The Equivalence Between Gauge and Non-Gauge Abelian Models

Gabriel Di Lemos Santiago Lima[1, *]

[1] *Centro Brasileiro de Pesquisas Físicas,*

Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil

Abstract

This work is intended to estabilish the equivalence between gauge and non-gauge abelian models.

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*Electronic address: glslima@cbpf.br

I. INTRODUCTION

In previous works, it was shown that gauge anomalous effective actions could be mapped into gauge invariant ones by some algebraic manipulations over the functional integral [1], [2]. Although such mapping is performed in the context of anomalous models, by the construction of such gauge invariant mapping it is clear that it does not need to be tied to this particular class of theories but, indeed, it may be generalized to other models that do not exhibit gauge symmetry. As the most illustrative example, we may cite the relation between the Proca model [3] and Stueckelberg's mechanism [4]. It can be shown that the second model may be derived from the first one by the same method [5].

On the other hand, at least for the particular case of abelian anomalous models, the gauge invariant mapping may raise two distinct models from the original one that reach the same gauge invariant effective action: the *standard* formulation, first proposed by Fadeev and Shatashvili [6] and derived from the referred mapping by Harada and Tsutsui [2], which includes a Wess-Zumino term into the original model; and the *enhanced* one, proposed in ref. [5], which adds up a coupling of the matter fields with the gradient of a scalar, called the Stueckelberg field due to its identification in the model proposed by Stueckelberg in ref. [4]. As the effective action of these two models is gauge invariant, one may conjecture that the anomaly disappears and, thus, that current may be conserved.

However, this being the case, such anomaly-free models coming from algebraic manipulations of the original anomalous one may raise a paradox: If one comes from the other by simple mathematical manipulations, which would mean that the two, in our case three models may be thought as essentially the same, why does it seem that one breaks current conservation while the other does not? May these models be considered as equivalent ones?

Indeed, the work of ref. [7] shows that, for the simplest case of the anomalous chiral Schwinger model [8], gauge-invariant correlation functions of the Harada and Tsutsui formulation coincide with those of the original anomalous theory. However, it was also shown that this is not the case of gauge dependent Green's functions, and that no choice of gauge condition of the standard formulation coincides with that of the original anomalous theory. This would mean that those formulations are not physically equivalent. On the other hand, it was also shown that, if in *both* formulations the gauge field is restricted to a special gauge condition, which means to *impose* current conservation to the original anomalous model, then they become equivalent. At this point, one may ask how one might impose current conservation in a non-gauge anomalous model. Is there any way that current conservation may arise from the original anomalous model? On the other hand, does the anomaly really disappear in its gauge invariant versions?

This work is intended to elucidate these questions for the case of abelian gauge models, and the relation between original abelian anomalous models, the standard formulation and the enhanced one is analyzed, as well as the relation between the Proca and Stueckelberg's models. In this sense, in section I, the *enhanced* version of Harada-Tsutsui gauge invariant mapping is derived, as well as the original *standard* one. In section II, the Stueckelberg model is derived by the enhanced mapping from the Proca one, and an analysis of both formulations shows their equivalence. In section III, the same kind of analysis is done, but comparing the enhanced version of abelian anomalous gauge models with the original ones, and it is shown that if one alternatively considers that current is conserved by the motion equation of the gauge field, as an analogue to the subsidiary condition arising in the Proca model, then both formulations become equivalent, since the first is reduced to the second by a gauge condition which turns to cancel the anomaly. The chiral Schwinger model is used as an example. It is also shown that the enhanced formulation of abelian anomalous models is free from anomalies.

Finally, the two examples analyzed lead us, naturally, to an equivalence statement related to gauge and non-gauge theories, which is done in section IV. Yet in this section, it is shown that the anomaly still remains in the standard formulation, and that this one cannot be equivalent to the other formulations. We, thus, conclude this work in section V.

