Gravitational Theory with Variable Rest Mass Defect

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ABSTRACT

The presentation here is based on the presumption that the total energy of a particle and photon is localized, and conserved on entering a static gravitational field. A mass particle thus entering a static gravitational field has an increasing velocity, but a decreasing rest mass, or a mass defect. On stopping at the surface of a gravitating body the kinetic energy of the particle is radiated away leaving the system with a mass defect, the same as in all other conservative particle field interactions. From Noether's theorem, it is known that energy in the Riemannian representation cannot be localized; it must be a property of the field. It is asserted here that a theory of gravitation can be formulated, that properly predicts known dynamic features, has proper covariant transformations, local conservation of energy, and an absence of Black Holes, without resorting to curved space. The Shapiro velocity of light can be deduced, and thus the proper stellar bending of light can be shown. The ideas presented here are not completely new, but the proposal will be shown to be all that is necessary to reproduce the phenomenology of GR. There are points of this development that are testable, and should prove or disproved validity in experiments on Black Holes and, Event horizons.

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INTRODUCTION

As is well known, but presumed unimportant, are aspects of the Ricci tensor representation that illustrate the theory is an approximation to the correct representation, but is not complete or accurate representation

The obvious shortfalls are that:

1. mass is represented as a continuous density function, when reality requires discrete point functions, and there is insufficient complexity in the tensor expressions to represent mass as a collection of point particles.

2. it does not scale down, nor properly function at microscopic level, though the theory recognizes no scaling limits.

3.it is a gauge field, with an infinite number of infinitesimal generators. Because of this, Noether's theorem illustrates that energy tensor is not covariant under general coordinate transformations, and there can be no local conservation of energy, meaning the flow of energy in and out of a spacetime volume is not conserved[9]. This leads directly the concept of black holes, since the source of the kinetic energy gained by a particle entering a gravitational field comes from the field and not the rest mass. There is no mass defect.

4. it is not covariant under general coordinate transformations. Local energy balance is dependent on the coordinates used for the calculation, consequently different results are obtained for different coordinate frames[4].

5. the distance between any two points in the defined curved space is ambiguous, and dependent on the path[4].

Researchers who do not view GR as an approximation do not consider these issues shortcomings, but the reality of physics.

The dynamic particle interactions presented here are formulated in covariant differential and algebraic relations between mass points, and only because of physical symmetry, would they apply to massive bodies of particles. It is shown that the phenomenology of GR can be reproduced without resorting to Riemannian space curvature and does not result in unphysical singularities. This development will adhere to a flat Minkowski space ($\delta\Delta t = 0$) and a variable speed of light.

The most likely test should come from observations of ray tracings of light rays following Fermat paths pass near the Schwarzschild radius, or neutron stars larger than allowed by GR. GR predicts rays cannot pass closer that the Schwarzschild radius without being captured however this presentation has a boundary at the gravitational radius.

Our first assumption will be that the total energy of a particle is localized in the volume of the particle, and that there is no static energy content in the fields related to the particle.

The rest mass being defined, has similarity to the Komar mass in GR, in that it is dependent on the Gravitational potential. The Komar mass is determined by integrating Einstein's Equation

$$R_t^t = 8\pi\left(T_t^t - 1/2\,T\right), \tag{1.1}$$

over a large volume in the 3-space generated with t = constant [10]. In some Kerr configurations this integral can become negative, which some researchers consider to be unphysical [11].

In this presentation, we will take the mass of a particle to be defined relative to an observer, noting that the mass of the observer is also dependent on the gravitational potential. The total energy is considered to reside locally at the point of a particle, and would be observed to have a different value for an observer located at a different gravitational potential.

Current views of photon energy in GR are contrary to Einstein's original view that the photon energy is constant and the shift is due slower clock associated with the emission. [12]. Current views of GR are that the photon looses energy to the field on rising.

