FRACTIONAL FIELD THEORY AND HIGH-ENERGY PHYSICS: NEW DEVELOPMENTS

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Abstract

During the last decade, a number of important developments have surfaced concerning fractional calculus and its applications in various branches of fundamental and applied science. In particular, *fractional field theory* (FFT) represents an active area of research in mathematical physics whose motivation stems, in part, from its ability to shed light on many of open questions surrounding Quantum Field Theory (QFT), Standard Model for particle physics (SM) and Quantum Gravity Theories (QG). We review here some recent developments of FFT that promise to recover the physics of SM in the low-energy limit and solve some of its seemingly intractable puzzles.

Key words: Fractional Field Theory, Standard Model, Renormalization Group, Conformal symmetry, Continuous dimension, Electroweak symmetry breaking, Hierarchy problem.

1. INTRODUCTION

The Standard Model of particle physics (SM) embodies our current knowledge of the strong and electroweak (EW) interactions. SM is a conceptual framework of remarkable predictive power whose fundamental degrees of freedom are the spin one-half quarks and leptons, the spin one gauge bosons and the spin-zero Higgs doublet. Symmetry constraints play a key role in fixing the dynamical structure of SM, which exhibits invariance under the combined $SU(3)_L \times SU(2)_Y \times U(1)_{EM}$ gauge group. SM contains a rich phenomenology able to account for a large variety of processes involving strong and EW interactions, confinement and electroweak symmetry breaking (EWSB), hadronic and leptonic flavor physics [ ]. Despite being confirmed in many independent tests, SM is considered an *incomplete* framework. At the time of writing, the root cause of EWSB remains elusive. The search for the physical source of EWSB has been one of the main drivers in both experimental and theoretical high-energy physics for the past 35 years. We still lack conclusive evidence for the Higgs boson that is alleged to break the electroweak $SU(2)_L \times U(1)_Y$ symmetry to its smaller electromagnetic $U(1)_{EM}$ subgroup. There is a fairly large slew of theoretical challenges facing SM, most cited ones including the origin of neutrino masses and mixing, the dark matter puzzle, the source of *CP* symmetry breaking and baryon asymmetry, the fine-tuning problem, the "triviality" problem and the LEP paradox associated with the minimal Higgs scenario, the flavor problem, the source of anomalous magnetic moments of charged leptons and the unknown connection to low-energy manifestations of QG [ ].

SM is built in strict compliance with a series of postulates called *consistency conditions*. The remarkable success of SM can be attributed to a unitary, local, renormalizable, gauge invariant and anomaly-free formulation of QFT [ ]. Since SM is based on a renormalizable gauge field theory, the prevailing opinion among theorists is that it can be extrapolated to energies above the EW scale. The underlying assumption is that QFT stays compliant to consistency conditions throughout all energy scales. Needless to say, this is a speculative conjecture whose validation awaits analysis of vast sets of data from the Large Hadron Collider (LHC) and other detector sites [ ].

Inspired by recent advances in *nonlinear dynamics* and *critical behavior outside equilibrium*, we develop here an approach to physics beyond SM that has received virtually no attention in mainstream research. The main tool at our disposal is the Renormalization group (RG) description of dynamics on fractal and multifractal structures, as embodied in the so-called *fractional field theories* (FFT). This class of theories represents an emerging topic in mathematical physics. Motivation for their appeal stems, in part, from the ability to offer surprising insights into many foundational questions involving QFT, SM and Quantum Gravity Theories (QG) [ ]. FFT is built on the key idea that, at some large energy scale, spacetime dimensionality turns into a continuous variable

$$D = 4 - \varepsilon, \quad \varepsilon << 1 \tag{1}$$

where $\varepsilon = 0$ recovers the familiar low-energy limit of both SM and General Relativity. In general, the deep ultraviolet regime of QFT may be described by a FFT where ε is not necessarily limited to a small deviation from zero but it assumes a scale-dependent range of values [ ].

Our contribution is organized as follows: next section is dedicated to a brief survey of FFT and some of their current applications. Section three presents few of existing hints in support of FFT. The dynamic role of continuous dimension (1) in ensuring consistency of QFT forms the topic of section 4. A novel mass and flavor generation mechanism based on (1) is formulated in section five. Section six deals with a straightforward solution to both gauge hierarchy and cosmological constant problems using (1) and dimensional regularization. Open questions, future developments and a summary of results are included in the last two sections.

We caution from the outset that our contribution is meant to be informal and introductory in nature. It reflects a body of ongoing theoretical work with many ideas under development.

Follow-up modeling efforts are needed to falsify, consolidate or expand our tentative conclusions.

2. BRIEF SYNOPSIS OF FRACTIONAL FIELD THEORIES

Fractional field theories are continuum field models built on spacetime endowed with non-integer dimensions. *Multi-fractional field theories* (MFT) are a subset of FFT in which the spacetime dimension *D* is variable and runs with the observation scale. By analogy with the behavior of RG equations, this property is called *dimensional flow.* A characteristic feature of both theories is that the measure entering the action functional is not the familiar *D*-dimensional entity $d^D x$, but a Lebesgue–Stieltjes measure $d\rho(x)$ whose form is determined by fractal geometry [ ].

Consider a multiplet of classical fields $\Phi^i = \Phi$, $i = 1, 2, ..., N$ embedded in Minkowski spacetime. Assuming that all fields are analytic functions of coordinates $x = x^\mu$, $\mu = 0,1,2,3$, a generic example for a FFT action in one dimension $x = x^1 \geq 0$ is given by

$$S[\Phi] = \int d\rho\, L(\Phi, \partial\Phi) \tag{2}$$

in which [ ]

$$d\rho = \frac{\alpha x^{\alpha-1}}{\Gamma(\alpha+1)} dx \tag{3}$$

and $0 < \alpha \leq 1$. In the limit of low-level fractality, we write $\alpha = 1-\varepsilon$ with $\varepsilon << 1$ and the measure becomes

$$d\rho = \frac{(1-\varepsilon)x^{-\varepsilon}}{\Gamma(2-\varepsilon)} dx \tag{4}$$

(3) is a natural generalization of differential dx for smooth spacetime ($\varepsilon = 0$).