II. ENHANCED VERSION OF HARADA-TSUTSUI GAUGE INVARIANT MAPPING

We consider an *anomalous* generic abelian effective action, defined by

$$\exp\left(iW[A_\mu]\right) = \int d\psi d\bar{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right), \tag{1}$$

where $I[\psi, \overline{\psi}, A_\mu]$ is invariant under local gauge transformations

$$A_\mu^\theta = A_\mu + \frac{1}{e}\partial_\mu \theta(x), \tag{2}$$

$$\psi^\theta = \exp\left(i\theta(x)\right)\psi, \tag{3}$$

$$\bar{\psi}^\theta = \exp\left(-i\theta(x)\right)\bar{\psi}, \tag{4}$$

that is,

$$I\left[\psi^\theta, \bar{\psi}^\theta, A^\theta\right] = I\left[\psi, \bar{\psi}, A\right], \tag{5}$$

while, by definition,

$$W[A_\mu^\theta] \neq W[A_\mu]. \tag{6}$$

The formulation with the addition of the Wess-Zumino term, first proposed by Fadeev and Shatashvilli [6], and then derived by Harada and Tsutsui [2], arises when one goes to the full quantum theory, by redefining the vacuum functional

$$Z = \int dAd\psi d\overline{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) = \int dA_\mu \exp\left(iW[A_\mu]\right) \tag{7}$$

multiplying it by the gauge volume

$$Z = \int d\theta dAd\psi d\overline{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) = \int d\theta dA \exp\left(iW[A_\mu]\right). \tag{8}$$

We, then, change variables in the gauge field so that

$$A_\mu \to A_\mu^\theta;\ dA_\mu \to dA_\mu^\theta, \tag{9}$$

and use translational invariance dA_μ, so that

$$dA_\mu^\theta = dA_\mu, \tag{10}$$

to reach the final gauge invariant effective action, which takes the $\theta - field$ into account, defined by

$$\exp\left(iW_{eff}[A_\mu]\right) \equiv \int d\theta \exp\left(iW[A_\mu^\theta]\right). \tag{11}$$

Using (1), it is evident that

$$\exp\left(iW[A_\mu^\theta]\right) = \int d\psi d\overline{\psi} \exp\left(iI_{st}[\psi, \overline{\psi}, A_\mu, \theta]\right), \tag{12}$$

where

$$I_{st}[\psi, \overline{\psi}, A_\mu, \theta] \equiv I[\psi, \overline{\psi}, A_\mu] + \alpha_1\left[A, \theta\right] \tag{13}$$

is called the standard action and

$$\alpha_1\,[A,\theta] \equiv W[A_\mu^\theta] - W[A_\mu] \tag{14}$$

is known as the Wess-Zumino term [9]. It can be seen that, besides the final effective action is made gauge invariant, the starting one (13) is not, since the Wess-Zumino term breaks gauge invariance. On the other hand, we may raise an alternative gauge invariant starting action by noticing that (11) can be also obtained by

$$\exp\left(iW[A_\mu^\theta]\right) = \int d\psi d\bar{\psi}\,\exp\left(iI_{en}[\psi,\overline{\psi},A_\mu,\theta]\right), \tag{15}$$

where

$$I_{en}[\psi,\overline{\psi},A_\mu,\theta] \equiv I[\psi,\overline{\psi},A_\mu^\theta]. \tag{16}$$

Moreover, this allows us to generalize the Harada-Tsutsui procedure to other non-anomalous models that do not exhibit gauge invariance by simply making the substitution $A_\mu \to A_\mu^\theta$ and integrating over the $\theta - field$. It is also evident that, to reach such a really gauge invariant formulation, we do not even need to proceed such substitution to the entire action. Indeed, one needs only to add up a gradient of a scalar to the gauge field in the parts of the initial action that does not remain gauge invariant *after* integrated out the fermions.

The inclusion of the $\theta - field$ in the enhanced formulation also transforms it into a modified gauge theory, even before the integration over the scalar. To see this, we notice that such formulation is invariant under simultaneous gauge transformations

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\Lambda \\ \theta &\to \theta - \Lambda. \end{aligned} \tag{17}$$

We shall distinguish between the scalar provided by the standard action from the one associated to the enhanced formulation, calling the first Wess-Zumino field and the second, Stueckelberg's one.