This development is cast in Minkowski space, and ascribes the frequency of a photon rising in a gravitational field to have a lower frequency at the elevated receptor due to the lower rest mass of the emitter not a change in the time scale of a loss of energy. After taking account of the rest mass change at different elevations, the results of the Pound-Rebka-Snider experiment indicates that the energy of the photon must be conserved. [6]

GENERAL DEVELOPMENT

Our initial assumption is that for a massive particle in a gravitational potential, the total mass of a particle at rest relative to an observer external to the field is defined by:

$$M^2 = M_0^2\left(1-\frac{\mu}{r}\right)^2 \tag{1.2}$$

Where M_0 is the rest mass external to the gravitational potential. The relativistic mass is then:

$$M^2\left(1-\frac{v^2}{c^2}\right) = M_0^2\left(1-\frac{\mu}{r}\right)^2 \tag{1.3}$$

Though similar this is a 2nd order departure from standard expressions. It is easy to show that this expression the same as the well known Lagrangian within measurable accuracy. i.e.

$$Mc^2 = \left(M_0c^2 - \frac{GMm}{r} + \frac{1}{2}Mv^2 \right) \tag{1.4}$$

Eq.(1.3) , is the fundamental relation, but there are subtle relativistic issues related to the interaction ϕ term that must be included.

$$\phi = \frac{GMm}{r}, \tag{1.5}$$

First to be noted is that the mass terms have to be the relativistic mass. This is obvious from the fact that, if the particles happen to be spinning the kinetic energy must be included, meaning the mass is relativistic mass. In addition each mass experience the other as if it is moving with their relative velocity.

From our knowledge of the Thomas precession, it is known that the distance a particle traveling the circumference of a circle around an attracting potential is shortened by the relativistic contraction. We would assert that if the circumference of a circle is contracted as the result of the relativistic velocity, the radius must also be contracted.

With those considerations the gravitation term in Eq.(1.4) , must be:

$$\frac{GMm}{r} \to \frac{G}{r_0\sqrt{1-v^2/c^2}}\frac{M_0}{\sqrt{1-v^2/c^2}}\frac{m_0}{\sqrt{1-v^2/c^2}} \tag{1.6}$$

or:

$$\frac{GMm}{r} \to \frac{GMm}{r\sqrt{1-v^2/c^2}^{3}} \to M\frac{\mu_{loc}}{r}\left(1+\frac{3}{2}\frac{v^2}{c^2}\right) \tag{1.7}$$

Orbital Mechanics

We now have a differential expression relating, the velocity, and the distance to the local gravitating mass. We should thus be able to solve for the orbital motion, without need to make assumptions about the force mass relation.

In the following it will be shown that the equations of motion produces orbital relations, equivalent to the weak field GR relations, with the same perihelion advance:

Noting. (1.7), and putting this into E. (1.3), we have:

$$M_0\left[1-\left(\frac{\mu_{loc}}{r}+\frac{\mu_{loc}}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \tag{1.8}$$

Noting that there is only one significant cross term this becomes:

$$M_0\left[1-\frac{\mu}{r}-\frac{3}{2}\frac{\mu}{r}\frac{v^2}{c^2}-\frac{\mu^2}{2r^2}-\left(\frac{1}{2}\frac{\mu}{r}\frac{v^2}{c^2}\right)+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\right]=M \tag{1.9}$$

We can separate this into:

$$-\frac{M-M_0}{M_0}c^2=c^2\left(-\frac{\mu}{r}-\frac{1}{2}\frac{\mu^2}{r^2}+\frac{1}{2}\frac{v^2}{c^2}\left(1-4\frac{\mu}{r}+\right)+\frac{3}{8}\frac{v^4}{c^4}\right) \tag{1.10}$$

Setting the left term in this to $\in$, we note that in a conservative system, this term is constant. This is because M_0 is a defined constant and the total energy is constant.

Using the procedures for finding as outlined in Robertson & Noonan,[4] the perihelion precession, in agreement with GR is:

$$\sigma=\left(\frac{1}{2}\mu u_0-\frac{3}{2}\mu u_0+2\mu u_0+2\mu u^2\right)=\left(\mu u_0+2\mu u^2\right)\sim 3\frac{\mu}{p} \tag{1.11}$$

The detailed calculations for this are included in Appendix I.

q.e.d. It has been shown that the proper orbital equations can be derived without resorting to Riemannian spacetime.

