

Triangle exact solution of 3-bodies problem.

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Here is presented a system of equations of 3-bodies problem in well-known *Lagrange's form* (*describing a relative motions of 3-bodies*). Analyzing of such a system, we obtain an exact solution in special case of *constant ratios* of relative distances between the bodies.

Above simplifying assumption reduces all equations of initial system *to a proper similar form*, which leads us to a final solution: initial triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, simultaneously rotating over the common center of masses as well as increasing or decreasing of it's size proportionally.

Let us consider the system of an ordinary differential equations for 3-bodies problem, at given initial conditions [1-3]:

$$m_1 \mathbf{q}_1'' = -\gamma \left\{ \frac{m_1 m_2 (\mathbf{q}_1 - \mathbf{q}_2)}{|\mathbf{q}_1 - \mathbf{q}_2|^3} + \frac{m_1 m_3 (\mathbf{q}_1 - \mathbf{q}_3)}{|\mathbf{q}_1 - \mathbf{q}_3|^3} \right\},$$

$$m_2 \mathbf{q}_2'' = -\gamma \left\{ \frac{m_2 m_1 (\mathbf{q}_2 - \mathbf{q}_1)}{|\mathbf{q}_2 - \mathbf{q}_1|^3} + \frac{m_2 m_3 (\mathbf{q}_2 - \mathbf{q}_3)}{|\mathbf{q}_2 - \mathbf{q}_3|^3} \right\},$$

$$m_3 \mathbf{q}_3'' = -\gamma \left\{ \frac{m_3 m_1 (\mathbf{q}_3 - \mathbf{q}_1)}{|\mathbf{q}_3 - \mathbf{q}_1|^3} + \frac{m_3 m_2 (\mathbf{q}_3 - \mathbf{q}_2)}{|\mathbf{q}_3 - \mathbf{q}_2|^3} \right\}.$$

- here $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ - means the radius-vector of bodies m_1, m_2, m_3 , accordingly.

For the purposes of exploring a relative motions of 3-bodies one to each other, let's rewrite the system above as below (by linear transformation of initial equations):

$$(\mathbf{q}_1 - \mathbf{q}_2)'' + \gamma(m_1 + m_2) \frac{(\mathbf{q}_1 - \mathbf{q}_2)}{|\mathbf{q}_1 - \mathbf{q}_2|^3} = \gamma m_3 \left\{ \frac{(\mathbf{q}_3 - \mathbf{q}_1)}{|\mathbf{q}_3 - \mathbf{q}_1|^3} + \frac{(\mathbf{q}_2 - \mathbf{q}_3)}{|\mathbf{q}_2 - \mathbf{q}_3|^3} \right\},$$

$$(\mathbf{q}_2 - \mathbf{q}_3)'' + \gamma(m_2 + m_3) \frac{(\mathbf{q}_2 - \mathbf{q}_3)}{|\mathbf{q}_2 - \mathbf{q}_3|^3} = \gamma m_1 \left\{ \frac{(\mathbf{q}_3 - \mathbf{q}_1)}{|\mathbf{q}_3 - \mathbf{q}_1|^3} + \frac{(\mathbf{q}_1 - \mathbf{q}_2)}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \right\},$$

$$(\mathbf{q}_3 - \mathbf{q}_1)'' + \gamma(m_1 + m_3) \frac{(\mathbf{q}_3 - \mathbf{q}_1)}{|\mathbf{q}_3 - \mathbf{q}_1|^3} = \gamma m_2 \left\{ \frac{(\mathbf{q}_1 - \mathbf{q}_2)}{|\mathbf{q}_1 - \mathbf{q}_2|^3} + \frac{(\mathbf{q}_2 - \mathbf{q}_3)}{|\mathbf{q}_2 - \mathbf{q}_3|^3} \right\}.$$

Let's designate as below:

$$\mathbf{R}_{1,2} = (q_1 - q_2), \quad \mathbf{R}_{2,3} = (q_2 - q_3), \quad \mathbf{R}_{3,1} = (q_3 - q_1) \quad (*)$$

Above designating causes the transformation of a previous system to another form:

$$\begin{aligned} \mathbf{R}_{1,2}'' + \gamma(m_1 + m_2) \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} &= \gamma m_3 \left\{ \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} + \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} \right\} \\ \mathbf{R}_{2,3}'' + \gamma(m_2 + m_3) \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} &= \gamma m_1 \left\{ \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} + \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} \right\} \\ \mathbf{R}_{3,1}'' + \gamma(m_1 + m_3) \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} &= \gamma m_2 \left\{ \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} + \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} \right\}. \end{aligned} \quad (1.1)$$