III. EQUIVALENCE BETWEEN THE PROCA AND STUECKELBERG MODELS

Consider a Proca field interacting with fermionic ones, whose action is

$$I_P[\psi,\overline{\psi},A_\mu] \equiv I_M[\psi,\overline{\psi},A_\mu] + W_P\,[A]\,, \tag{18}$$

where $I_M[\psi, \overline{\psi}, A_\mu]$ is the matter action minimally coupled to the abelian field A_μ, that exhibits local gauge symmetry, and $W_P[A]$ is the pure Proca action, defined by

$$W_P[A] \equiv \int d^n x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu \right).$$

Evidently, the action above has no gauge symmetry, since the massive term breaks it. The classical equations of motion lead us to

$$\frac{\delta I_M}{\delta \psi} = \frac{\delta I_M}{\delta \overline{\psi}} = 0 \tag{19}$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = e J^\nu, \tag{20}$$

where

$$J^\mu = -\frac{1}{e} \frac{\delta I_M}{\delta A^\mu} \tag{21}$$

is the conserved matter current obtained by global invariance. If we take the divergence of eq. (20), then we just arrive with

$$\partial_\mu A^\mu = 0 \tag{22}$$

as a *subsidiary* condition.

On the other hand, one could apply the Harada-Tsutsui gauge invariant mapping in the enhanced version, by gauge transforming only the massive part of the action to obtain

$$I_{P(en)}\left[\psi, \overline{\psi}, A, \theta\right] = I_M[\psi, \overline{\psi}, A] + W_{P(en)}[A, \theta], \tag{23}$$

where $W_{P(en)}[A]$ is just the pure enhanced Proca action, given by

$$W_{P(en)}[A, \theta] \equiv W_P\left[A^\theta\right] = -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} \int d^4x \left(A^\mu + \frac{1}{e} \partial^\mu \theta \right) \left(A_\mu + \frac{1}{e} \partial_\mu \theta \right). \tag{24}$$

It is straightfoward to notice that $W_{P(en)}[A]$ is just the Stueckelberg action. To see this, we notice that if we rename the $\theta - field$ so as

$$B(x) \equiv \frac{m}{e} \theta(x), \tag{25}$$

then (24) takes the exact form of the gauge invariant[1] Stueckelberg action [4]

$$W_{Stueck}[A, B] = -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \int d^4x \left(m A^\mu + \partial^\mu B \right) \left(m A_\mu + \partial_\mu B \right). \tag{26}$$

[1] It is invariant in the sense of Stueckelberg's gauge transformations, which takes the Scalar into account, just as in (17).

It is clear that the Stueckelberg model is reducible to the original Proca's one by the gauge choice where the Stueckelberg field is set constant. But the result that is of our interest would be to show the equivalence between Proca's model and its gauge invariant version after integrated out the $\theta - field$. To this end, at this point we may integrate (24) over the gauge orbits to find the gauge invariant version of Proca model coupled to the fermions

$$\exp\left(iI'_P\left[\psi,\bar{\psi},A\right]\right) \equiv \exp\left(iI_M\left[\psi,\bar{\psi},A\right]\right)\int d\theta \exp\left(iW_{P(en)}\left[A,\theta\right]\right). \tag{27}$$

To do this, we notice that

$$\int d\theta \exp\left(iW_{P(en)}\left[A,\theta\right]\right) = \exp\left(iW_P\left[A\right]\right)\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right), \tag{28}$$

and that

$$\begin{aligned}&\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right)\\ &= \exp\left(-\frac{i}{2}m^2\int d^nxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right)\int d\theta \exp\left(-i\frac{m^2}{2e}\int d^nx\left[\left(\frac{e}{\Box}\partial^\mu A_\mu+\theta\right)\Box\left(\frac{e}{\Box}\partial^\nu A_\nu+\theta\right)\right]\right).\end{aligned} \tag{29}$$

Performing the following change of variables in the $\theta - field$:

$$\theta \to \theta' = \theta + \frac{e}{\Box}\partial^\mu A_\mu, \tag{30}$$

it is straightforward to find

$$\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right) \sim \exp\left(-\frac{i}{2}m^2\int d^nxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right), \tag{31}$$

and, thus

$$I'_P\left[\psi,\bar{\psi},A\right] = I_M\left[\psi,\bar{\psi},A\right] + \int d^nx\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2A_\mu\left(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu\right\}. \tag{32}$$

