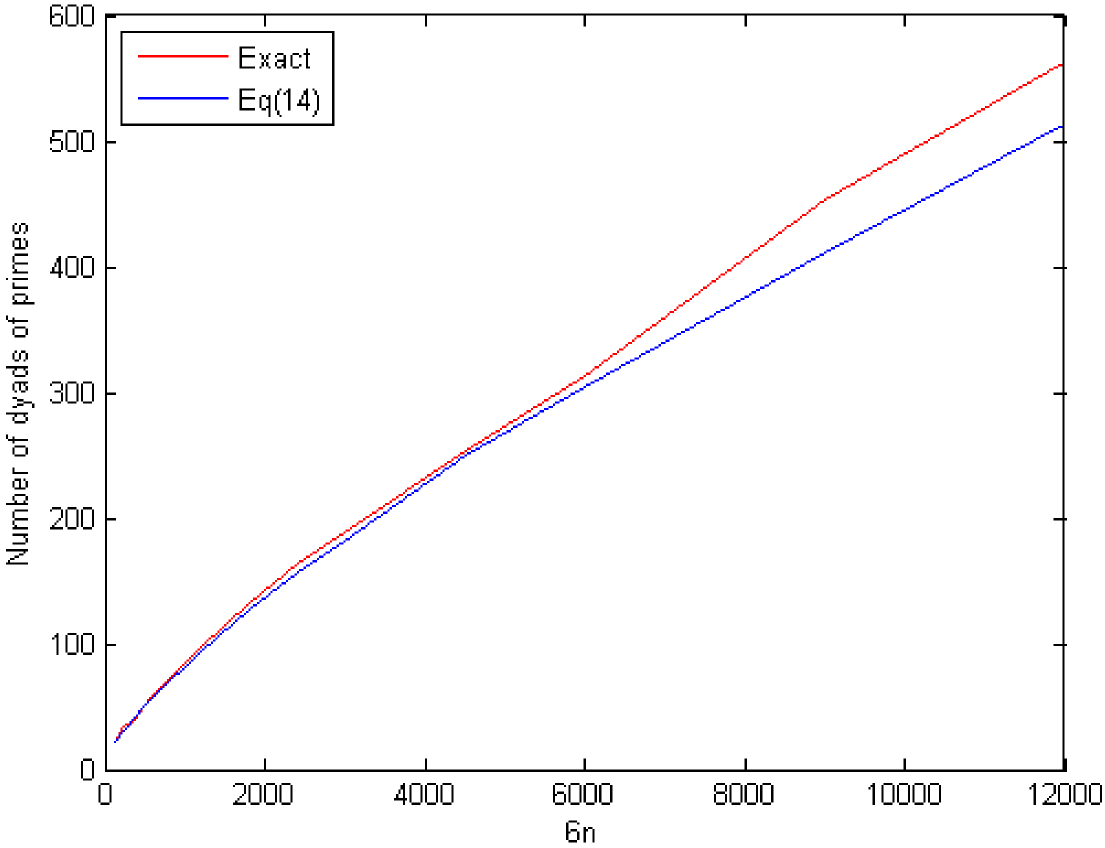


Figure 4: Number of pairs (dyads) of primes that fulfill Goldbach Conjecture for the smallest 12000 numbers (the red line represents the real number of dyads of primes that fulfill Goldbach Conjecture, while the blue line corresponds to the relation (14)). The graph is based on Table 6 only.

Also in this second sample of 12,000 numbers (6 × 2000 = 12000) the growth rate of α_1 and β_1 is greater than the growth rate of the ordered pairs n_s, so the number of verifications n_{pp} of (14) increases continuously and removes from the unity ("1"), as is the case with actual verifications.

Comment: Among a probabilistic experiment and the creation of ordered pairs when decomposing an even number in the sum of two odd numbers, there is a *substantial difference*. In a probabilistic experiment, the *likelihood that one out of the four possibilities* of Table 2 occurs is a matter of coincidence, or luck. The order in which the various possibilities they appear in the potential space is completely random. There is *no rule which specifies the type of possibility* in a particular test. Completely different is the case of an even number's decomposition in a sum of two odd numbers with the creation of ordered pairs, where the potential for each of the pairs are arranged, is *strictly predetermined*. Therefore the analysis of an even number in sum of two odd numbers with the creation of ordered pairs is *purely deterministic*.

4. The Critical Question

4.1 General

A very important process in mathematics is the generalization. We take a problem and examine its behavior in a limited area, and then the conclusions arising from the study of this sample, we try to expand to cover larger areas.

We studied the behavior of a sample of all the even numbers up to 12,000 and found that the number of pairs that verify the conjecture of Goldbach, have a *clear upward trend*, which has a close relationship with the probabilistic equation (14a).

Equation (14a) imposes no restriction on the size of the sample.

Neither the deterministic decomposition of an even number in odd pairs, based on equations (9) and (10), has a similar problem.

The *crucial question* that arises here is, whether for sufficiently large even numbers the distribution of primes in the arithmetic line continues to be such as to ensure similar behavior in deterministic ordered pairs in which sufficiently large even numbers are analyzed, to that even numbers of the relatively small sample that we previously looked, so as to *legitimize* to generalize to all the even numbers.

Our belief is strongly 'YES', since, as we explain below, the creation of composite numbers on the numeric (arithmetic) line, determined by strictly defined rules, *uniform for the whole crowd of natural numbers*. Hence the distribution of prime numbers, which is formed by the relationship

Prime numbers = Natural numbers – Composite numbers

to follow the same inevitability and it is strictly prescribed.

The view that the distribution of primes along the arithmetic line is random and chaotic is wrong and misleading.

4.2 Basic rule for the creation of Composite numbers

As the prime numbers (except 2) are odd, we will seek the prime numbers in the set of odd numbers $\mathbb{N}_1 = \{\chi/\chi = 2n+1,\ n \in \mathbb{N}\}$. This set which can be written as $\mathbb{N}_1 = \{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \ldots\}$ is a countable set equivalent to $\mathbb{N}$.