Analysing system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

$$\mathbf{R}_{1,2}'' + \mathbf{R}_{2,3}'' + \mathbf{R}_{3,1}'' = 0 .$$

If we also sum all the equalities (*) one to each other, we should obtain

$$\mathbf{R}_{1,2} + \mathbf{R}_{2,3} + \mathbf{R}_{3,1} = 0 \quad (**)$$

Besides, if we substitute an expression for $\mathbf{R}_{3,1} / |\mathbf{R}_{3,1}|^3$ - from 2-nd to 1-st equation of (1.1), then $\mathbf{R}_{1,2} / |\mathbf{R}_{1,2}|^3$ to the 3-d - we should obtain below:

$$\begin{aligned}
& \left\{ \mathbf{R}_{1,2}'' + \gamma(m_1 + m_2 + m_3) \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} \right\} \frac{1}{m_3} = \mathbf{F}(t), \\
& \left\{ \mathbf{R}_{2,3}'' + \gamma(m_1 + m_2 + m_3) \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} \right\} \frac{1}{m_1} = \mathbf{F}(t), \quad (1.2) \\
& \left\{ \mathbf{R}_{3,1}'' + \gamma(m_1 + m_2 + m_3) \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} \right\} \frac{1}{m_2} = \mathbf{F}(t) .
\end{aligned}$$

So, the linear recombining of equations (1.1) let us define some vector function $\mathbf{F}(t)$ which seems to be unique for all equations of (1.2). Otherwise, taking into consideration (**), we also obtain

$$\begin{aligned}
& \mathbf{R}_{1,2}'' + \mathbf{R}_{2,3}'' + \mathbf{R}_{3,1}'' + \gamma(m_1 + m_2 + m_3) \left\{ \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} + \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} + \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} \right\} = \\
& = \mathbf{F}(t) \cdot (m_1 + m_2 + m_3), \quad \Rightarrow \\
& \Rightarrow \mathbf{F}(t) = \gamma \left\{ \frac{\mathbf{R}_{1,2}}{|\mathbf{R}_{1,2}|^3} + \frac{\mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} + \frac{\mathbf{R}_{3,1}}{|\mathbf{R}_{3,1}|^3} \right\} .
\end{aligned}$$

It is well-known fact [1-3] that there are existing only 5 cases of exact (1.1) solutions (below $\mathbf{R}_i = (x_i, y_i, z_i)$, $i = 1,2; 2,3; 3,1$):

- 3 Lagrange's linear cases, when $\mathbf{R}_{1,2} \sim \mathbf{R}_{2,3} \sim \mathbf{R}_{3,1}$
- 2 Euler's cases of equipotential triangle, when

$$|\mathbf{R}_{1,2}| = |\mathbf{R}_{2,3}| = |\mathbf{R}_{3,1}| \quad \Leftrightarrow \quad (**) \quad \Rightarrow \quad \mathbf{F}(t) = 0 .$$

Let's consider a solutions of (1.2) for which is valid an assumption below

$$\frac{|\mathbf{R}_{2,3}|}{|\mathbf{R}_{1,2}|} = a = \text{const} , \quad \frac{|\mathbf{R}_{2,3}|}{|\mathbf{R}_{3,1}|} = b = \text{const} \quad (***)$$

It means that proportions of absolute meanings of relative distances $\mathbf{R}_{1,2}$, $\mathbf{R}_{2,3}$, $\mathbf{R}_{3,1}$ between the bodies *should be the same all the time & should be equal to the initial proportions (which are given by special initial conditions)*.

It could be possible only if such a triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, rotating over the center of masses as well as increasing or decreasing the lengths of sides of such a triangle proportionally.

Besides, we obtain:

$$\mathbf{F}(t) = \gamma \cdot \left(\frac{a^3 \cdot \mathbf{R}_{1,2} + \mathbf{R}_{2,3} + b^3 \cdot \mathbf{R}_{3,1}}{|\mathbf{R}_{2,3}|^3} \right), \quad \mathbf{R}_{3,1} = -\mathbf{R}_{2,3} - \mathbf{R}_{1,2} ,$$

$$\Rightarrow \mathbf{F}(t) = \gamma \cdot \left(\frac{(a^3 - b^3) \cdot \mathbf{R}_{1,2} + (1 - b^3) \cdot \mathbf{R}_{2,3}}{|\mathbf{R}_{2,3}|^3} \right).$$

So, in case $a = b = 1$ we obtain *Euler's cases of equipotential triangle*, but in the case $\mathbf{R}_{1,2} \sim \mathbf{R}_{2,3} \sim \mathbf{R}_{3,1}$ all the equations of system (1.2) could be reduced to one of *Lagrange's linear cases* [1].

In according with assumption (***) above, such a solution of (1.2) should be factorized as below ($\mathbf{R}_o = \mathbf{R}(t_o)$):

$$\mathbf{R} = \frac{\mathbf{R}_o}{|\mathbf{R}_o|} \cdot R(t) \cdot \sin(\omega t + \varphi_o) ,$$

- here $\mathbf{R} = \mathbf{R}_i$ – is a vector of general motion, which describes *an identical character* of evolution for each of 3-bodies relative distances $\mathbf{R}_{1,2}, \mathbf{R}_{2,3}, \mathbf{R}_{3,1}$ ($i = 1,2, 2,3, 3,1$), besides ($\varphi_o = \varphi(t_o)$ – *the initial angle when triangle of bodies began to rotate*):

$$R(t) = \|\mathbf{R}\| \cdot \max \{ \sin(\omega t + \varphi_o) \}$$

– is the scale factor or measure of appropriate relative distances between the bodies.