Another generic example is based on using *fractional differential and integral operators* in field theory [ ]. Equations containing such operators are used to analyze the behavior of systems characterized by

- Power-law nonlinearity,
- Power-law long-range spatial correlations or long-term memory,
- Fractal or multi-fractal properties.

In the last decade, the number of applications of fractional operators in science and engineering has been steadily growing. They include models of fractional-relaxation effects, anomalous transport in fluids and plasma, wave propagation in complex media, viscoelastic materials, universal response in dielectric media, non-Markovian evolution of quantum fields, networks of fractional oscillators, dynamics of non-extensive statistical systems, QFT, QG and so on. The reader is referred to [ ] for an informative (albeit incomplete) update of how fractional operators are used in various contexts.

Let $f(x,\lambda) \in L_p(E^1)$ an arbitrary function of x defined on a one-dimensional Euclidean space $x = x^1 \geq 0$ where λ is a parameter and $1 < p < 1/\alpha$. Fractional integration of order α on $(-\infty, y)$ and $(y, +\infty)$ is described by [ ]

$$(I_+^\alpha f)(y,\lambda) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{y} \frac{f(x,\lambda)\,dx}{(y-x)^{1-\alpha}}$$

(5a)

$$(I_-^\alpha f)(y,\lambda) = \frac{1}{\Gamma(\alpha)} \int_{y}^{+\infty} \frac{f(x,\lambda)\,dx}{(x-y)^{1-\alpha}}$$

An alternate formulation is given by the left (*L*) and right (*R*) Riemann-Liouville operators,

$$
{}_0D_L^{\alpha} f(y,\lambda) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dy} \int_0^y (y-x)^{-\alpha} f(x,\lambda)\, dx
$$

(5b)

$$
{}_0D_R^{\alpha} f(y,\lambda) = \frac{1}{\Gamma(1-\alpha)} (-\frac{d}{dy}) \int_y^0 (x-y)^{-\alpha} f(x,\lambda)\, dx
$$

Fractional operators describe dynamics on fractal and multi-fractal structures and are a natural generalization of momentum-energy operators for energy scales far beyond the SM scale [ ]. Given the non-local nature of fractional operators, which is incompatible with underlying principles of QFT, connecting FFT with QFT in the low-energy limit turns out to be a non-trivial challenge [ ].

Current formulation of QFT contains several hints that point toward fractal space-time and FFT. We examine them in the next section.

3. HINTS FOR FRACTAL SPACETIME IN FIELD THEORY

3.1) The first hint stems from the deep analogy between the Euclidean Path Integral formulation of QFT and critical phenomena [ ]. This implicit connection was first made clear by Wilson's seminal work on RG program [ ]. As the underlying geometry of critical phenomena is manifestly fractal, it follows that QFT lives on a fractal foundation [ ]. Surprisingly, with few isolated exceptions, the dynamic ramifications of this connection remain largely unexplored to-date [ ]. In what follows we glance upon the relationship between RG, critical phenomena and the concept of continuous dimension.

Fluctuations stemming from environmental interactions and the uncertainty principle, as well as correlations in statistical physics and QFT, are known to become unbounded near a critical point. In this region, fluctuations lead to singular thermodynamic behavior characterized by universal critical exponents and scaling functions [ ]. These power-law singularities bring to light an underlying emerging symmetry associated with critical phenomena, namely the manifest *scale invariance* of the theory: at the critical point, the physical system has no characteristic scale and the correlation length diverges. In the language of QFT, divergence of the correlation length is equivalent to a massless theory.

RG provides a natural framework for explaining the onset of critical phenomena, the roots of universality and the classification of various systems in terms of universality classes. In the context of RG, the process of integrating out fluctuations and the short-distance degrees of freedom is made systematic. For instance, if there is a single mass scale M in the microscopic theory, RG proceeds by building an *effective field theory* whose content may be understood as a power expansion in $1/M$. RG is based on the premise that the renormalization technique absorbs all relevant fluctuations above M. There are two implicit premises behind this technique: a) fluctuations have a finite average and b) renormalization process is carried out at a fixed dimensionality of space-time.

A key consequence of RG in both statistical physics and QFT is that universal properties near second-order phase transitions depend strongly on the space-time dimensionality. Consider, for instance, the traditional one-component Ising model consisting of an orthogonal lattice of spins experiencing nearest neighbor coupling. It can exhibit an infinite number of multi-critical points in $D = 2$, a critical Wilson-Fisher or a tri-critical point in $D = 3$ and a Gaussian fixed point for

$D=4$ [ ]. Percolation, random walks and formation of fractal clusters in critical systems undergoing second order phase transitions are also typical examples of processes whose outcome depends on D [ ]. The relevant literature on statistical physics of phase transitions points out that continuity in the dimensionality of space is an essential ingredient for the correct description of critical phenomena. As mentioned below, extrapolation from $D=4$ to an infinitesimally lower dimension $D=4-\varepsilon$ is the basis for dimensional regularization in field theory and represents one frequent method in the non-perturbative study of the RG flow near non-trivial fixed points [ ]. Recent work on field theories formulated in continuous dimension asserts that a new type of critical behavior develops at a fixed energy RG scale μ as a result of incremental changes in the dimensional parameter ε [ ].