Photon Energy

From the defining relation of this theory Eq.(1.3), the view of the Pound-Rebka-Snider[6], Mossbauer effect experiment (1960–1965)[6] changes. Instead of the photon losing energy as the photon rises in the tower, the emission of the photon at the bottom of the tower is from a less massive generator, and at a lower frequency. The generated frequency plus the added Doppler frequency provided by the velocity of the source in the experiment equals the frequency at the top, thus the photon loses no energy in the flight up the tower.

$$\nu_B\left(1-\frac{\mu}{r}\right)+\nu_D=\nu_T\,. \tag{1.12}$$

This is a departure from General Relativity. GR requires a photon escaping from a gravitational field to lose energy to the field, and in the case of a black hole the entirety of the energy is lost before escapement. Since the energy in discussed here is localized and not lost to the gravitational field, the Schwarzschild radius is no barrier.

Proper Deflection and Velocity of Light

The purpose of the following thought experiment is to deduce the change in value of the relative velocity of light inside a gravitational field, based on the fact that the rest mass is dependent on the elevation in the gravitational potential.

Assumptions

1) The emitted frequencies of photons from an atom, and thus extended to an atomic clock, is proportional to the rest mass. This assumption correspondes to the potential time dilation of General Relativity, however in this case, the change is the result, not of a change in the time scale, but a change in the rest mass.

2) The physical dimensions of a material object at rest are invariant in a gravitational potential.

In order to determine the speed of light shift in a gravitational potential a thought experiment based on the above assumptions can be devised.

1) A laser interferometer is set up in an elevator on the top floor of a building, with a standing wave, having an integral number of wavelength across a resonating cavity.

2) The apparatus is lowered to the bottom floor.

3) We will make the assumption that there is no observable difference in the number of standing waves in the resonating cavity. This is also required by the equivalence principle

Using our assumptions, and Eq(1.12). the frequency has decreased as a result of the decreased rest mass of the system at the lower position, and is:

$$\nu = \nu_0\left(1-\frac{\mu}{r}\right) \tag{1.13}$$

If the frequency has declined by the potential factor then the wavelength would extend beyond the interferometer space, if there were not an equivalent reduction of the wavelength by the same factor.

$$\lambda = \lambda_0\left(1-\frac{\mu}{r}\right) \tag{1.14}$$

Since the product of $\nu\lambda$ is the velocity of light, we have for a change in the velocity:

$$c \;=\; \lambda_0\nu_0\left(1-\frac{\mu}{r}\right)\left(1-\frac{\mu}{r}\right) \;=\; c_0\left(1-\frac{\mu}{r}\right)^2 = 1-2\frac{\mu}{r}+\frac{\mu}{r}^2 \tag{1.15}$$

The first two terms of this expression are the correct values by way of the measured Shapiro effect, and thus by way of Fermat's principle the proper stellar deflection of light has been has been deduced and is in agreement with GR, as shown by Blandford et al [7]. It is the third term μ^2/r that is not present in GR and yet to be measured that distinguishes this theory.

From the velocity of light in shown in Eq.(1.15), and the fact that the photon rising in a gravitational field does not lose energy to the field, shown in Eq.(1.12), it is apparent that the Fermat photon trajectories for this theory would be established by the index of refraction:

$$\eta = \eta_0/\left(1-\mu/r\right)^2 \tag{1.16}$$

Whereas the GR trajectories would be established by the Shapiro Velocity and index of refraction:

$$\eta = \eta_0 / \left(1 - 2\mu / r\right) \tag{1.17}$$

The second order differences in these two expressions should be soon measurable by deflection experiments, either by black holes (Event Horizon Telescope) or by solar experiments (LATOR mission).