The two factors in the product ($\beta.\upsilon$) in which every odd composite is analyzed, should be *odd*, because only the product of odd numbers results in an odd number. Therefore, the complex numbers can be partitioned into subsets of the form:

$$E_{\rm i} = \{\chi/\chi = (2n+1) \times [(2n+1), (2n+3),\ (2n+5), (2n+7), \ldots (2n+2k+1) \ldots]\} \qquad (15)$$

Such composite *odd* numbers are those of **Table 7**, where we see that the smallest of the infinite odd composite numbers generated in each row is the product of the *first number in the series, by itself*, i.e. 3×3, 5×5, 7×7, 9×9, 11×11, …

Table 7: Decomposition of Composite Numbers (CN) in a product of two odd numbers

$E_3 =$	3 ×	3	5	7	9	11	13	15	17	19	21	…	$2n+1$	…
$E_5 =$	5 ×	5	7	9	11	13	15	17	19	21	23	…	$2n+1$	…
$E_7 =$	7 ×	7	9	11	13	15	17	19	21	23	25	…	$2n+1$	…
$E_9 =$	9 ×	9	11	13	15	17	19	21	23	25	27	…	$2n+1$	…
$E_{11} =$	11×	11	13	15	17	19	21	23	25	27	29	…	$2n+1$	…
$E_{13} =$	13×	13	15	17	19	21	23	25	27	29	31	…	$2n+1$	…

In each of the above infinite series of composite odds, which are multiples of 3, 5, 7, 9, etc., we have the first number to be a "*square*" (3^2, 5^2, 7^2, …), whereas everyone else is a "*rectangular*". From these squares, the 'squares' of prime numbers cannot be transformed into 'rectangles'.

These squares will be called the "*original squares*". Unlike the squares of prime numbers, squares of composite numbers, such as the 9×9, can be also transformed into rectangles 3×27, which we will call "*abusive squares*".

Easily see that the elements of the set E_3=3×(3,5,7,…2n+1) in the first row of **Table 7**, which are multiples of 3, is a *periodic phenomenon* in the set of integer numbers and may be derived from formula (16a):

$$\boldsymbol{E_3: \quad (CN)_3 = x = 9 + 6\mu = 3^2 + 2\times 3\mu}, \quad \mu=0,1,2,\ldots \tag{16a}$$

For $\mu = 0$: $x = 9 = 3\times3$
For $\mu = 1$: $x = 15 = 3\times5$
For $\mu = 2$: $x = 21 = 3\times7$
For $\mu = 3$: $x = 27 = 3\times9$, e.t.c.

As a periodic phenomenon, it can also be represented as a transverse wave, as follows. In a rectangular system of axes xOy, we identify the straight line that represents the set of natural numbers, with the axis Ox. For the sake of clarity of shape, we ignore the even numbers, and we indicate only the odd ones. If you construct a transverse wave that *starts at the number 9, of wavelength* $\boldsymbol{\lambda=12}$ *units of length* in the set of natural numbers $\mathbb{N}$ or $\boldsymbol{\lambda' = 12/2 = 6}$ units of length in the set of odd numbers $\mathbb{N}_1$, the Composite Numbers $(CN)_3$ described by equation (16a) coincide with the intersections of the transverse wave and the axis Ox (zero point deviation from the axis Ox in the direction of y axis, as shown in Figure 5b). The appearance of composite numbers that are multiples of 3 on the arithmetic line is the most common of any other odd number; it divides the sum of odd pairs $\mathbb{N}_1$ in infinite pairs of consecutive odd numbers, starting with the square of 3 and reaching the utmost ends of the arithmetic line of Figure 5a. These successive pairs, of which, as we have explained, their first number is of the form (6λ-1) while the second form (6λ+1), will be either primes or multiple of primes > 3. All these pairs are candidates to become the ***Twin Primes***[2], as long as none of the two numbers of the pair is crossed by a subsequent wave, thus remain to be primes, as shown in Figure 5b.

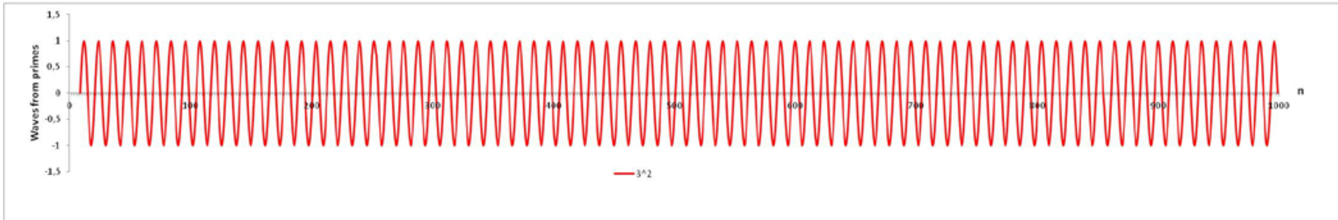
But also the elements of the set E_5 = 5×(5, 7, 9, 11, 13, 15, 17, 19, 21 …, 2κ+1…) constitute a periodic phenomenon in the set of numbers starting from 25, with a wavelength λ = 45-25 = 20 or λ' = 20/2 = 10 in the set $\mathbb{N}_1$. The formula (16b) gives the multiples of 5 that are transverse sections of the wave with the axis Ox, which apparently are composite numbers

$$E_5: \quad (CN)_5 = 25 + 10\mu = 5^2 + 2\times5\mu, \quad \mu=0,1,2,\ldots \tag{16b}$$

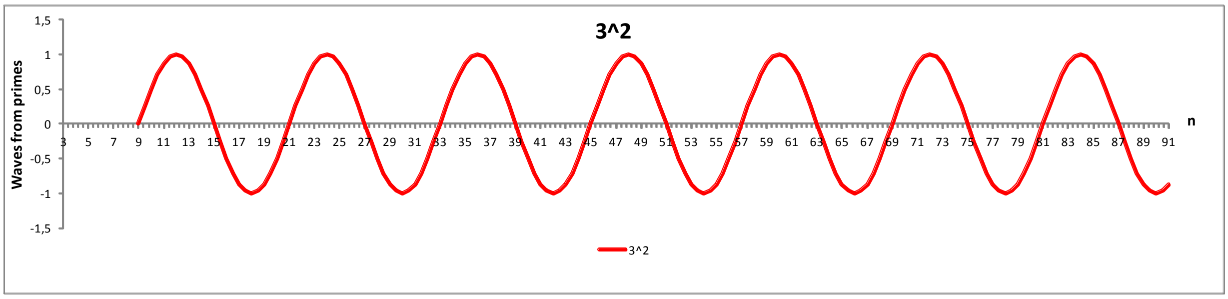
Quite similarly, The elements of E_7 = 7×(7, 9, 11,13, 15, 17, 19, 21, 23…,2κ+1…) are also periodic at a frequency 14μ, where $\mu \in \mathbb{N}_1$. The first term is 49 and the formula that gives the multiples of 7 is

$$E_7: \quad (CN)_7 = 49 + 14\mu = 7^2 + 2*7\mu, \quad \mu=0,1,2,\ldots \tag{16c}$$

[2] A **twin prime** is a prime number that differs from another prime number by two, for example the twin prime pair (3, 5).