3.2) One of the earliest proposals for non-integer dimensionality of space-time was put forward in [ ] where it was shown that vacuum fluctuations surrounding an electron in QED have lesser influence in $D<4$ dimensions. As a result, the first order correction to the anomalous magnetic of the electron in $D=4-\varepsilon$ dimensions becomes

$$\delta g = |\varepsilon| \frac{\alpha(\gamma_E + \ln \pi)}{4\pi} \tag{6}$$

where α represents the fine structure constant, γ_E is Euler's constant and the numerical value of parameter ε is found to be

$$|\varepsilon| = (5.3 \pm 2.5) \times 10^{-7} \tag{7}$$

Reference [ ] points out that numerical bounds on ε may be taken from the literature of dimensional regularization models. Using measurements of anomalous magnetic moment of muon and electron, one obtains

$$|\varepsilon| < 10^{-8}, \quad l = 10^{-15}\, m \tag{8}$$

in which l stands for the characteristic length scale for the onset of fractal space-time. Likewise, experimental determination of the Lamb shift in hydrogen yields

$$|\varepsilon| < 10^{-11}, \quad l = 10^{-11} m \tag{9}$$

whereas derivations based on astrophysical observations lead to

$$|\varepsilon| < 10^{-9}, \quad l = 10^{11} m \quad \text{(planetary precession)} \tag{10}$$

$$|\varepsilon| < 10^{-5},\ l = 14.4\, Gpc \ \text{(cosmic microwave background)} \tag{11}$$

3.3) As it is known, dimensional regularization is a key computational tool for removing infinities in perturbative QFT. The deviation from space-time dimension (1) is treated as a regulator of Feynman integrals and meaningful results are obtained at the end of calculations as $\varepsilon \to 0$. An important property of dimensional regularization is that it complies with gauge and Lorentz invariance, in contrast with other regularization methods (e.g. the cutoff schemes) [ ].

In general, the technique of renormalization in perturbative QFT consists in a two-step program: *regularization* and *subtraction*. One first controls the divergence present in momentum integrals by inserting a suitable regulator, and then brings in a set of counter-terms to cancel out the divergence. Momentum integrals in QFT have the generic form

$$I = \int_0^\infty d^4 q F(q) \tag{12}$$

Two regularization techniques are frequently employed to manage (12), namely “momentum cutoff” and “dimensional regularization”. When the momentum cutoff scheme is applied for regularization in the UV region, the upper limit of (12) is replaced by a finite cutoff Λ_{UV},

$$I \to I_{\Lambda_{UV}} = \int_0^{\Lambda_{UV}} d^4 q F(q) \tag{13}$$

Explicit calculation of the convergent integral (13) amounts to a sum of three polynomial terms

$$I_{\Lambda_{UV}} = A(\Lambda_{UV}) + B + C(1/\Lambda_{UV}) \tag{14}$$

Dimensional regularization proceeds instead by shifting the momentum integral (12) from a four-dimensional space to a continuous D - dimensional space

$$I \to I_D = \int_0^{\infty} d^D q F(q) \tag{15}$$

Introducing the dimensional parameter $\varepsilon = 4 - D$ leads to

$$I_D \to I_\varepsilon = A'(\varepsilon) + B' + C'(1/\varepsilon) \tag{16}$$

The connection between dimensional and cutoff regularization techniques is given by [ ]

$$\log \frac{\Lambda_{UV}^2}{\mu^2} = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi + \frac{5}{6} \tag{17a}$$

in which μ is the sliding RG scale [ ]. We find it convenient to present (17a) is a slightly different form, that is,

$$\varepsilon \propto \frac{1}{\log(\Lambda_{UV}^2/\mu^2)} \tag{17b}$$

It is apparent from the above that the four-dimensional spacetime is recovered in either one of these limits:

a) $\Lambda_{UV} \to \infty$ and $0 < \mu << \Lambda_{UV}$,

b) $\Lambda_{UV} < \infty$ and $\mu \to 0$.

However, both limits are in conflict with our current understanding of deep ultraviolet and deep infrared boundaries of field theory. Theory and experimental observations alike tell us that the notions of infinite *or* zero energy are, strictly speaking, meaningless. This is to say that either infinite energies (point-like objects) or zero energies (infinite distance scales) lead to divergences whose removal requires the machinery of RG program. Indeed, there is always a finite cutoff at both ends of either energy or energy density scale (deep ultraviolet = Planck scale, deep infrared = finite radius of the observable Universe or the non-vanishing energy density of the vacuum set by cosmological constant). It follows from these considerations that the limit $\varepsilon \to 0$ works exclusively as reasonable approximation and realistic models near or beyond the SM scale must account for space-time geometries having continuous dimensionality.

3.4) A consistency condition that is violated by SM is invariance under *conformal transformations* in D = 4 dimensions[1]. This condition constrains the action functional to be independent from the choice of measurement units. Conformal symmetry is broken in field

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[1] Hereafter, *scale* and *conformal symmetry* are used interchangeably, with the caveat that there are systems that display scale invariance but fail to be conformal invariant [ ]. We ignore this distinction here for the sake of simplicity and clarity.

theory by dimensionful parameters in the Lagrangian *or* as result of quantization. The latter requires regularization of amplitudes and introduction of an arbitrary RG scale in the theory [ ]. However, it can be shown that conformal symmetry may be restored if the theory is placed on spacetime having continuous dimensionality [ ]. To this end, consider the Lagrangian of massless electrodynamics. It reads,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} i \gamma^{\mu} D_{\mu} \psi \tag{18}$$