Although we went far away going to the full quantum model to derive (32), we now use its classical version and derive the equations of motion. Then, we just obtain

$$\frac{\delta I_M}{\delta\psi} = \frac{\delta I_M}{\delta\bar{\psi}} = 0 \tag{33}$$

$$eJ^\nu = \partial_\mu F^{\mu\nu} + m^2\left(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu, \tag{34}$$

and it turns obvious that the motion equations of this gauge invariant version of massive vector model coincides with the Proca one if we fix the Lorentz gauge $\partial_\mu A^\mu = 0$, showing equivalence between both formulations. It can be seen that such gauge choice is equivalent to choose θ constant before integration over the scalar. We shall return to this point next sections.

This illustrative example is just a guideline to reach a rather more interesting and less *common sense* result presented in next section.

IV. EQUIVALENCE BETWEEN THE ORIGINAL AND ENHANCED VERSIONS OF ABELIAN ANOMALOUS MODELS

Now, we return to the anomalous generic gauge model defined in (1), where

$$I\left[\psi, \bar{\psi}, A\right] = I_{M(Ano)}\left[\psi, \bar{\psi}, A\right] + I_S\left[A\right], \tag{35}$$

with $I_{M(Ano)}\left[\psi, \bar{\psi}, A\right]$ being the anomalous matter action and $I_S\left[A\right]$ is the gauge invariant free bosonic one.

The local gauge invariance breakdown of the effective action (6) is used to be referred with current nonconservation. To understand this, we see that since the effective action is not gauge invariant we may say that we do not have the *Noether* identity $\partial_\mu\left(-\frac{1}{e}\frac{\delta W[A]}{\delta A_\mu(x)}\right) \equiv 0$, *i. e.*, identically

$$\mathcal{A} \equiv \partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right) \neq 0. \tag{36}$$

The quantity defined by (36) is used to be referred as an *anomaly.* To understand the relation between (36) and current divergence, we notice that

$$\begin{aligned}
\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right)\exp\left(iW\left[A\right]\right) &= \partial_\mu\left\{\frac{i}{e}\frac{\delta}{\delta A_\mu}\left[\exp\left(iW\left[A\right]\right)\right]\right\} \\
&= \partial_\mu\left\{\frac{i}{e}\frac{\delta}{\delta A_\mu}\left[\int d\psi d\bar{\psi}\exp\left(iI\left[\psi, \bar{\psi}, A\right]\right)\right]\right\} \\
&= \int d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi, \bar{\psi}, A\right]}{\delta A_\mu(x)}\right)\exp\left(iI\left[\psi, \bar{\psi}, A\right]\right).
\end{aligned} \tag{37}$$

Since $I_S\left[A^\theta\right] = I_S\left[A\right]$, we have

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta I_S\left[A\right]}{\delta A_\mu(x)}\right) \equiv 0, \tag{38}$$

and, therefore,

$$\int d\psi d\bar{\psi}\partial_\mu J^\mu(x)\exp\left(iI\left[\psi,\bar{\psi},A\right]\right)=\mathcal{A}\exp\left(iW\left[A\right]\right), \tag{39}$$

where

$$J^\mu(x)\equiv-\frac{1}{e}\frac{\delta I_{(Ano)M}\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)} \tag{40}$$

is the classical conserved current that may be obtained by global invariance of the action. If $\mathcal{A}$ is considered non-null, then eq. (39) means current conservation breakdown at quantum level, representing one of the most intriguing problems in quantum field theory. In this sense, to be very precise in our purposes, we *define* the anomaly by (39), generalizing it to the mean expectation value of the classical current divergence over the remaining fields beside the gauge one,

$$\int d\varphi d\psi d\bar{\psi}\partial_\mu J^\mu(x)\exp\left(iI\left[\psi,\bar{\psi},A,\varphi\right]\right)=\mathcal{A}\exp\left(iW\left[A\right]\right), \tag{41}$$

where φ represents all other fields that may enter the theory beside the ones being considered, and an *anomalous model* as being the one whose anomaly defined in (41) is not *identically* null.