(a)



(b)

Figure 5: Formation of composite numbers starting with the square of 3 until (a) 1000 and (b) detail until 91.

Generalizing, we get a similar rationale in the calculation of the formula which gives us all the multiples of the odd number $(2n+1)$:

$$E_{(2n+1)}: \qquad (CN)_{(2n+1)} = (2n+1)^2 + 2(2n+1)\mu\,, \quad \mu=0,1,2,\ldots \tag{16d}$$

From (16d) we conclude that, as we move to increasingly odd numbers, so the composite numbers (CN) set up by their respective primes are ever fewer because, firstly the start of a wave is shifted to the right of the numerical line, and secondly the wavelength increases.

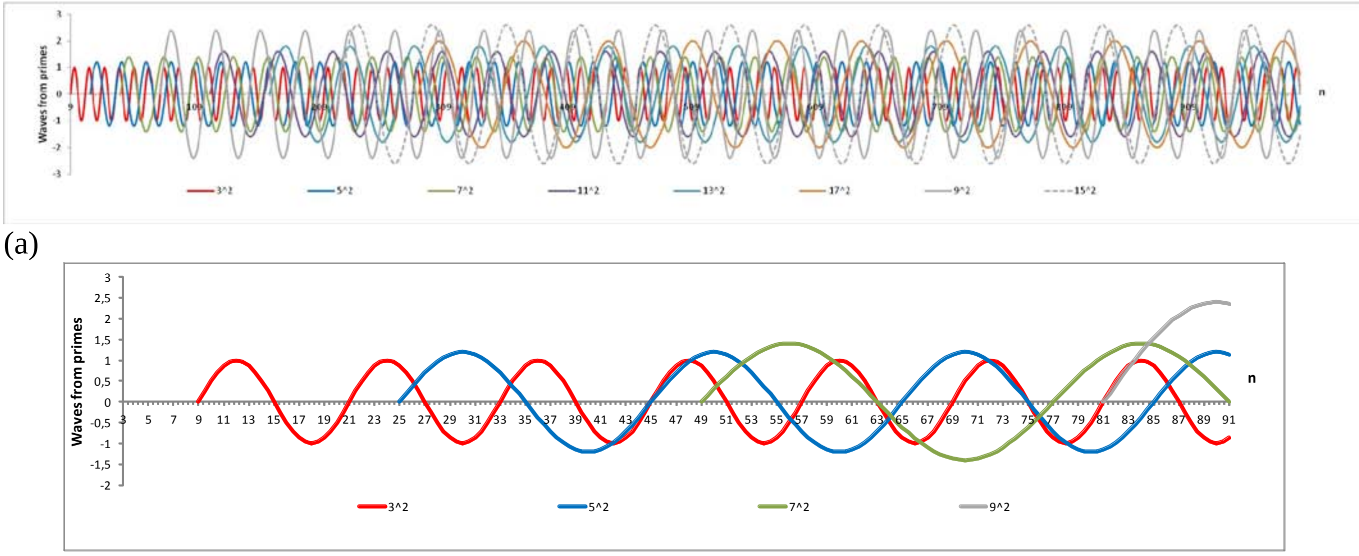
Through a set of thousands, millions, (even for infinite remote parts of arithmetic line) waves with different wavelengths and the phase difference, some numbers of the numerical line, which is located in the heart of the package, are left untouched by all this multitude of waves. This means that there are some N_P natural numbers which remain primes, because apparently the following relationship is satisfied:

$$(N_P - P_i^2)/2P_i\mu_i \notin \mathbb{Z} \tag{16e}$$

where P_i consecutive primes $\leq \sqrt{N}$. That means all the conditions are fulfilled:

$$(N_P\text{-}25)/10\mu_i,\ (N_P\text{-}49)/14\mu_i,\ (N_P\text{-}121)/22\mu_i,\ldots\ (N_P - P_i^2)/2P_i\mu_i \notin \mathbb{Z}$$

*These numbers, being infinite and endless, as Euclid proved using the "Reductio ad absurdum" method, are the Prime Numbers (PN) that appear in **Figure 6**.*



(a)

(b)

Figure 6: (a) Waves proceeding from the primes (3, 5, 7, 11, 13, 17) as well as the composites (9,15).
(b) Detail for the primes (3, 5, 7) and the composite number 9.

As we can see from *Figure 6*, only the waves starting at the squares of *primes* (original squares) create composite numbers. In contrast, the waves starting from composite numbers (abusive squares), having a growing density as we move to larger numbers, do not create new composite numbers. That's why we call them "*sterile waves*".

If therefore from the set $\mathbb{N}_1$ of odd numbers we create subsets A_i, having as first term of each successive subset the squares of all odd numbers and last term the odd number which is the next smallest square of the next odd, namely:

$$A_i = \{ x / x: (2i+1)^2 \leq x < (2i+3)^2, \ x=\text{odd}\}, \tag{17}$$

Such a subset will include $(2i+3)^2 - (2i+1)^2 = 8(i+1)$ terms that belong to the set of natural numbers $\mathbb{N}$. Therefore, the number of terms in the set $\mathbb{N}_1$ of odd ones will be:

$$N_1 = 8(i+1)/2 = 4(i+1) \ \text{όπου} \ i \in \mathbb{N}. \tag{18}$$

Due to the way of their creation (from the square of the odd $2i+1$ up to the previous odd number than the square of $2i+3$), the subsets A_i are disjoint, and the union of all these subsets equals the set of the odd numbers $\mathbb{N}_1$:

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup \ldots = \mathbb{N}_1 \tag{19}$$

where (card = cardinality)

$$\text{card}A_0 = 4, \ \text{and} \ \text{card}A_{i+1} - \text{card}A_i = 4, \ i = 1,2,\ldots \tag{20}$$

In **Table 8** we present these subsets of the odd numbers $\mathbf{A_i}$ from 1 to 1087 (for n=0 to n=15), which correspond to the squares of the odd numbers that are smaller or equal to $\sqrt{\mathbf{1087}}$, that is **31.** In these subsets for easier distinction, the composites are encoded in 'turquoise' while the primes in 'black' colour.