An arbitrary change in coordinate scale $x \to x' = \lambda x$ along with the corresponding field transformations

$$\psi(x) \to \psi'(x) = \lambda^{3/2} \psi(x) \tag{19}$$

$$A_{\mu}(x) \to A'_{\mu}(x) = \lambda A_{\mu}(x) \tag{20}$$

leave the action unchanged [ ]. The Noether current associated with the change of scale is given by

$$J^{\mu}_{scale} = x_{\nu} \theta^{\mu\nu} \tag{21}$$

in which $\theta^{\mu\nu}$ represents the conserved energy-momentum tensor , $\partial_{\mu} \theta^{\mu\nu} = 0$. The conservation of scale current (21) amounts to the vanishing of the trace of the energy-momentum tensor, that is,

$$\partial_{\mu} J^{\mu}_{scale} = \theta^{\mu}_{\mu} = 0 \tag{22}$$

In D space-time dimensions the trace of massive theory may be cast in the form

$$\theta_{\mu}^{\mu} = \frac{\varepsilon}{4} F^{\eta\sigma} F_{\eta\sigma} + m\overline{\psi}\psi + R(\varepsilon, m, \psi, \overline{\psi}, F^{\eta\sigma}, F_{\eta\sigma}) \quad (23)$$

where the first two terms explicitly highlight the contribution of electron mass and the deviation from four-dimensionality. All terms vanish in the limiting case $m = 0$ and $\varepsilon = 0$. The residual term in (23) embodies correction effects not included in the first two terms. Enforcing conformal invariance defined by a vanishing trace in (23) implies that electrons may gain mass *on account of* deviations from $D = 4$.

4. LOCAL CONFORMAL SYMMETRY IN CONTINUOUS DIMENSION

Demanding that a conformal invariant field theory be defined in continuous dimension (1) means that all variables and descriptors of that theory (coordinates, dimensional measures, fields, propagators and gauge charges) behave as *fractal functions*. This viewpoint echoes the observation that collective phenomena at criticality approach scale-invariance and live on a fractal support [ ]. On this basis, taking $s = \Lambda_{UV}/\mu$ and starting from (1) and (17b),

$$\varepsilon \propto s^{-2} \quad (24)$$

it is natural to conjecture that the following scaling relations hold near criticality:

a) coordinates: $x \propto s^{\nu_x(x)} = \varepsilon^{-\frac{\nu_x(x)}{2}}$ (25)

b) fermions: $\psi \propto s^{\nu_\psi(x)} = \varepsilon^{-\frac{\nu_\psi(x)}{2}}$

c) gauge bosons: $A \propto s^{\nu_A(x)} = \varepsilon^{-\frac{\nu_A(x)}{2}}$

d) dimensional measure: $\rho_x \propto s^{\nu_\rho(x)} = \varepsilon^{-\frac{\nu_\rho(x)}{2}}$

e) dimensional measure in field space: $\rho_\varphi \propto s^{\nu_\varphi(x)} = \varepsilon^{-\frac{\nu_\varphi(x)}{2}}$

Two observations are in order at this point:

1) Similar scale transformations describe conformal invariant behavior of propagators in FFT [ ].

2) Since (25) represents a set of parametric relations, conformal invariance under (25) is necessarily a *comprehensive* symmetry involving concomitant transformations of coordinates, fields, propagators and dimensional measures. This is in contrast with conventional QFT where the symmetry under coordinate transformations and symmetry under internal transformations in field space cannot be trivially mixed (Coleman - Mandula theorem [ ]).

Consider now a generic classical or quantum field theory defined on arbitrary space-time dimension *D*:

$$S[\Phi_i] = \int d^D x (g_i \Phi_i) \tag{26}$$

in which Φ_i denotes a product of fields and/or their derivatives and g_i are coupling constants or masses. Due to (24) and (25), each term of (26) contains a multiplication of power functions

$$f(x, g, \Phi) = d^D x (g\Phi) \propto \varepsilon^{a(x)} \varepsilon^{b(x)} \varepsilon^{c(x)} = \varepsilon^{a(x)+b(x)+c(x)} \tag{27}$$

Here, $a(x)$, $b(x)$ and $c(x)$ are linearly dependent on the set of $\nu(x)$ exponents defined in (25). An arbitrary scale transformation applied to (27) amounts to

$$f(\lambda^{\alpha(x)}x, \lambda^{\beta(x)}g, \lambda^{\gamma(x)}\Phi) = \lambda^{\alpha(x)a(x)+\beta(x)b(x)+\gamma(x)c(x)} f(x, g, \Phi) \tag{28}$$

with $\lambda \neq 1$. There are two cases of interest here, namely:

1) If $\alpha(x)a(x) + \beta(x)b(x) + \gamma(x)c(x) = 1$ then (28) corresponds to ordinary scale invariance of homogeneous functions [ ].

2) The case $\alpha(x)a(x) + \beta(x)b(x) + \gamma(x)c(x) = 0$ defines a condition of scale invariance that is insensitive to λ.

Consider again the action functional (26). We seek to cast the requirement of conformal symmetry applied to (26) in a form that explicitly highlights the role played by dimensional flow. The mass dimensions of coordinates, couplings and fields are given by

$$[g] = [M]^{d_g(D)}, \; [\Phi] = [M]^{d_\Phi(D)}, \; [x] = [M]^{-d_x(D)}, \; [d^D x] = [M]^{-D\, d_x(D)}$$

By dimensional analysis, action (26) represents a Lorentz scalar if

$$d_g(D) + d_\Phi(D) - D\, d_x(D) = 0 \tag{29}$$

Now, demanding that coupling is a scalar quantity ($d_g = 0$), implies that the rest of dimensions need to be rescaled through a suitable flow $d_x(D) \to d_x^{'}(D')$, $d_\Phi(D) \to d_\Phi^{'}(D')$ and $D \to D'$ such that

$$d_\Phi^{'}(D') - D'\, d_x^{'}(D') = 0 \tag{30}$$

In terms of ε (29) becomes

$$d_g(\varepsilon) + d_\Phi(\varepsilon) + (\varepsilon - 4) d_x(\varepsilon) = 0 \tag{31}$$

with

$$d_g(0) + d_\Phi(0) - 4(d_x(0)) = 0 \tag{32}$$

Let

$$\Delta(\varepsilon) = (\varepsilon - 4)[d_x(\varepsilon)] \tag{33}$$

Assuming that all mass dimensions are analytic functions of (1) leads to the following relations