Although such theories may bring theoretical problems, we may alternatively face an anomalous model as a faithful one, take the gauge field equation of motion from the effective action

$$\frac{\delta W\left[A\right]}{\delta A_\mu(x)}=0, \tag{42}$$

and, in analogy with the Proca model, obtain the nullity of anomaly as a *subsidiary* condition

$$\mathcal{A}\equiv\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right)=0. \tag{43}$$

However, such nullity of anomaly means constraints into the theory. It remains to be proved, though, the internal consistency of a theory leading with such constraints. In this sense, we shall analyze a concrete example, the anomalous chiral Schwinger model, whose action is

$$I\left[\psi,\bar{\psi},A\right]=\int d^2x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}i\gamma^\mu\left[\partial_\mu-ieA_\mu P_+\right]\psi\right\}, \tag{44}$$

where

$$P_+\equiv\frac{1}{2}\left(1+\gamma_5\right). \tag{45}$$

This action is gauge invariant and the classical conserved current obtained by its symmetry is given by

$$J^{\mu}(x) = \bar{\psi}\gamma^{\mu}P_{+}\psi. \tag{46}$$

The effective action is exactly soluble [8], and given by

$$W[A] = \int d^2x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2}{8\pi}A_{\mu}\left[ag^{\mu\nu} - (g^{\mu\alpha} + \epsilon^{\mu\alpha})\frac{\partial_{\alpha}\partial_{\beta}}{\Box}\left(g^{\beta\nu} - \epsilon^{\beta\nu}\right)\right]A_{\nu}\right\}, \tag{47}$$

where $g^{\mu\nu}$ is the $2-D$ Minkowski metric, $\epsilon^{\mu\alpha}$ is the Levi-Civita tensor and a is an arbitrary regularization parameter.

Now, it is easy to see that $W\left[A^{\theta}\right] \neq W[A]$ [2]. Indeed,

$$\begin{aligned}\alpha_1[A,\theta] &= W\left[A^{\theta}\right] - W[A] \\ &= \frac{1}{4\pi}\int d^2x\left\{\frac{1}{2}(a-1)\partial_{\mu}\theta\partial^{\mu}\theta - e\theta\left[(a-1)\partial_{\mu}A^{\mu} + \epsilon^{\mu\nu}\partial_{\mu}A_{\nu}\right]\right\}.\end{aligned} \tag{48}$$

Therefore, the chiral Schwinger model *is* anomalous, with the anomaly being

$$\mathcal{A} = -\frac{e}{4\pi}\left\{(a-1)\partial_{\mu}A^{\mu} + \epsilon^{\mu\nu}\partial_{\mu}A_{\nu}\right\}. \tag{49}$$

On the other hand, by the alternative point-of-view above explained, we may impose the variational principle to the effective action (47), and we just find the motion equation of the vector field

$$\partial_{\mu}F^{\mu\nu} + \frac{e^2}{4\pi}\left(aA^{\nu} - \frac{\partial^{\nu}\partial^{\mu}}{\Box}A_{\mu} + \epsilon^{\alpha\mu}\frac{\partial^{\nu}\partial_{\alpha}}{\Box}A_{\mu} - \epsilon^{\nu\alpha}\frac{\partial_{\alpha}\partial^{\mu}}{\Box}A_{\mu} + \epsilon^{\nu\alpha}\epsilon^{\beta\mu}\frac{\partial_{\alpha}\partial_{\beta}}{\Box}A_{\mu}\right) = 0. \tag{50}$$

Taking the divergence of (50) and using the fact that

$$\epsilon^{\mu\alpha}\epsilon^{\beta\nu} = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu}, \tag{51}$$

we just arrive with the subsidiary condition that cancels the anomaly

$$(a-1)\partial_{\mu}A^{\mu} + \epsilon^{\mu\nu}\partial_{\mu}A_{\nu} = 0. \tag{52}$$

Substituting it back to (50), it is straightforward to find the Proca gauge invariant version of the massive $2-D$ vector field's equation of motion

$$\partial_{\mu}F^{\mu\nu} + \frac{e^2}{4\pi}\frac{a^2}{(a-1)}\left(\eta^{\mu\nu} - \frac{\partial^{\nu}\partial^{\mu}}{\Box}\right)A_{\mu} = 0, \tag{53}$$

but with the vector field restricted to the condition (52).