By the term "*wave number*" we mention in the following subsets we mean the number of waves which propagate through the particular subset. The first subset, 1 to 7 (n = 0), is not traversed by any wave. The second, 9 to 23, (n = 1), is traversed by a wave that gives us the multiples of 3. The third, 25 to 47 (n = 2), by two waves that give us the multiples of 3 and 5, and so on.

According to the above, the conversion of some of the natural numbers to composites and the distribution of the arithmetic line is governed by specific rules that are uniform for all natural numbers. The procedure is deterministic and therefore independent of sample size.

Table 8: Subsets of odd numbers A_i (from n = 0 to n = 15), the composites encoded in turquoise colour while the primes in black. The vertical bar (|) appears only for visual purposes, every five numbers.

For n = 0 => A_0 : Sum of odd terms = 4(n +1) = 4 => Number of waves **n = 0**: **{001 003 005 007}**
For n = 1 => A_1 : Sum of odd terms = 4(n +1) **= 8** => Number of waves **n = 1: {009 011 013 015 017 |** **019 021 023}**
For n = 2 => A_2: Sum of odd terms = 4(n +1) **=12** => Number of waves **n = 2** **{025 027 029 031 033 | 035 037 039 041 043 | 045 047}**
For n = 3 => A_3: Sum of odd terms = 4(n +1) **=16** Number of waves **n = 3** **{049 051 053 055 057 | 059 061 063 065 067 | 069 071 073 075 077 | 079**
For n = 4 => A_4: Sum of odd terms = 4(n +1) **=20** Number of waves **n = 4** **{081 083 085 087 089 | 091 093 095 097 099 | 101 103 105 107 109 | 111 113 115 117 119}**
For n = 5 =>A_5: Sum of odd terms = 4(n +1) **=24** Number of waves **n = 5** **{121 123 125 127 129 | 131 133 135 137 139 | 141 143 145 147 149 | 151 153 155 157 159 |** **161 163 165 167}**
For n = 6 =>A_6: Sum of odd terms = 4(n +1) **=28** Number of waves **n = 6** **{169 171 173 175 177 | 179 181 183 185 187 | 189 191 193 195 197 | 199 201 203 205 207** **209 211 213 215 217 | 219 221 223}**
For n = 7 => A_7: Sum of odd terms = 4(n +1) **=32** Number of waves **n = 7** **{225 227 229 231 233 |235 237 239 241 243 | 245 247 249 251 253 | 255 257 259 261 263 |** **265 267 269 271 273 | 275 277 279 281 283| 285 287 }**
For n = 8 => A_8: Sum of odd terms = 4(n +1) **=36** Number of waves **n = 8** **{289 291 293 295 297| 299 301 303 305 307| 309 311 313 315 317|319 321 323 325 327** **329 331 333 335 337 | 339 341 343 345 347 | 349 351 353 355 357 | 359}**
For n = 9 => A_8: Sum of odd terms = 4(n +1) **=40** Number of waves **n = 9** **{361 363 365 367 369 | 371 373 375 377 379 | 381 383 385 387 391 | 389 393 395 397 399 |** **401 403 405 407 409 | 411 413 415 417 419 | 421 423 425 427 429 | 431 433 435 437 439}**
For n = 10 =>A_{10}: Sum of odd terms = 4(n +1) **=44** Number of waves **n = 10** **{441 443 445 447 449 | 451 453 455 457 459 | 461 463 465 467 469 | 471 473 475 477 479 |** **481 483 485 487 489| 491 493 495 497 499 | 501 503 505 507 509| 511 513 515 517 519 |** **521 523 525 527}**
For n = 11 =>A_{11}: Sum of odd terms = 4(n +1) **=48** Number of waves **n = 11** **{529 531 533 535 537| 539 541 543 545 547| 549 551 553 555 557|559 561 563 565 567 |** **569 571 573 575 577| 579 581 583 585 587| 589 591 593 595 597| 599 601 603 605 607|** **609 611 613 615 617* 619 621 623 }**
For n = 12=>A_{12}: Sum of odd terms = 4(n +1) **=52** Number of waves **n = 12** **{625 627 629 631 633| 635 637 639 641 643| 645 647 649 651 653| 655 657 659 661 663|** **665 667 669 671 673| 675 677 679 681 683|685 687 689 691 693 | 695 697 699 701 703|** **705 707 709 711 713* 715 717 719 721 723* 725 727}**
For n =13 => A_{13}: Sum of odd terms = 4(n +1) **=56** Number of waves **n = 13** **{729 731 733 735 737| 739 741 743 745 747| 749 751 753 755 757|759 761 763 765 767|** **769 771 773 775 777| 779 781 783 785 787| 789 791 793 795 797| 799 801 803 805 807|** **809 811 813 815 817| 819 821 823 825 827| 829 831 833 835 837| 839 }**
For n =14 => A_{14}: Sum of odd terms = 4(n +1) **=60** Number of waves **n = 14** **{841 843 845 847 849| 851 853 855 857 859| 861 863 865 867 869|871 873 875 877 879|** **881 883 885 887 889| 891 893 895 897 899| 901 903 905 907 909| 911 913 915 917 919|** **921 923 925 927 929|931 933 935 937 939| 941 943 945 947 949| 951 953 955 957 959| }**
For n =15 => A_{15}: Sum of odd terms = 4(n +1) **=64** Number of waves **n = 15** **{961 963 965 967 969| 971 973 975 977 979| 981 983 985 987 989| 991 993 995 997 999|** **1001 003 005 007 009| 011 013 015 017 019| 021 023 025 027 029| 031 033 035 037 039|** **041 043 045 047 049| 051 053 055 057 059| 061 063 065 067 069| 071 073 075 077 079|** **081 083 085 087}**

From what we have presented here the validity of the statements below is obvious.

Sentence-1
The relation between the first term χ_0 of every subset A_n and its ascending number *n* is: $\chi_0 = (2n+1)^2$. At the same time, the symbol *n* represents also the number of waves that transverse the concrete subset (A_n).

Sentence-2
Every new subset A_i has 4 terms (odd numbers) more than its previous subset, A_{i-1}. See also Eq(20).

Sentence-3
In every new subset A_i only one new wave acts, starting from $(2n + 1)^2$ and step (half wave) $2(2n + 1)$. Since the term $(2n + 1)$ is a prime number, every new wave creates composite numbers using the formula (16d), that is, $(CN)_{(2n+1)} = (2n + 1)^2 + 2(2n + 1)\mu$.