$$d_\Phi^{'}(\varepsilon') = d_\Phi(\varepsilon) + \delta\varepsilon \frac{\partial d_\Phi}{\partial \varepsilon}$$

$$\Delta(\varepsilon') = \Delta(\varepsilon) + \delta\varepsilon \frac{\partial \Delta}{\partial \varepsilon}$$

$$d_\Phi(0) + \varepsilon \frac{\partial d_\Phi}{\partial \varepsilon} + \Delta(0) + \varepsilon \frac{\partial \Delta}{\partial \varepsilon} = \varepsilon\left(\frac{\partial d_\Phi}{\partial \varepsilon} + \frac{\partial \Delta}{\partial \varepsilon}\right) - d_g = 0$$

$$\boxed{\varepsilon\left(\frac{\partial d_\Phi}{\partial \varepsilon} + \frac{\partial \Delta}{\partial \varepsilon}\right) = d_g}$$

where $\varepsilon \propto \delta\varepsilon << 1$. Also let

$$d_\Phi \propto \varepsilon^{\eta_\Phi},\ \Delta \propto \varepsilon^{\eta_\Delta}$$

Following (25), the above condition can be presented as

$$\boxed{\eta_\Phi(x) d_\Phi(x) + \eta_\Delta(x)\Delta(x) = d_g(x)} \tag{34}$$

Two prime examples of field theories having dimensionful coupling constants are General Relativity (GR) and the four-fermion model of weak interactions. It is known that quantized versions of such “effective” theories face serious challenges because different diagrams no longer can be added together to cancel non-renormalizable behavior. However, this may no longer the case if spacetime dimension is treated as a continuous variable. For instance, in light of (30) to (34), one can start from the Einstein-Hilbert action of classical gravity

$$S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \tag{35}$$

and come up with a scalar Newton constant by letting coordinates and the components of the metric tensor $g_{\mu\nu}$ flow their dimensions according to

$$D(d_x) \to D'(d_x^{'}) \tag{36a}$$

$$d_{g_{\mu\nu}}(D) \to d_{g_{\mu\nu}}^{'}(D') \tag{36b}$$

$$d_G(D) \to d_{G'}^{'}(D') \tag{36c}$$

This substitution unveils the deep and counterintuitive connection between *conformal symmetry in continuous dimension* and GR [ ]. As shown in the Appendix, this finding lines up well with theories where local scale symmetry is the source of classical gravity, dark matter sector and the mass generation mechanism in SM [ ].

We close this section with a brief remark on how the Lebesgue-Stieltjes measure (3) and (4) may be used to reinforce consistency in field theory.

First, under any scale transformation the measure behaves as [ ]

$$\rho(\lambda x) = \lambda^{(1-\varepsilon_0)+(1-\varepsilon_x)+(1-\varepsilon_y)+(1-\varepsilon_z)}\rho(x), \quad \lambda > 0 \tag{37}$$

where $\varepsilon_\mu = 4 - D_\mu$ ($\mu = 0,1,2,3$) and $\rho(x) = d^4x$. The measure becomes insensitive to λ if deviations from integer dimensions are large ($\varepsilon_\mu = O(1)$) and if

$$\varepsilon_0 + \varepsilon_x + \varepsilon_y + \varepsilon_z = 4 \tag{38}$$

This constraint allows for a *non-commutative* description of spacetime since, in general,

$$\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z \neq \varepsilon_0 \tag{39}$$

Second, when extended to the Path Integral (PI) analysis, (37) can set the stage for *anomaly cancellation* in field theory following the Fujikawa criterion [ ]. To sketch this point, consider the case

$$\varepsilon_{x,y,z}(x) \approx \varepsilon_0(x) = \varepsilon(x) \tag{40}$$

$$\lambda = 1 - \sigma\,, \quad \sigma < 1 \tag{41}$$

(37) takes the form

$$\rho(\lambda x) \approx \{1 - 4[1 - \varepsilon(x)]\sigma\}\,\rho(x) \tag{42}$$

In the PI treatment pioneered by Fujikawa, the symmetry of a quantum theory can be tested by starting from the generating functional. Consider the functional describing a vector field A_μ^a and an axial current source a_μ in four-dimensional spacetime [ ]

$$W[a_\mu, A_\mu^a] = \int [d\psi][d\overline{\psi}]\exp i\int d^4x[L(\psi,\overline{\psi}, A_\mu^a) - a_\mu j_A^\mu] \tag{43}$$

Define an infinitesimal scaling of fermion fields ($\eta(x) << 1$) as in

$$\psi' = (1 - i\eta\gamma_5)\psi \tag{44a}$$

$$\overline{\psi'} = \overline{\psi}(1 - i\eta\gamma_5) \tag{44b}$$

The corresponding change of functional (40) is

$$W[a_\mu - \partial_\mu \eta, A_\mu^a] = J\, W[a_\mu, A_\mu^a] \tag{45}$$

Here, J represents the Jacobian associated with the transformation of PI measure. Its explicit form is given by

$$J = \exp[-i\int d^4x\, \eta(x)\partial^\mu j_{A,\mu}(x)] \tag{46}$$

where, by definition,

$$\int [d\psi][d\overline{\psi}] = \int [d\psi'][d\overline{\psi}']J \tag{47}$$

If the same analysis is carried out in continuous dimension, coordinates are expected to scale according to (25) and (46) turns into

$$J = \exp\{-i\int d^4x[1 - 4(1 - \varepsilon(x))\sigma]\,\eta(x)\partial^\mu j_{A,\mu}(x)\} \tag{48}$$

The theory is anomaly-free if the Jacobian is unitary, that is, if

$$[1 - 4(1 - \varepsilon(x))\sigma]\,\eta(x)\partial^\mu j_{A,\mu}(x) = 0 \tag{49}$$

In particular, for any $\varepsilon(x)$ given by (40), one can always conveniently choose a scale $\sigma = \lambda - 1$ that satisfies

$$\sigma \propto \frac{1}{4[1-\varepsilon(x)]} \tag{50}$$

Although far from being either rigorous or complete, this derivation points out that is possible, at least in principle, to remove anomalies from an underlying theory by embedding it in a spacetime with continuous dimensionality.