As for the integral of momentum, for such a case it describes *a harmonic character* of rotation of triangle m_1, m_2, m_3 over the center of masses (ω – *is the angle velocity of triangle rotation, I – is the proper moment of inertia of such a triangle*):

$$I \cdot \omega = \text{const}, \quad I \sim R^2, \quad \Rightarrow \quad \omega = \frac{C}{R^2}, \quad C = \text{const} \quad (1.3)$$

Thus, if an assumption below is valid

$$\frac{|\mathbf{R}_{2,3}|}{m_1} = \frac{|\mathbf{R}_{1,2}|}{m_3}, \quad \frac{|\mathbf{R}_{2,3}|}{m_1} = \frac{|\mathbf{R}_{3,1}|}{m_2}$$

$$\Rightarrow \quad \frac{m_1}{m_3} = a, \quad \frac{m_1}{m_2} = b,$$

- above (1.2) system of vector equations could be reduced *to only one* ODE below:

$$\left(R(t) \cdot \sin(\omega t + \varphi_0) \right)'' + \gamma \left\{ m_1 + m_2 + m_3 \cdot \left(\frac{b^3 + (b^3 - 1) \cdot \alpha_{xi}}{a^3} \right) \right\} \frac{\text{sign} [\sin(\omega t + \varphi_0)]}{\left(R(t) \cdot \sin(\omega t + \varphi_0) \right)^2} = 0,$$

- where α_{xi} – are the coefficients of proportionality between the initial coordinates of vectors $\mathbf{R}_{2,3} / |\mathbf{R}_{2,3}|$, $\mathbf{R}_{1,2} / |\mathbf{R}_{1,2}|$. We also should take into consideration the expression (1.3) for angle velocity: $\omega = C / R^2$, $C = \text{const}$.

Such an ordinary differential equation for finding the function $R(t)$ is very complicated to solve by analytical methods, so it should be solved by numerical math methods.

Besides, according to the *Brun's theorem* [4], we know that there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*).

Let's summarise:

First of all, we represent the equations of 3-bodies problem in appropriate *Lagrange form* (1.2), describing *a relative motions* of 3-bodies. Such a system is proved to describe a motions of *similar character* for evolution of each of 3-bodies relative distances $\mathbf{R}_{1,2}$, $\mathbf{R}_{2,3}$, $\mathbf{R}_{3,1}$.

Then we consider a solutions of (1.2) for which is valid an assumption below:

$$\frac{m_1}{m_3} = \frac{|\mathbf{R}_{2,3}|}{|\mathbf{R}_{1,2}|} = a, \quad \frac{m_1}{m_2} = \frac{|\mathbf{R}_{2,3}|}{|\mathbf{R}_{3,1}|} = b.$$

Besides, we assume that triangle of 3-bodies m_1, m_2, m_3 is rotating on circle orbit around the common center of masses as well as increasing or decreasing the size of above triangle proportionally. Size (*radius*) of such an orbit is determined by masses m_1, m_2, m_3 as well as by parameters a, b . It means that proportions of absolute meanings of the relative distances $\mathbf{R}_{1,2}, \mathbf{R}_{2,3}, \mathbf{R}_{3,1}$ should be *the same* all the time & should be equal to the initial proportions (*which are given by special initial conditions*).

Thus, such a solution of (1.2) should be factorized as below ($\mathbf{R}_o = \mathbf{R}(t_o)$):

$$\mathbf{R} = \frac{\mathbf{R}_o}{|\mathbf{R}_o|} \cdot R(t) \cdot \sin(\omega t + \varphi_o) ,$$

- here $\omega = C / R^2$, $\varphi_o = \varphi(t_o)$ – the initial angle when triangle of bodies began to rotate, $\mathbf{R} = \mathbf{R}_i$ – is a vector of general motion, which describes *the identical character* of evolution for each of 3-bodies relative distances $\mathbf{R}_{1,2}, \mathbf{R}_{2,3}, \mathbf{R}_{3,1}$ ($i = 1,2, 2,3, 3,1$), where:

$$R(t) = |\mathbf{R}| \cdot \max \{ \sin(\omega t + \varphi_o) \}$$

– is the scale factor or measure for appropriate relative distances between the bodies.

Finally, all vector equations of (1.2) could be reduced *only to one* ODE below:

$$\left(R(t) \cdot \sin(\omega t + \varphi_o) \right)'' + \gamma \left\{ m_1 + m_2 + m_3 \cdot \left(\frac{b^3 + (b^3 - 1) \cdot \alpha_{xi}}{a^3} \right) \right\} \frac{\text{sign} [\sin(\omega t + \varphi_o)]}{\left(R(t) \cdot \sin(\omega t + \varphi_o) \right)^2} = 0 ,$$

- where α_{xi} – are the coefficients of proportionality between the proper coordinates of initial vectors $\mathbf{R}_{2,3} / |\mathbf{R}_{2,3}|$, $\mathbf{R}_{1,2} / |\mathbf{R}_{1,2}|$.

The last ordinary differential equation - *in regard to the function $R(t)$* - is very complicated to solve by analytical methods, but it could be solved properly only by numerical math methods.

References:

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