Sentence-4
Easily proved that for any *n*, in the concrete wave μ may take only the values $\mu = 0$, 1 and 2 (three intersections of the transverse wave with the axis Ox for each new subset). Since the term $(2n + 1)$ is a composite number, it does not create new composites based on the formula:

$$(CN)_{(2n+1)} = (2n + 1)^2 + 2(2n + 1)\mu.$$

Sentence-6
In this way the complex numbers are created, each new wave will form 0, 1, or at most 2 new composite numbers, in the first subset it acts. The value of μ=2 coincides with the values that are multiples of 3.

Sentence-7
From the above analysis, it is showed that the creation of composite numbers follows a uniform determinism, from the first up to the last subset A_i we created, using the formula $A_i = \{\chi /\chi = (2n+1)^2 , \chi < (2n+3)^2\}$ (17) . Therefore, both the crowd and the distribution of prime numbers will also be deterministically defined and uniform throughout the set of natural numbers.

Finding
As long as therefore the distribution of prime numbers, in the extended sample that we examined, (and which we can expand indefinitely) is such that it follows very closely the probabilistic relationship (14a) and this distribution is strictly deterministically defined for all of natural numbers, *we have the right* to extend the validity of (14a) for the entire set of natural numbers.

5 Extension of equation (14α) for the set of natural numbers

Theorem-4: If we call $p(N)$ the function that counts the number of ordered pairs of primes that fulfill Goldbach Conjecture for a natural number *N*, this crowd is approximated by the formula:

$$p(N) = N/[\log(N/2)]^2 \quad (21)$$

PROOF

Let us consider a sufficiently large *even number N*. According to the Prime Number Theorem (PNT), the primes that exist in the set from 1 to *N*, are approximated by:

$$\pi(N)=N/\log(N) \quad (22)$$

If we decompose this even number in ordered pairs of odd numbers according to equations (9) and (10), the odd numbers that correspond to the column A, and are in increasing order, will be all less than $N/2$. Therefore, the primes α_1, which exist in column A, will be given by:

$$\alpha_1= \pi(N/2)=N/2/[\log(N/2)] \quad (22a)$$

Therefore the primes β_1 that exist in the column B, will be the subtraction of (22a) from (22)

$$\beta_1= N/\log(N)- (N/2)/\log(N/2) \quad (23)$$

But $N/\log(N)$ can be written as $N/\log[2\times(N/2)]$ or $N/[\log(N/2) + \log2]$, and for a sufficiently large N, is approximated by

$$N/\log(N/2) \quad (23a)$$

Substituting (23a) into (23) we obtain:

$$\beta_1= N/\log(N/2) - (N/2)/\log(N/2)= N/2/[\log(N/2)] \quad (24)$$

Thus we notice that, for very large even numbers N, the primes α_1 in column A and the primes β_1 in column B, tend to become equal ($\alpha_1 \cong \beta_1$).

From what we have mentioned for the crowd n_s of ordered pairs in which the even number N is decomposed, if it is set in the form $6n$-2 and $6n$ it is $(N/4)$-1, while if it is in the form $6n$+2 it is $N/4$. For very large N, we can consider all three cases with $n_s=N/4$, without inserting any serious mistake.

Substituting the values of α_1, β_1 and n_s in eq(13a), we have:

$$n_{pp} = (N/2)/[log(N/2)]^2/(N/4) \text{ or}$$

$$P(N) = N/[\log(N/2)]^2 \quad (21)$$

The above relation completes the proof of theorem 4.■

The graphical representation of (21) is illustrated by the green line in **Figure 7.**

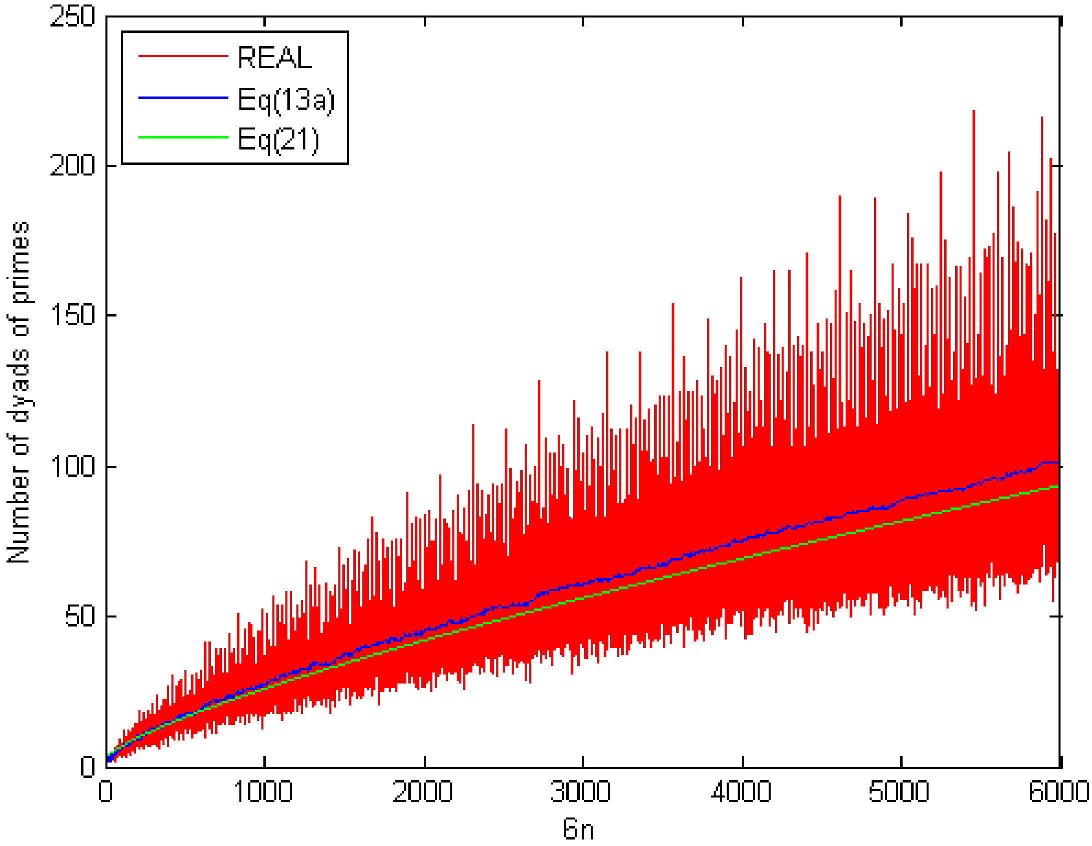


Figure 7: Comparison between the real number of pairs of prime numbers that fulfill Goldbach conjecture and the approximation using eq(13a) and eq(21).