5. MASS AND FLAVOR STRUCTURE FROM CONTINUOUS DIMENSION

The goal of this section is to explore two questions: How is local conformal symmetry in continuous dimension related to the physics of SM? In particular, is it possible to derive the mass and flavor structure of SM from a mechanism that mimics electroweak symmetry breaking (EWSB) but occurs in continuous dimension?

Before going into details, let us recall that RG is a powerful framework for understanding the approach to critical behavior in statistical physics and to scale invariance in field theory. In the Wilson picture, RG equations describe the trajectories of operators towards or away from a functional attractor set. According to this model, the flow of masses, gauge couplings, fields and mixing angles is given by the corresponding set of β -functions [ ].

A standard assumption in perturbative QFT is that the attractors of the RG flow consist of a finite number of *isolated fixed points* [ ]. There is preliminary evidence that the end of the RG flow is a limit cycle or an attractor with a more complex structure [ ]. There is also evidence that scale-invariant RG trajectories lead to periodic and quasi-periodic attractors [ ]. These attractors are prone to become unstable under perturbations and, in some cases, acquire a fine structure and turn into *strange attractors*. A straightforward example is the Landau-Ginzburg-Wilson theory and the instability of its Wilson-Fisher attractor. The role played by the degenerate nature of this attractor in generating the mass and flavor structure of SM has been discussed in [ ]. Conformal symmetry is preserved there as the spectrum of masses and gauge charges develops *on the critical surface*. Unlike the conventional scenario of EWSB, this model is free from any fine-tuning (or vacuum stabilization) mechanisms. The relationship between ε , particle masses and gauge charges is given by:

$$\boxed{(g^*)^2 M^2 \sim \overline{\mu_{EW}}^2 = const}$$

$$\boxed{(g^*)^2 \sim m_f^* \sim \varepsilon} \tag{51}$$

Here $\overline{\mu_{EW}}$ denotes the reduced EW scale, g^* the gauge charge on the critical surface, M the vacuum expectation value of the vector boson on the critical surface and m_f^* is the normalized fermion mass[2]. It can be shown that the WF attractor of the RG flow is unstable and changes from a single isolated point to a distribution of points [ ].

One obtains from (51)

$$\frac{M_Z^2}{M_W^2} = \frac{g_2^2 + e^2}{g_2^2} = 1 + \frac{\alpha_{EM}}{\alpha_2} \tag{52}$$

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[2] Parameters entering (51) are defined in [ ]. They have arbitrary magnitude being based on the behavior of the LGW model near a *generic* critical scale $\overline{\mu_{EW}}$. Numerical values of SM parameters may be obtained after applying a suitable rescaling operation in (51) such as $g^2 = \lambda_g (g^*)^2$, $M_b^2 = \lambda_M M^2$ and $\mu_{EW}^2 = \lambda_{EW} \overline{\mu}_{EW}^2$, with $\lambda_{EW} / (\lambda_g \lambda_M) = O[\mu_{EW}^2 / (g^2 M_b^2)]$.

in which $\alpha_{EM} = e^2/4\pi$ is the fine-structure constant and $\alpha_2 = g_2^2/4\pi$ the strength of the weak interaction. The rationale for (52) lies in the fact that the charged gauge boson $W^{\pm}$ carries a superposition of weak and electromagnetic charges, whereas the neutral gauge boson Z^0 carries only the weak isospin charge. Inverting (52) and taking into account the last rows of Table 1 below, leads to

$$\frac{M_W^2}{M_Z^2} = \frac{1}{1+\dfrac{\alpha_{EM}}{\alpha_2}} = \frac{1}{1+\dfrac{1}{\overline{\delta}}} \approx 1 - \frac{1}{\overline{\delta}} = \cos^2\theta_W \tag{53}$$

where $\overline{\delta}$ stands for the Feigenbaum constant [ ]. (53) suggests a natural explanation for the *Weinberg angle* θ_W. We may write (52) as

$$\frac{g_2^2}{M_W^2} = \frac{g_2^2 + e^2}{M_Z^2} = const \tag{54}$$

This relation offers a straightforward interpretation for both Fermi constant and the mass of the *hypothetica*l Higgs boson. Indeed, in SM we have [13]

$$\frac{g_2^2}{M_W^2} = 4\sqrt{2}G_F \tag{55}$$

and

$$\mathrm{v}(\varphi^0) \propto \sqrt{\frac{1}{G_F\sqrt{2}}} \approx 246.22\ GeV \tag{56}$$

where $\mathrm{v}(\varphi^0)$ denotes the vacuum expectation value for the neutral component of the “would-be” Higgs doublet.