We now turn back to the *general* case and proceed the enhanced mapping, obtaining (16). From the gauge invariance of $W_{eff}\left[A\right]$ defined in (11), and following the analogue steps that lead to eq. (37), it is straightforward to find

$$\int d\theta d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right)$$
$$=\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_\mu(x)}\right)\exp\left(iW_{eff}\left[A\right]\right). \tag{54}$$

Since $W_{eff}\left[A^\theta\right]=W_{eff}\left[A\right]$, we have the Noether identity

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_\mu(x)}\right)\equiv 0, \tag{55}$$

and thus

$$\int d\theta d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right)\equiv 0. \tag{56}$$

Since in fermionic theories the gauge fields are used to be coupled linearly to the matter ones, and the difference between A_μ and A_μ^θ is just a translation, we may be sure that

$$\frac{\delta I_{M(Ano)}\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}=\frac{\delta I_{M(Ano)}\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)}. \tag{57}$$

By (38) we obtain, therefore

$$\int d\theta d\psi d\bar{\psi}\partial_\mu J^\mu\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right)\equiv 0, \tag{58}$$

which means that the abelian enhanced formulation is anomaly-free.

As already discussed, the enhanced formulation, before integration over the scalar, may be viewed as an anomalous analogue of the Stueckelberg mechanism [5], and it obviously reduces to the original one by the gauge choice where θ is set constant. We now return to the example of chiral Schwinger model and get its enhanced version. Then we have, after integrated the fermions,

$$W\left[A^\theta\right]=\alpha_1\left[A,\theta\right]+W\left[A\right]. \tag{59}$$

Therefore, one needs only to consider the Wess-Zumino term (48) in the integration over θ. Thus,

$$\exp\left(iW_{eff}\left[A\right]\right)$$
$$=\exp\left(iW\left[A\right]\right)\int d\theta\exp\left(\frac{i}{4\pi}\int d^2x\left\{\frac{1}{2}\left(a-1\right)\partial_\mu\theta\partial^\mu\theta-e\theta\left[\left(a-1\right)\partial_\mu A^\mu+\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}\right) \tag{60}$$

Using that

$$\frac{i}{4\pi}\int d^2x\left\{\frac{1}{2}(a-1)\,\partial_\mu\theta\partial^\mu\theta-e\theta\left[(a-1)\,\partial_\mu A^\mu+\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}$$
$$=-\frac{i}{8\pi}(a-1)\int d^2x\left[\frac{1}{\Box}e\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right)\right.$$
$$\left.+\theta\right]\Box\left[\frac{1}{\Box}e\left(\partial_\alpha A^\alpha+\frac{1}{(a-1)}\epsilon^{\alpha\beta}\partial_\alpha A_\beta\right)+\theta\right]$$
$$-\frac{e^2}{\Box}\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right)\left(\partial_\alpha A^\alpha+\frac{1}{(a-1)}\epsilon^{\alpha\beta}\partial_\alpha A_\beta\right), \tag{61}$$

performing the following translation over the $\theta-field$:

$$\theta'=\theta+\frac{1}{\Box}e\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right);d\theta'=d\theta, \tag{62}$$

and proceeding integration over the new parameter θ' in (70), it is straightforward to find

$$W_{eff}\left[A\right]=W\left[A\right]+\frac{e^2}{8\pi}(a-1)\int d^2x\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right)\frac{1}{\Box}\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right). \tag{63}$$

Using (47) and (51), we finally obtain

$$W_{eff}\left[A\right]=\int d^2x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\frac{e^2}{4\pi}\frac{a^2}{(a-1)}A_\mu\left[g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right]A_\nu\right\}. \tag{64}$$

The effective action (64) is exactly the Proca $2-D$ gauge invariant action that gives the equation of the anomalous original model (53) but without the restriction (52) over the gauge field. Therefore, analogously to the *Proca/Stueckelberg* case, if we fix the gauge by (52) in the gauge invariant effective anomalous action, then the enhanced model reduces to the original anomalous one, showing equivalence between both formulations.