CONCLUSION

With simple assumptions regarding the relation between rest mass, and relativistic mass, proper gravitational dynamics and stellar deflection phenomena can be predicted. The proposed theory yields the proper orbital equations, with the proper perihelion advance, deflection of light and gravitational red shift. The gravitational potential exchanges no energy with photons, thus photons are not bound in a gravitational field, and there are no black holes, a belief often expressed by Einstein [13]. Precision light deflection experiments near large masses, or discoveries of neutron star masses larger than GR allows, will validate or invalidate this theory.

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Appendix I

Details of Perihelion Advance

The general rest mass velocity relation proposed is:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-2\frac{\mu}{r}\right) \tag{2.1}$$

Where the velocity invariant potential is:

$$M_0^2\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^3}\right)=M^2\left(1-\frac{v^2}{c^2}\right) \tag{2.2}$$

Taking square root:

$$M_0\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^3}\right)^{1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}=M \tag{2.3}$$

Binomial expansions:

$$
\begin{aligned}
(1-x)^{1/2} &= 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16} \\
(1+x)^{-1/2} &= 1-\frac{x}{2}+\frac{3}{8}x^2 \\
(1-x)^{-1/2} &= 1+\frac{x}{2}+\frac{3}{8}x^2+\frac{5}{16}x^3
\end{aligned}
\tag{2.4}
$$

The simple expansion would be:

$$
\left[1-2\frac{\mu_{loc}}{r_{loc}}\right]^{1/2} = \left[1-\frac{\mu_{loc}}{r_{loc}}-\frac{1}{2}\frac{\mu_{loc}}{r_{loc}}\frac{\mu_{loc}}{r_{loc}}\right] \tag{2.5}
$$

Expanding all the terms in Eq. (2.3).

$$
M_0\left(\left(1-\frac{\mu_{loc}}{r_{loc}}\left(1-v_2^2/c_2^2\right)^{-3/2}-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right)\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{2}\left(\frac{v}{c}\right)^4\right)\right)=M\,, \tag{2.6}
$$

and:

$$
M_0\left(\left[1-\left(\frac{\mu_{loc}}{r_{loc}}+\frac{\mu_{loc}}{r_{loc}}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]\right)=M \tag{2.7}
$$

There is only one cross term of significant value.

v

$$
M_0\left[1-\frac{\mu}{r}-\frac{\mu}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}-\left[\frac{\mu}{r}\frac{1}{2}\left(\frac{v}{c}\right)^2\right]+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \tag{2.8}
$$

Simplifying and separating the mass terms:

$$
\begin{aligned}
&M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4-\frac{4}{2}\frac{\mu}{r}\left(\frac{v}{c}\right)^2\right]=M \\
&M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M \\
&M_0+M_0\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M \\
&\frac{M-M_0}{M_0}-\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=0
\end{aligned}
\tag{2.9}
$$

multiplying by c^2, & noting that in a conservative system where the total energy is constant, the mass term is constant.

$$c^2 \frac{M - M_0}{M_0} = \in \tag{2.10}$$

Thus:

$$2 \in - \left[-2c^2 \left(\frac{\mu}{r} + \frac{1}{2} \frac{\mu}{r} \frac{\mu}{r} \right) + v^2 \left(1 + \frac{3}{4} \left(\frac{v}{c} \right)^2 - 4 \frac{\mu}{r} \right) \right] \tag{2.11}$$

The corresponding GR term per Robertson & Noonan.

$$2 \in + \frac{2\mu c^2}{r} - v^2 + \frac{2h^2\mu}{r^3} \tag{2.12}$$

Some conventional coordinate transformations:

$$\begin{bmatrix} u - 1/r & u^2 \left(r^2 \dot{\theta} \right)^2 = u^2 h^2 & \left(\frac{dr}{dt} \right)^2 = h^2 \left(\frac{du}{d\theta} \right)^2 \\ v^2 = \left[\left(\frac{dr}{dt} \right)^2 + u^2 \left(r^2 \dot{\theta} \right)^2 \right] & = & \left[h^2 \left(\frac{du}{d\theta} \right)^2 + u^2 h^2 \right] \end{bmatrix} \tag{2.13}$$

making the substitutions, we have:

$$2 \in - \left(-2c^2 \left(1 + \frac{1}{2} \mu u \right) \mu u + v^2 \left(1 - 4\mu u \right) + v^2 \frac{3}{4} \left(\frac{v}{c} \right)^2 \right) \tag{2.14}$$