In **Figure 7** we observe that, the graph representing the equation (21) and representing the number of verifications of the Goldbach conjecture, is below the number of verifications using (13a) and is located just below the average line of actual verifications (red line). Note that the red line of Figure 7 does *not* correspond to the red line in Figure 2, but the union of sets of values corresponding together to the purple and red line in Fig.2, which corresponds to the *sum* of real pairs.

This is a natural consequence of the fact that the number of prime numbers calculated from (PNT), is below the actual number of primes $\pi(N)$, as shown in **Figure 8**.

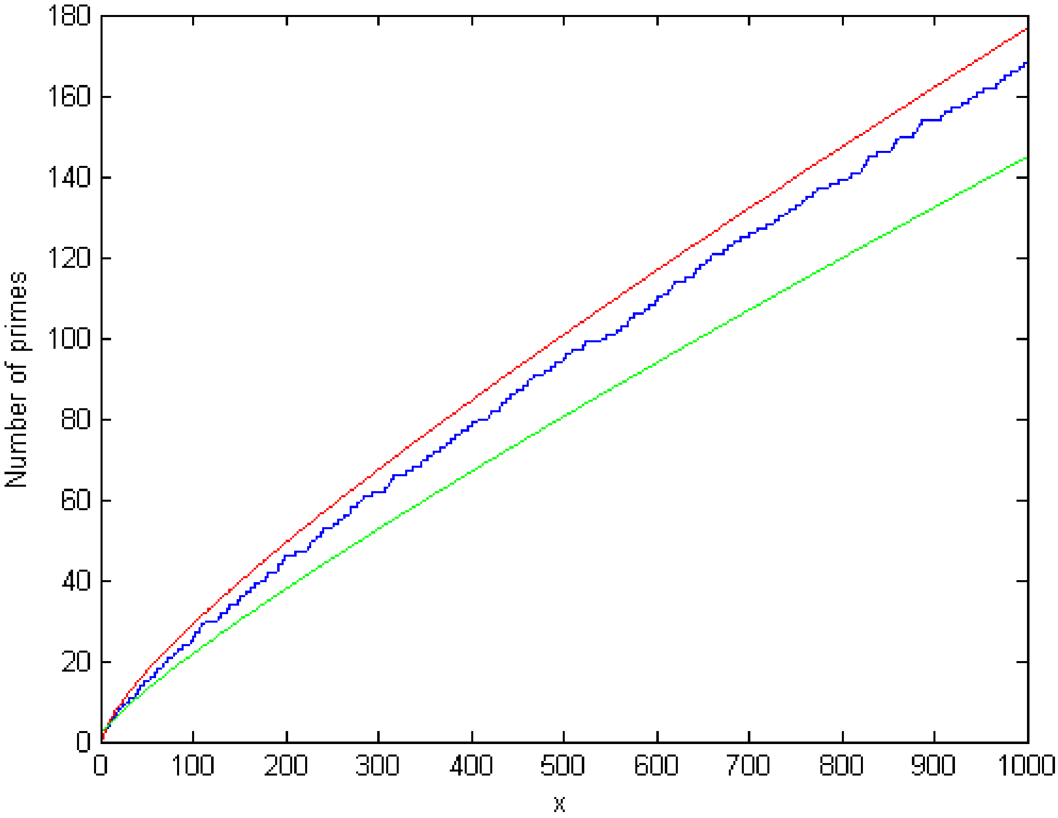


Figure 8: Prime Number Theorem
[red line: according to the function Li(x),
Blue line: true number,
Green line: according to *x*/log(*x*)]

6. Monotonically increasing function $p(N)$

Theorem-5

We shall prove that the estimator of the verifications *P*(*N*) of Goldbach conjecture is a monotonically increasing function of *N*.

PROOF

We consider the continuous function $p(x) = x/[\log(x/2)]^2$. The first and second derivatives are given by:

$$p'(x) = \frac{1}{\left[\log(x/2)\right]^2} - \frac{2}{\left[\log(x/2)\right]^3} \equiv \frac{\log(x/2) - 2}{\left[\log(x/2)\right]^3} \tag{25}$$

$$p''(x) = \frac{6}{x\left[\log(x/2)\right]^4} - \frac{2}{x\left[\log(x/2)\right]^3} \equiv -2\frac{\log(x/2-3)}{x\left[\log(x/2)\right]^4} \tag{26}$$

Since for the first derivative it obviously holds that:

$$p'(x) > 0, \ \ \forall x > 2e^2 \ , \tag{27}$$

while the second derivative never becomes zero :

$$p''(x) > 0, \ \forall x > 2e^3, \tag{29}$$

it is finally concluded that the function $p(x)$ is monotonically increasing and convex.■

Note: The proof of Theorem-5 can be also achieved on the basis of elementary argumentation, which is cited in **Appendix B**.

7. Conclusion

We proved a stronger sentence than the original Goldbach conjecture, which can be formulated as follows:

Final sentence

The strictly predetermined distribution of primes along the arithmetic line is such as to ensure that as we move towards larger and larger even numbers N, the growth of the product $(\alpha_1\beta_1)$ is greater than the growth rate of the number of ordered pairs of odd numbers n_s in which the even number under question is analyzed using equations (9) and (10), where α_1 is the number of prime numbers in column A and β_1 is the number of prime numbers in column B of ordered pairs.

The direct result of this property of prime numbers is that:

Not only the Goldbach conjecture is true, but the number of verifications has a clearly increasing trend, as we move to larger even numbers, which is determined by the relation (21).

REFERENCES

[1] Christian Goldbach, Letter to L. Euler, June 7, 1742.

[2] David Hilbert, "Mathematical Problems", *Bulletin of the American Mathematical Society*, vol. 8, no. 10 (1902), pp. 437-479. Earlier publications (in the original German) appeared in *Göttinger Nachrichten*, 1900, pp. 253-297, and *Archiv der Mathematik und Physik*, 3dser., vol. 1 (1901) 44-63 and 213-237.

[3] János Pintz, Landau's problems on primes, *Journal de théorie des nombres de Bordeaux* 21 (2) (2009) 357-404.

[4] Richard K. Guy, *Unsolved problems in number theory*, third edition, Springer-Verlag, 2004.

[5] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes, *Acta Mathematica* 44 (1) (1923) 1-70.