V. DISCUSSION

The examples mentioned above may lead us to the following statement: *A gauge theory is equivalent to a non-gauge one if the first is reducible to the second one by a gauge choice.* By the modified Stueckelberg gauge conditions point-of-view (17), the original and enhanced formulations are obviously equivalent, since the second reduces to the first by the gauge choice where the Stueckelberg scalar θ is set constant.

By the canonical gauge theory point-of-view, on the other hand, our examples show us that the integrated effective models are reducible one to the other by the Lorentz gauge choice

$$\partial_\mu A^\mu = 0 \tag{65}$$

in the Proca case, and the rather distinct one

$$(a-1)\,\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu = 0 \tag{66}$$

in the chiral Schwinger model. We see that, to achieve these gauge conditions, we have to perform the following transformations over a not restricted generic gauge field A_μ:

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda \tag{67}$$

taking the divergence of A'_μ in the Proca case in (67), we have

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \frac{1}{e}\Box \Lambda_P = 0 \tag{68}$$

which means that

$$\Lambda_P = -\frac{e}{\Box}\partial_\mu A^\mu . \tag{69}$$

Doing the same for the chiral Schwinger model and adding $\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu$, it is straightforward to find

$$\Lambda_{Sch} = -\frac{e}{\Box}\left(\partial_\mu A^\mu + \frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right). \tag{70}$$

If we compare (69) and (70) with (30) and (62), respectively, we see that the translation over the $\theta - field$ to reach the pure gauge invariant action is just

$$\theta \rightarrow \theta' = \theta - \Lambda. \tag{71}$$

This suggests that the enhanced gauge condition $\theta = const.$ that ensures equivalence between both models is transferred to the gauge fields after integrated out the Stueckelberg, as manifested in (65) and (66), in such a way that it turns to be the subsidiary conditions of the original models.

We now turn our attention to the standard formulation. The work of ref. [7], in particularly analyzing the standard version of the chiral Schwinger model, shows that its gauge invariant correlation functions coincide with those of the original anomalous theory, but it also shows that it is not the case for gauge dependent Green's functions, and no choice of gauge conditions exists for which the generating functional of the standard formulation coincides with that of the original theory. The conclusion is, thus, that its physical contents are quite different. However, it was also shown that the action with the addition of the Wess-Zumino term is equivalent to the original anomalous model if the gauge condition (66) is *imposed* to both models. As it was shown, this condition may arise from the original model as a subsidiary condition. On the other hand, besides the final effective action is made gauge invariant, the starting one is not, since the Wess-Zumino term breaks it. To understand what it means, we consider a model with the standard action (13) and try to obtain the conserved quantity given by the gauge invariance of the effective action, it is straightforward to find

$$\begin{aligned} &\partial_\mu \left(-\frac{1}{e} \frac{\delta W_{eff}[A]}{\delta A_\mu(x)} \right) \exp\left(iW_{eff}[A] \right) \\ &= \int d\theta d\psi d\bar{\psi} \partial_\mu \left(-\frac{1}{e} \frac{\delta I_{st}\left[\psi, \bar{\psi}, A, \theta\right]}{\delta A_\mu(x)} \right) \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right) = 0 \end{aligned} \tag{72}$$

or, by (13) and (40)

$$\int d\theta d\psi d\bar{\psi} \partial_\mu J^\mu \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right) = \int d\theta d\psi d\bar{\psi} \partial_\mu \left(\frac{1}{e} \frac{\delta \alpha_1[A, \theta]}{\delta A_\mu(x)} \right) \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right). \tag{73}$$

If we integrate the right hand side of (73) and use (55), we will just obtain

$$\int d\theta d\psi d\bar{\psi} \partial_\mu J^\mu \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right) = \mathcal{A} \exp\left(iW_{eff}[A] \right), \tag{74}$$

with $\mathcal{A}$ given by (36), which means, by our definition (41), that the standard formulation is still anomalous. We can notice that in this model, unlike the original anomalous one, no subsidiary condition arises in order to cancel the anomaly. To be more precise, a kind of subsidiary condition arises if we use the Dyson-Schwinger equation for the $\theta - field$. To see this, we writte