Now taking the derivative with respect to the angular coordinate:

$$\frac{d}{d\theta} \left[2 \in - \left(-2c^2 \left(1 + \frac{1}{2} \mu u \right) \mu u + \left[\left(\frac{dr}{dt} \right)^2 + u^2 \left(r^2 \dot{\theta} \right)^2 \right] \left(1 - 4\mu u \right) + v^2 \frac{3}{4} \left(\frac{v}{c} \right)^2 \right) \right], \tag{2.15}$$

or:

$$\frac{d}{d\theta} \left[2 \in - \begin{pmatrix} -2c^2 \left(1 + \frac{1}{2} \mu u \right) \mu u \\ + \left[\left(\frac{dr}{dt} \right)^2 + u^2 h^2 \right] \left(1 - 4\mu u \right) \\ + v^2 \frac{3}{4} \left(\frac{v}{c} \right)^2 \end{pmatrix} \right] \tag{2.16}$$

Making some substitutions.

$$0=\frac{d}{d\theta}\left[2\in-\left(\begin{array}{l}-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u\\ +h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right](1-4\mu u)\\ +\frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2\right]^2\end{array}\right)\right] \qquad \frac{\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2}{v^2=\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2h^2\right]} \tag{2.17}$$

Differentiating the three terms, designating each as A,B, & C:

$$\left[\frac{d}{d\theta}\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu\right)= -2\mu c^2(1+\mu u)\frac{du}{d\theta} = -2h^2\frac{du}{d\theta}\frac{\mu c^2}{h^2}(1+\mu u)\right] \quad (A)$$

Parts of the B term:

$$\left[\begin{array}{l}\frac{d}{d\theta}h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]\\ = h^2\left[2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right)+2u\left(\frac{du}{d\theta}\right)\right]\\ =\left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right)+u\right]\\ \\ \frac{d}{d\theta}(1-4\mu u)= -4\mu\left(\frac{du}{d\theta}\right)\\ \end{array}\right] \tag{2.18}$$

So the B term is:

$$\left[\begin{array}{l}\frac{d}{d\theta}\left\{h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right](1-4\mu u)\right\} \\ \quad = \left[\begin{array}{l}-h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]4\mu\left(\frac{du}{d\theta}\right) \\ +\left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right)+u\right](1-4\mu u)\end{array}\right] \\ \quad = 2h^2\left(\frac{du}{d\theta}\right)\left[\begin{array}{l}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right](-2\mu) \\ +\left[\left(\frac{d^2u}{d\theta^2}\right)+u\right](1-4\mu u)\end{array}\right]\end{array}\right] \quad (B) \qquad (2.19)$$

And the C term:

$$\left[\begin{array}{l}\frac{d}{d\theta}\ \frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2\right]^2 \\ \quad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\frac{d}{d\theta}\left(\left(\frac{du}{d\theta}\right)^2+u^2\right) \\ \quad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\left(2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right)+2u\left(\frac{du}{d\theta}\right)\right) \\ \quad = 2h^2\left(\frac{du}{d\theta}\right)\frac{3}{2}\frac{1}{c^2}h^2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\left(\left(\frac{d^2u}{d\theta^2}\right)+u\right) \\ \quad = \left(2h^2\left(\frac{du}{d\theta}\right)\right)\frac{3}{4}\left[2\frac{1}{c^2}h^2u^2\right]\left(\left(\frac{d^2u}{d\theta^2}\right)+u\right) \qquad \left(\frac{du}{d\theta}\right)^2\sim 0\end{array}\right] \quad (C)(2.20)$$