[6] Viggo Brun, Le crible d'Eratosthene et le theoreme de Goldbach, *C. R. Acad. Sci.*, Paris, 168 (1919) 544-546.

[7] L. G. Schnirelman, On additive properties of numbers. *Izv. Donsk. Politehn. Inst*. 14 (1930), 3–28 (Russian). Also: L. G. Schnirelman, Über additive Eigenschaften von Zahlen. *Math. Ann*. 107 (1933) 649–690.

[8] I. M. Vinogradov, Some theorems in analytic theory of numbers, *Dokl. Akad. Nauk* SSSR, 4 (1934) 185-187. Also: I. M. Vinogradov, *Elements of Number Theory*. Mineola, NY: Dover Publications, 2003.

[9] K. G. Borozdkin, On a problem of Vinogradov's constant, *Trudy Mat. Soc*, SSSR, 1 (1956) 3.

[10] Jingrun Chen, On the representation of a large even integer as the sum of a prime and the product of at most 2 primes, Kexue Tongbao 17 (1966) 385-386.

[11] M. C. Liu, T. Z. Wang, On the Vinogradov's bound in the three primes Goldbach conjecture, *Acta Arithmetica*, 105 (2) (2002) 133-175.

[12] N. Lygeros, F. Morain, O. Rozier. http://www.lix.polytechnique.fr/~morain/Primes/myprimes.html

[13] Tomás Oliveira e Silva, http://www.ieeta.pt/~tos/goldbach.html

[14] Deshouillers, Effinger, Te Riele and Zinoviev, A complete Vinogradov 3-primes theorem under the Riemann hypothesis. *Electronic Research Announcements of the American Mathematical Society* **3** (15) (1997) 99–104.
[15] Kaniecki, Leszek., On Šnirelman's constant under the Riemann hypothesis, *Acta Arithmetica* **4** (1995) 361–374.
[16] Yuan, Wang. *Goldbach Conjecture.* 2nd edn., World Scientific Publishing, New Jersey, Singapore, 2000.
[17] Chudakov, Nikolai G. On the Goldbach problem, *Doklady Akademii Nauk SSSR* **17**: (1937) 335–338.
[18] Van der Corput, J. G., Sur l'hypothèse de Goldbach, *Proc. Akad. Wet. Amsterdam* **41** (1938) 76–80.
[19] T. Estermann, On Goldbach's problem: proof that almost all even positive integers are sums of two primes, *Proc. London Math. Soc*. 2 **44** (1938): 307–314.
[20] Ramaré, Olivier, On Šnirel'man's constant, *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Sér. 4, 22 (4) (1995) 645-706.
[21] Sinisalo, Matti K.. "Checking the Goldbach Conjecture up to 4 10^{11}". *Mathematics of Computation* **61** (204) (1993) 931–934.
[22] J. R.Chen, On the representation of a larger even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica* **16** (1973) 157–176.
[23] H.L.Montgomery, R.C.Vaughan, The exceptional set in Goldbach's problem, *Acta Arith*. 27 (1975) 353-370.
[24] Yu. V. Linnik, "Prime numbers and powers of two", Collection of articles. To the sixtieth birthday of academician Ivan Matveevich Vinogradov, *Trudy Mat. Inst. Steklov*., **38**, Acad. Sci. USSR, Moscow, 1951, 152–169 (Mi tm1114)
[25] Heath-Brown, D. R.; Puchta, J. C., Integers represented as a sum of primes and powers of two, *Asian Journal of Mathematics* **6** (3) (2002) 535–565.
[26] J.Pintz, I. Z. Ruzsa, On Linnik's approximation to Goldbach's problem, I, *Acta Arithmetica* **109** (2) (2003) 169–194.
[27] Jörg Richstein, Verifying the Goldbach conjecture up to $4{\cdot}10^{14}$, *Mathematics of Computation*, 70 (236) (2000) 1745-1749.
[28] I. Mittas, Generalization of Goldbach's conjecture and some special cases, Italian Journal of Pure and Applied Mathematics, 27 (2010) 241-254.
[29] http://en.wikipedia.org/wiki/Goldbach%27s_weak_conjecture
[30] Eric W. Weisstein, Goldbach Conjecture. From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/GoldbachConjecture.html
[31] http://jtnb.cedram.org/item?id=JTNB_2009__21_2_357_0
[32] http://jtnb.cedram.org/cedram-bin/article/JTNB_2009__21_2_357_0.pdf
[33] Neil Sheldon, A statistician's approach to Goldbach's Conjecture, Teaching Statistics 25 (1) (2003) 12-13.
[34] Song Y. Yan, A simple verification method for the Goldbach conjecture, International Journal of Mathematical Education in Science and Technology 25 (5) (1994) 681-688.
[35] John Derbyshire, *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Joseph Henry Press, Washington, D.C., 2003.
[36] Apostolos Doxiadis, *Uncle Petros and Goldbach's Conjecture*. New York: Bloomsbury. 2001.
[37] Marcus du Sautoy, *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*, HarperCollins Publishers, New York, 2003.
[38] Harrison John, Formalizing an Analytic Proof of the Prime Number Theorem, J Autom Reasoning (2009) 43:243–261
[39] Newman, D. J., Simple Analytic Proof of the Prime Number Theorem, The American Mathematical Monthly, Vol. 87, No. 9 (Nov., 1980), pp. 693-696

APPENDIX A

Prime number theorem

Let π(*x*) be the *prime-counting function* that gives the number of primes less than or equal to *x*, for any real number *x*. The prime number theorem then states that the limit of the quotient of the two functions π(*x*) and $x/\ln(x) \equiv x/\log(x)$ as *x* approaches infinity is 1, which is expressed by the formula

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{A-1}$$

known as **the asymptotic law of distribution of prime numbers**. Using asymptotic notation this result can be restated as

$$\pi(x) \simeq \frac{x}{\log(x)} \tag{A-2}$$

This theorem does *not* say anything about the limit of the *difference* of the two functions as *x* approaches infinity. Instead, the theorem states that *x*/ln(*x*) approximates π(*x*) in the sense that the relative error of this approximation approaches 0 as *x* approaches infinity.

The prime number theorem is equivalent to the statement that the *n*th prime number p_n is approximately equal to *n* ln(*n*), again with the relative error of this approximation approaching 0 as *n* approaches infinity.

For analytical proofs we refer to [38,39] among others.

Corrolary: The average density of primes is given by: $\pi(N) \sim \frac{1}{\log N}$. Obviously, it decreases by increasing *N*. As a result, the density of primes in the column A is greater than that of column B and at the same time the density of primes in the two columns is different.