$$\begin{aligned} &\int d\theta d\psi d\bar{\psi} \frac{\delta I_{st}}{\delta \theta} \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right) \\ &= \int d\theta d\psi d\bar{\psi} \frac{\delta \alpha_1}{\delta \theta}[A, \theta] \exp\left(iI_{st}\left[\psi, \bar{\psi}, A, \theta\right] \right) = 0, \end{aligned} \tag{75}$$

but,

$$\frac{\delta\alpha_1}{\delta\theta}[A,\theta]=\frac{\delta W}{\delta\theta}\left[A^\theta\right]=\int d^ny\frac{\delta W\left[A^\theta\right]}{\delta A_\mu^\theta(y)}\frac{\delta A_\mu^\theta(y)}{\delta\theta(x)}$$
$$=\int d^nx\frac{1}{e}\frac{\delta W\left[A^\theta\right]}{\delta A_\mu^\theta}\partial_\mu\delta(x-y)=\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A^\theta\right]}{\delta A_\mu^\theta}\right)=\mathcal{A}^\theta \tag{76}$$

and, therefore

$$\int d\theta d\psi d\bar{\psi}\mathcal{A}^\theta\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=0. \tag{77}$$

We see that, if the anomaly is made gauge invariant, which means to set $a=1$ in the chiral Schwinger model, then the left hand side of (77) reduces to (74) and the anomaly cancels out. However, in our simplest example, the choice $a=1$ represents a gauge parameter (70), to be used in order to integrate the scalar by the translation in (62), which is infinite. It is easy to see, by eq. (60), that such a choice also represents a functional Dirac delta that has the anomaly as its parameter, that is, if $a=1$, then

$$\exp\left(iW_{eff}[A]\right)$$
$$=\exp\left(iW[A]\right)\int d\theta\exp\left(-\frac{i}{4\pi}\int d^2x\left\{e\theta\epsilon^{\mu\nu}\partial_\mu A_\nu\right\}\right)$$
$$=\delta\left(\mathcal{A}[A]\right)\exp\left(iW[A]\right). \tag{78}$$

On the other hand, a distinct value of a clearly turns the anomaly cancellation impossible. Moreover, the condition that turns to cancel a gauge invariant anomaly is not an allowed gauge choice, as required in order to ensure the equivalence between a gauge and a non-gauge model.

Although the anomaly may not be invariant, the final effective action is still made gauge invariant. In this sense, if we were allowed to "choose" a gauge condition such as $\mathcal{A}=0$, then the anomaly would cancel out, the current would become conserved and the standard formulation would turn to be the original one. We may also see that the standard formulation reduces to the original one if we set $\theta=const$. However, obviously the situation imposed by such condition is not physically equivalent to leaving the anomaly non-null, since we have no current conservation in one situation, and have it conserved in another one. Therefore, we have a breaking of the physical equivalence between distinct gauge configurations in the final gauge invariant effective action of the standard formulation by the fact that the standard action is not gauge invariant. This may also be related to the fact that the only way to reduce one model to the other is by imposing the nullity of a gauge invariant anomaly, which is not an equivalent configuration. These considerations may explain the results found in ref. [7].

VI. CONCLUSION

This work has shown up a rather contra-intuitive idea: that a gauge invariant model may be equivalent to a non-invariant one. Besides being against the comon sense, it is perfectly possible, as it was shown, as long as one is reduced to the other by some gauge condition. The strangeness about this result, nevertheless, may be related to the idea that current conservation is due to gauge symmetry. However, at least in the classical case, Noether theorem ensures current conservation through *global* gauge invariance and the variational principle, instead of a rather stronger condition, which is local gauge symmetry, as it becomes evident in the Proca case. Work is in progress to clarify the relation between local gauge symmetry and current conservation in the context of quantum models.

On the other hand, this idea becomes manifest by an interesting procedure of recovering gauge symmetry from non-gauge theories, that can be thought as a generalization of the Stueckelberg mechanism. It is well-known that the Stueckelberg model is renormalizable and unitary by a special gauge choice, making possible the quantization of a massive abelian vector model with no Higgs boson [10]∼[12], . By bringing this idea to the context of abelian anomalous models, by means of the enhanced Harada-Tsutsui procedure, the freedom of choosing conditions over the gauge field and the Stueckelberg scalar may also open the possibility of quantizing models with non-trivial fermionic Jacobian.

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