Parameter ratio	Behavior	Actual	Predicted
m_u/m_c	$\bar{\delta}^{-4}$	3.365×10^{-3}	4.323×10^{-3}
m_c/m_t	$\bar{\delta}^{-4}$	3.689×10^{-3}	4.323×10^{-3}
m_d/m_s	$\bar{\delta}^{-2}$	0.052	0.066
m_s/m_b	$\bar{\delta}^{-2}$	0.028	0.066
m_e/m_μ	$\bar{\delta}^{-4}$	4.745×10^{-3}	4.323×10^{-3}
m_μ/m_τ	$\bar{\delta}^{-2}$	0.061	0.066
M_W/M_Z	$(1-\frac{1}{\bar{\delta}})^{1/2}$	0.8823	0.8623
$(\alpha_{EM}/\alpha_W)^2$	$\bar{\delta}^{-2}$	0.053	0.066
$(\alpha_{EM}/\alpha_{QCD})^2$	$\bar{\delta}^{-4}$	4.034×10^{-3}	4.323×10^{-3}

Tab 1: Actual versus predicted ratios of SM parameters (except neutrinos)

A similar comparison may be drawn on *neutrinos*. Since neutrino oscillation experiments are only sensitive to neutrino mass squared differences and not to the absolute neutrino mass scale (m_ν^0), they can only supply lower limits for two of the neutrino masses, that is, $(m_{ATM}^2)^{1/2} \approx 5\times10^{-2}$ eV and $(m_{SOL}^2)^{1/2} \approx 1\times10^{-2}$ eV [ ]. As a result, it is more relevant to consider experimentally constrained bounds on m_ν^0 reported from beta decay, neutrinoless double beta decay as well as from cosmological observations [ ].

Based on these inputs, it makes sense to set the upper (U) and lower (L) limit values for the absolute neutrino mass scale as $(m_\nu^0)_U = 2$ eV and $(m_\nu^0)_L = 0.1$ eV. According to Tab. 1, ratios of charged lepton masses scale as $\overline{\delta}^{-2}$ and $\overline{\delta}^{-4}$, which suggests that m_ν^0 should naturally follow a $\overline{\delta}^{-8}$ or $\overline{\delta}^{-16}$ pattern . Table 2 displays a side-by-side comparison on the mass ratio m_ν^0 / m_e for $(m_\nu^0)_U$ and $(m_\nu^0)_L$, respectively, and shows that numerical predictions line up fairly well with current observations.

Parameter ratio	**Behavior**	**Actual**	**Predicted**
m_ν^0 / m_e	$\overline{\delta}^{-8}$	$< 2\times10^{-7}$ $< 4\times10^{-6}$	1.87×10^{-5}
m_ν^0 / m_e	$\overline{\delta}^{-16}$	$< 2\times10^{-7}$ $< 4\times10^{-6}$	3.5×10^{-10}

Tab. 2: Actual vs. predicted ratios of neutrino mass scales.

6. A natural solution for the hierarchy problem

In section 3 we briefly addressed the issue of dimensional regularization which can be applied at both infrared (IR) and ultraviolet (UV) boundaries of the energy scale. The goal of this section is to show that the concept of continuous dimension offers a straightforward solution to the so-called *hierarchy problem* of field theory [ ].

To regularize field theory in the IR one needs to first redefine the limits of integration in (15). If Γ is taken to represent the lowest limit and μ_0 an arbitrary RG scale, a strictly positive ε on less than four dimensions ($D < 4$) requires taking the reciprocal of the logarithm of (15) - (17b) to comply with $\mu_0 > \Gamma$. The IR version of (17b) accordingly reads:

$$\varepsilon' = 4 - D \propto \frac{1}{\log(\mu_0^2 / \Gamma^2)} \tag{58}$$

We proceed with the following assumptions:

6.1) The deep IR cutoff of field theory is set by the cosmological constant scale

$$\Gamma = (\Lambda_{cc})^{1/4} \tag{59}$$

where Λ_{cc} represents the cosmological constant.

6.2) The deep UV cutoff of field theory is set by the Planck scale:

$$\Lambda_{UV} = \Lambda_{Pl} \tag{60}$$

Combining 6.1) and 6.2) implies that, as the EW scale is approached from above or below via inherent statistical fluctuations, (17b) and (58) naturally converge to a common value. Taking $\mu_0 = \mu_{EW}$ and substituting in (17b) and (58) yields

$$\frac{\mu_{EW}}{\Gamma} = \frac{\Lambda_{Pl}}{\mu_{EW}} \rightarrow (\Lambda_{cc})^{1/4} = \frac{\mu_{EW}^2}{\Lambda_{Pl}} \tag{61}$$

Several conclusions may be drawn from (61), namely,

- Asymptotic approach to four-dimensional space-time explains the existence of the deep IR cutoff (Λ_{cc}) and deep UV cutoff (Λ_{Pl}). Stated differently, fractal space-time description supplied by the condition $\varepsilon > 0$ and $\varepsilon' > 0$ appears to be linked to these natural bounds [20].
- Fixing two out of the three scales involved in (61) automatically determines the third one.
- The gauge hierarchy problem, cosmological constant problem and the existence of the EW phase transition appear to be deeply interconnected.
- The derivation presented here stands in sharp contrast with sophisticated approaches to the hierarchy problem based on Supersymmetry (SUSY), Technicolor, Extra-dimensions, Anthropic arguments or gauge unification near the Planck scale.

7. Open questions and future developments

Here is a partial list of open questions that need to be answered in future extensions of our work:

7.1) We have seen that FFT brings up an intriguing picture on phenomena that may unfold beyond SM, in particular, in the deep Terascale sector of high-energy physics. But, since at least in its basic formulation, FFT is not *unitary* and *local*, it is at odds with fundamental principles of QFT. FFT may be however converted to a unitary theory by taking advantage of non-local attributes of fractional differential and integral operators [ ]. As indicated in the text, the key symmetry here is *local conformal invariance in continuous dimension* displayed by dynamics on fractal structures. When associated with self-similarity, this property blurs the distinction between locality and non-locality and makes room for a meaningful connection between FFT and the low-energy regime of SM [ ].

7.2) As suggested in section four, dimensional flow on fractal spacetime enables one to transform dimensionful parameters into scalars and improve consistency of the theory. It remains to be explained why Nature favors this mechanism in the first place.

7.3) The analysis introduced in section five (as well as [ ]) suggests that RG flow in continuous dimension (1) offers a natural framework for understanding the mechanism of EWSB. Under conditions that allow use of Landau-Ginzburg-Wilson theory, one is able to retrieve the entire structure of SM from nonlinear analysis of RG equations. Among many questions related to this topic, we mention the following:

- Are there additional generations of gauge bosons and fermions or is there a stability limit of RG trajectories constraining the number of these flavors? [ ].
- Can flavor mixing and the absence of flavor changing neutral currents be explained using mixing of RG trajectories near transition to chaos?
- Can all electroweak precision observables be correctly recovered?.