APPENDIX B

The function log N

The logarithmic function log$N$ that interferes in the formula $\pi(N)=N/\log(N)$ for the calculation of the primes (shown in **Figure 9**), has the property to increase very slowly. The latter property plays a significant role in our study.

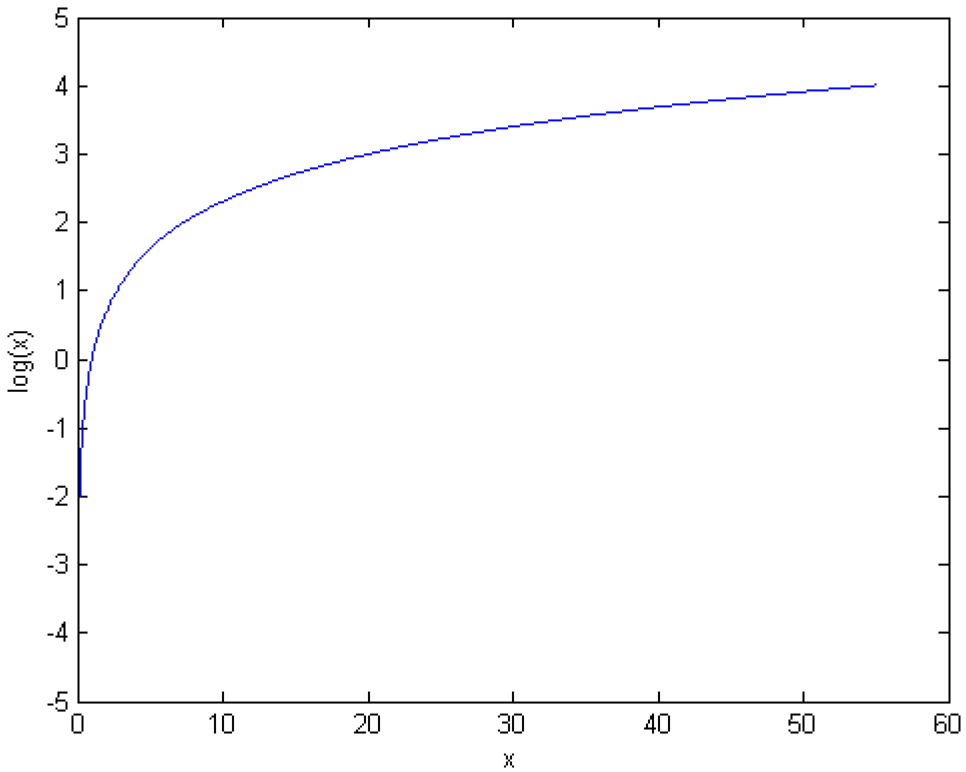


Figure 9: The logarithmic function (natural logarithm, with base *e*)

In **Figure 10** we see the graphs of x^{α} for small values of α, such as α = 0.1, 0.2, 0.3, 0.4, 0.5 and the logarithmic function with the sign (×) in thick red, for a comparison. As we can notice, the smaller α is the more the graph of x^{α} approaches the horizontal line passing through 1 (for $\alpha \rightarrow 0$, $x^{\alpha} \rightarrow x^{0} \equiv 1$).

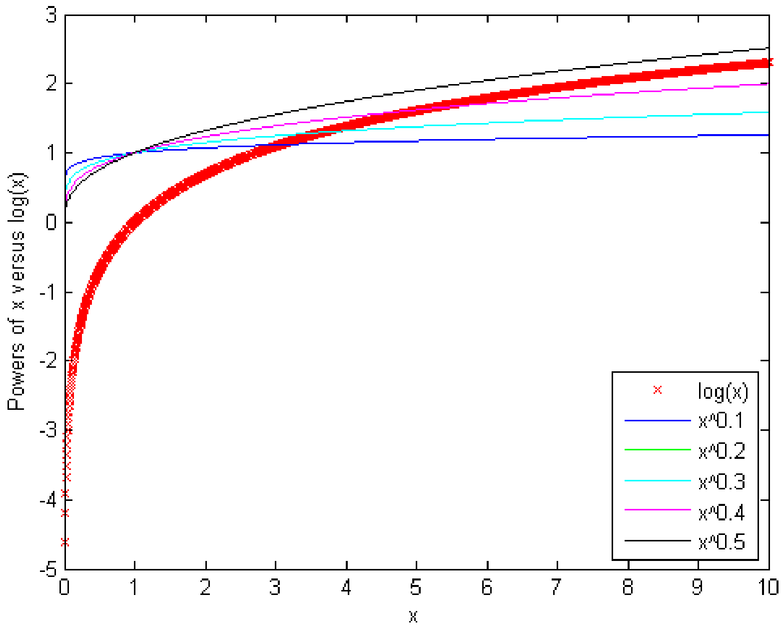


Figure 10: Graphical representation of the powers of x (x^{α}) for small positive values of α, in comparison with the logarithmic function.

For values α <0.3 (concretely for $\alpha < 1/e$) the logarithmic curve intersects the curves x^{α} near the left part of the figure from down to upwards. However, when we go sufficiently right, we shall see that the curve log(x) intersects again x^{α} and remains underneath forever. In more detail, the logarithmic curve intersects for the first time $x^{0.3}$ at $x \cong 5.1107$ and again the same curve (i.e. $x^{0.3}$) at the value $x \cong 379.0962$ (where $\log x \cong x^{0.3} \cong 5.9378$), the curve $x^{0.2}$ close to $x \cong 3.3211\text{e}+05$ (where $\log x \cong x^{0.2} \cong 12.7132$) and the function $x^{0.1}$ close to x=3.4306e+015 (where $\log x \cong x^{0.1} \cong 35.7715$).

In other words, the function log$x$ is as to try to "coincide" with x^0 of course without being able to achieve it. However, despite the fact that log$x$ never equals to x^0, it achieves to sink below x^{ε} and to remain underneath, for every positive number ε, no matter how small it is, when x becomes sufficiently large. In brief, when x becomes sufficiently large the function log$x$ *increases slower than every power of* x. Obviously, in some manner, since log$x$ increases slower than every power of x, the same happens for every power of log$x$, such as $(\log x)^2$, $(\log x)^3$, $(\log x)^4$, and so on. This property of the function log$x$ has the result that the verifications of Goldbach conjecture $P(N)$ in eq (21), *increase* as N increase, because the nominator increase faster than the denominator.