Collecting and factoring a common term gives:

$$0 = -2h^2\left(\frac{du}{d\theta}\right)\begin{bmatrix} - & \frac{\mu c^2}{h^2}\left(1+\mu u\right) \\ + & \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(-2\mu\right) \\ + & \left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\left(1-4\mu u\right) \\ + & \left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] \end{bmatrix} \tag{2.21}$$

collecting common terms reduces the number of terms:

$$0 = \begin{bmatrix} - & \frac{\mu c^2}{h^2}\left(1+\mu u\right) \\ + & \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(-2\mu\right) \\ + & \left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\left(1 + \frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] - 4\mu u\right) \end{bmatrix} \tag{2.22}$$

Dividing by the coefficient of the second order term gives:

$$0 = \begin{bmatrix} -\frac{\mu c^2}{h^2}\left(1+\mu u\right) / \left(1 + \frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] - 4\mu u\right) \\ -\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(2\mu\right) / \left(1 + \frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] - 4\mu u\right) \\ +\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right] \end{bmatrix} \tag{2.23}$$

or:

$$0=\begin{bmatrix} -\dfrac{\mu c^2}{h^2}\left(1+\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ -\left(2\mu u^2\right)\left(1-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ +\left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right] \end{bmatrix} \qquad \left(\frac{du}{d\theta}\right)^2 \sim 0 \tag{2.24}$$

The equation for a circle is:

$$\left(\frac{d^2u}{d\theta^2}\right)+u=u_0+f \tag{2.25}$$

Where f is a perturbation of the orbit.

The precession, per the procedure of Robertson & Noonan is $\sigma=\dfrac{1}{2}\dfrac{\partial}{\partial u}f$.

Where in this case f is:

$$f=\begin{bmatrix} u_0\left(+\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ \left(2\mu u^2-3\mu\left[\dfrac{h^2u^4}{c^2}\right]+8\mu^2u^3\right) \end{bmatrix} \tag{2.26}$$

Where: $\dfrac{\mu c^2}{h^2}= u_0=\dfrac{1}{p\left(1-e^2\right)}$.

So:

$$\sigma=\frac{1}{2}\frac{\partial}{\partial u}\begin{bmatrix} u_0\left(+5\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+2\mu u^2\right) \\ +\left(-3\mu\left[\dfrac{h^2u^4}{c^2}\right]\ +8\mu^2u^3\right) \end{bmatrix} \tag{2.27}$$

Where σ is a ratio of the perihelion advance to the orbit circumference

$$\sigma = \frac{1}{2}\left[\left(\begin{matrix} \left(+5\mu u_0 - 3\left[\frac{h^2}{c^2}\right]u_0 u + 4\mu u\right) \\ +\left(-12\mu\left[\frac{h^2 u^3}{c^2}\right] + 24\mu^2 u^2\right) \end{matrix}\right)\right] \qquad \frac{\left[\frac{h^2}{c^2}\right] = \frac{\mu}{u_0}}{\left[-\frac{1}{2}12\mu\left[\frac{\mu}{u_0}\right]u^3 + \frac{1}{2}24\mu^2 u\right] \sim 0} \tag{2.28}$$

Then we have for the precession:

$$\sigma = \left(\frac{1}{2}\mu u_0 - \frac{3}{2}\mu u_0 + 2\mu u_0 + 2\mu u^2\right) = \left(\mu u_0 + 2\mu u^2\right) \sim 3\frac{\mu}{p} \tag{2.29}$$

The units are the ratio of the advance to the orbital circumference.

Comparing with the GR value from Robertson & Noonan:

$$\sigma = \frac{1}{2}\frac{\partial}{\partial u}\left(\left(\frac{dr}{dt}\right)^2 2\frac{u}{c^2} + 2\frac{uu^2}{c^2} h^2\right) = 3\frac{u^2}{c^2} h^2 = 3\mu u\left(\frac{u}{u_0}\right) \tag{2.30}$$

Thus our procedure yields the proper perihelion precession.