- Can all decay channels of particle physics be understood as the result of *chaotic mixing* and *diffusion* of RG trajectories on strange attractors?.

- Is it possible to devise a *non-perturbative formulation of chiral gauge theory* starting from FFT, where breaking of discrete symmetries arises naturally?

8. Conclusion

FFT represent an emerging research topic in mathematical physics. These theories are founded on two premises:

a) spacetime has an intrinsic fractal or multifractal structure at energy scales sufficiently far from the ones describing SM and General Relativity,

b) spacetime dimensionality either stays fixed and arbitrarily close to $D=4$ *or* flows with the probed scale in such a way as to approach $D=4$ at ordinary energies.

Drawing on the properties fractal differential and integral operators, we conjectured herein that the most comprehensive invariance of FFT near $D=4$ is *local conformal symmetry in continuous dimension.* It combines both spatial and internal symmetries, at variance with the constraints imposed by the Coleman-Mandula theorem of conventional QFT. There are several theoretical benefits of this conjecture, namely:

- Under certain conditions, dimensional flow can render a theory dimensionless and set the stage for its renormalizability.
- It can also lead to anomaly cancellation and to a consistent renormalization scheme for QFT [ ].
- It can naturally account for the mass and flavor hierarchies of SM, including neutrinos.

- It can explain in a straightforward manner the hierarchy problem and the cosmological constant problem.

- It does not lead to a strongly coupled field theory and does not require fine-tuning or "ad-hoc" introduction of additional symmetries and/or fields to ensure consistency.

We close by noting that FFT may also explain the existence of broken discrete symmetries in EW interactions (*C* and *CP*) [ ] and provide a sensible interpretation on the composition of non-baryonic "dark matter" [ ].

APPENDIX A

LOCAL CONFORMAL SYMMETRY AS MASS GENERATION MECHANISM

Let $M^{3,1}$ denote a pseudo-Riemannian spacetime with metric $g_{\alpha\beta}$ having signature (+, - , - , -). Let $\Omega(x)=s^{\nu_x(x)}$ be a strictly positive function on defined on $M^{3,1}$ which has an inverse. The local conformal transformation in $M^{3,1}$ is defined through the following change of metric,

$$g_{\mu\nu}(x)\to\overline{g}_{\mu\nu}(x)=\Omega^2(x)g_{\mu\nu}(x) \qquad \text{(A.1)}$$

The set of all local conformal transformations forms the multiplicative abelian infinite-dimensional group *C*. The effect of a *local conformal transformation* is to redefine the length scale according to

$$dl(x)=\sqrt{-g_{ij}dx^i dx^j}\to d\overline{l}(x)=\Omega(x)dl(x) \qquad \text{(A.2)}$$

The meaning of the symmetry associated with group *C* is that the structure of physical laws must be independent from the units chosen to measure length, time and mass. In general, a field theory that depends on *dimensionful* rather than *dimensionless* variables not only fails to comply with local conformal invariance, but typically breaks requirements of gauge symmetry and renormalizability. It also leads to anomalies when classical theory is quantized and radiative corrections are accounted for [ ].

Let Φ be a local tensor or spinor field of arbitrary spin and consider the map

$$\Omega(x) \to U(\Omega(x)) \tag{A.3}$$

whose operation is described by

$$\overline{\Phi}(x) = U(\Omega(x))\Phi(x) = s^{\nu_\Phi(x)}\Phi(x) \tag{A.4}$$

The number

$$\kappa(x) = \frac{\nu_x(x)}{\nu_\Phi(x)} \tag{A.5}$$

is called the *conformal weight* of Φ and map (A.3, A.4) defines the representation of C in field space. Using (A.5) in a global rather than local sense, it can be shown that the Maxwell, Yang-Mills tensors and Dirac field in four dimensional spacetime (D = 4) have conformal weights $\kappa = 0$ and $\kappa =$ - 3/2, respectively [ ].

As with the standard arguments for the existence of gauge fields, demanding that the theory stays invariant to local conformal transformations (A.1), (A.2) and (A.4) implies that there is a gauge field S (called the *Weyl boson)* and a corresponding covariant derivative defined through [ ]

$$\partial_\mu \Phi(x) \to [\partial_\mu + \kappa(x) S_\mu]\,\Phi(x) \tag{A.6}$$

$$S(x) \to S_\mu(x) - \partial_\mu \kappa(x) \tag{A.7}$$

The field tensor associated with S is given by

$$H_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu \tag{A.8}$$

The scalar component of *S* is called the *Weyl scalar* (σ). It has conformal weight -1 in $D = 4$ and satisfies $H_{\mu\nu} = 0$. Since $\kappa = \kappa(x(\varepsilon))$ on account of (24) and (25), the constraint of local conformal transformation induced by (A.6) shows that space-time dimension takes on the role of a *local gauge coupling*. This conclusion is consistent with the content of sections five, where the entire flavor structure of SM emerges from the properties of RG flow in continuous dimension $D = 4 - \varepsilon$.

Weyl boson has several remarkable features, namely [ ]:

a) it does not couple to either one of SM particles. It can only form a Bose-Einstein condensate under the effect of classical gravity and can be thus interpreted as a likely candidate for *dark matter*.

b) Weyl scalar represents the Goldstone boson arisen from breaking of local scale invariance and turns *S* into a massive vector particle.

c) Weyl scalar is related to classical gravity and its vacuum is linked to Newton's constant (*G*).

d) Higgs scenario of EWSB represents a particular embodiment of the Weyl boson model.

REFERENCES (to follow)